# A1 Classical primitives: computations and methods

For a function $f : I \to \mathbb{R}$ defined on a connected interval of $I$, let us denote by $F$ one of its primitives. We recall that primitives are not unique, but defined up to an additive constant. In the following table, you may find the primitives of some elementary functions.

| $f(x)$ | $F(x)$ |
|---|---|
| $x^\alpha, \alpha \neq -1$ | $\frac{x^{\alpha+1}}{\alpha+1}$ |
| $x^{-1}$ | $\ln\lvert x\rvert$ |
| $e^x$ | $e^x$ |
| $\sin x$ | $-\cos x$ |
| $\cos x$ | $\sin x$ |

| $f(x)$ | $F(x)$ |
|---|---|
| $\frac{1}{\cos^2 x}$ | $\tan x$ |
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\frac{1}{\sqrt{1-x^2}}$ | $\arcsin x$ |
| $\frac{1}{x^2+1}$ | $\arctan x$ |

Classical methods to compute primitives include change of variables (Proposition 5.3.33), integration by parts (Proposition 5.2.11), by induction (Exercise A1.2), etc. We present some other common methods below, depending on the form that the integrand takes.

## A1.1 Rational fractions

Let $f$ be a rational fraction over $\mathbb{R}$, that is $f = \frac{P}{Q} \in \mathbb{R}(X)$, where $P, Q \in \mathbb{R}[X]$ and $Q \neq 0$. It can be viewed as a map $x \mapsto f(x)$, which is well defined for $x \in \mathbb{R}$ with $Q(x) \neq 0$. The polynomial $Q$ can be factorized in $\mathbb{R}[X]$ as a product of irreducible polynomials, that is,

$$Q(x) = \text{cst} \cdot \prod_{i=1}^{m} (x^2 + b_i x + c_i)^{m_i} \prod_{j=1}^{n} (x - r_j)^{n_j},$$

where $m, n \in \mathbb{N}_0$ and

- for every $1 \leqslant i \leqslant m$, we have $b_i, c_i \in \mathbb{R}$ satisfying $b_i^2 - 4c_i < 0$ and $m_i \in \mathbb{N}$,
- for every $1 \leqslant j \leqslant n$, we have $r_j \in \mathbb{R}$ and $n_j \in \mathbb{N}$.

Therefore, we may decompose $f$ into partial fractions,

$$\frac{P(x)}{Q(x)} = T(x) + \sum_{i=1}^{m} \frac{M_i(x)}{(x^2 + b_i x + c_i)^{m_i}} + \sum_{j=1}^{n} \frac{N_i(x)}{(x - r_j)^{n_j}},$$

where

- $T \in \mathbb{R}[X]$ is a polynomial,
- for every $1 \leqslant i \leqslant m$, $M_i \in \mathbb{R}[X]$ is a polynomial of degree $\deg(M_i) \leqslant 2m_i - 1$,
- for every $1 \leqslant j \leqslant n$, $N_i \in \mathbb{R}[X]$ is a polynomial of degree $\deg(N_j) \leqslant n_j - 1$.

# 附錄一 經典原函數：計算與方法

對於定義在連通區間 $I$ 上的函數 $f : I \to \mathbb{R}$，我們把他的其中一個原函數記作 $F$。不要忘記，原函數並沒有唯一性，兩個相同函數的原函數只會相差一個加法常數。在下面表格，你可以找到一些基礎函數的原函數。

| $f(x)$ | $F(x)$ |
|---|---|
| $x^\alpha, \alpha \neq -1$ | $\frac{x^{\alpha+1}}{\alpha+1}$ |
| $x^{-1}$ | $\ln\lvert x\rvert$ |
| $e^x$ | $e^x$ |
| $\sin x$ | $-\cos x$ |
| $\cos x$ | $\sin x$ |

| $f(x)$ | $F(x)$ |
|---|---|
| $\frac{1}{\cos^2 x}$ | $\tan x$ |
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\frac{1}{\sqrt{1-x^2}}$ | $\arcsin x$ |
| $\frac{1}{x^2+1}$ | $\arctan x$ |

計算原函數的經典方法當中有：變數變化（命題 5.3.33）、分部積分（命題 5.2.11）、數學歸納法（習題 A1.2）等等。下面我們會根據被積分函數的長相，介紹一些其他常見的方法。

## 第一節 有理函數

令 $f$ 為 $\mathbb{R}$ 上的有理函數，也就是說 $f = \frac{P}{Q} \in \mathbb{R}(X)$，其中 $P, Q \in \mathbb{R}[X]$ 而且 $Q \neq 0$。他可以被看作函數 $x \mapsto f(x)$，只有當 $x \in \mathbb{R}$ 且 $Q(x) \neq 0$ 時是有定義的。多項式 $Q$ 可以在 $\mathbb{R}[X]$ 中分解為不可約多項式的乘積，也就是說：

$$Q(x) = \text{cst} \cdot \prod_{i=1}^{m} (x^2 + b_i x + c_i)^{m_i} \prod_{j=1}^{n} (x - r_j)^{n_j},$$

其中 $m, n \in \mathbb{N}_0$，而且

- 對於每個 $1 \leqslant i \leqslant m$，我們有 $b_i, c_i \in \mathbb{R}$ 滿足 $b_i^2 - 4c_i < 0$ 和 $m_i \in \mathbb{N}$；
- 對於每個 $1 \leqslant j \leqslant n$，我們有 $r_j \in \mathbb{R}$ 且 $n_j \in \mathbb{N}$。

因此，我們可以把 $f$ 分解成部份分式：

$$\frac{P(x)}{Q(x)} = T(x) + \sum_{i=1}^{m} \frac{M_i(x)}{(x^2 + b_i x + c_i)^{m_i}} + \sum_{j=1}^{n} \frac{N_i(x)}{(x - r_j)^{n_j}},$$

其中

- $T \in \mathbb{R}[X]$ 是個多項式；

This can be further rewritten as

$$\frac{P(x)}{Q(x)} = T(x) + \sum_{i=1}^{m}\sum_{r=1}^{m_i} \frac{M_{i,r}(x)}{(x^2+b_ix+c_i)^r} + \sum_{j=1}^{n}\sum_{s=1}^{n_j} \frac{N_{i,s}}{(x-r_j)^s},$$

where

- $T \in \mathbb{R}[X]$ is a polynomial,
- for every $1 \leqslant i \leqslant m$ and $1 \leqslant r \leqslant m_i$, $M_{i,r} \in \mathbb{R}[X]$ is a polynomial of degree at most $1$,
- for every $1 \leqslant j \leqslant n$ and $1 \leqslant s \leqslant n_j$, $N_{i,s} \in \mathbb{R}[X]$ is a polynomial of degree at most zero, that is a constant.

Therefore, it is enough to determine the primitives of rational fractions of the forms,

$$\frac{1}{(x-r)^n}, n \in \mathbb{N} \quad \text{and} \quad \frac{sx+t}{(x^2+bx+c)^m}, b^2-4c<0, \quad m \in \mathbb{N}.$$

For the first type of rational fraction, we easily find

$$\int \frac{\mathrm{d}x}{(x-r)^n} = \begin{cases} \frac{1}{(1-n)(x-r)^{n-1}} + \text{cst} & \text{if } n \neq 1, \\ \ln|x-r| + \text{cst} & \text{if } n = 1. \end{cases}$$

For the second type, we rewrite in the following form

$$\frac{sx+t}{(x^2+bx+c)^m} = \frac{2\alpha(x-p)}{\left[(x-p)^2+q^2\right]^m} + \frac{\beta}{\left[(x-p)^2+q^2\right]^m}$$

The first term has primitives in the following form,

$$\int \frac{2\alpha(x-p)}{\left[(x-p)^2+q^2\right]^m}\,\mathrm{d}x = \begin{cases} \frac{\alpha}{(1-m)\left[(x-p)^2+q^2\right]^{m-1}} + \text{cst} & \text{if } m \geqslant 2, \\ \alpha \ln\left[(x-p)^2+q^2\right] + \text{cst} & \text{if } m = 1. \end{cases}$$

For the second term, we perform a change of variables $x - p = q\tan\theta$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then, we find

$$\int \frac{\beta}{\left[(x-p)^2+q^2\right]^m}\,\mathrm{d}x = \frac{\beta}{q^{2m-1}} \int \cos^{2m-2}\theta\,\mathrm{d}\theta,$$

which can be evaluated recursively (see Wallis' integral in Exercise A1.2), or Section A1.2.

**Example A1.1.1 :** Find a primitive of $\frac{x}{(x^2+x+1)^2}$. We write

$$\frac{x}{(x^2+x+1)^2} = \frac{x+\frac{1}{2}}{((x+\frac{1}{2})^2+\frac{3}{4})^2} - \frac{1}{2}\frac{1}{((x+\frac{1}{2})^2+\frac{3}{4})^2}$$

- 對於每個 $1 \leqslant i \leqslant m$，$M_i \in \mathbb{R}[X]$ 是個度數滿足 $\deg(M_i) \leqslant 2m_i - 1$ 的多項式；
- 對於每個 $1 \leqslant j \leqslant n$，$N_i \in \mathbb{R}[X]$ 是個度數滿足 $\deg(N_j) \leqslant n_j - 1$ 的多項式。

這還可以被重新寫成

$$\frac{P(x)}{Q(x)} = T(x) + \sum_{i=1}^{m}\sum_{r=1}^{m_i} \frac{M_{i,r}(x)}{(x^2+b_ix+c_i)^r} + \sum_{j=1}^{n}\sum_{s=1}^{n_j} \frac{N_{i,s}}{(x-r_j)^s},$$

其中

- $T \in \mathbb{R}[X]$ 是個多項式；
- 對於每個 $1 \leqslant i \leqslant m$ 還有 $1 \leqslant r \leqslant m_i$，$M_{i,r} \in \mathbb{R}[X]$ 是個度數最多為 $1$ 的多項式；
- 對於每個 $1 \leqslant j \leqslant n$ 還有 $1 \leqslant s \leqslant n_j$，$N_{i,s} \in \mathbb{R}[X]$ 是個度數最多為 $0$ 的多項式，也就是說是個常數。

因此，我們只需要找出下面這些類型有理函數的原函數即可：

$$\frac{1}{(x-r)^n}, n \in \mathbb{N} \quad \text{以及} \quad \frac{sx+t}{(x^2+bx+c)^m}, b^2-4c<0, \quad m \in \mathbb{N}.$$

對於第一種類型的有理函數，我們不難得到：

$$\int \frac{\mathrm{d}x}{(x-r)^n} = \begin{cases} \frac{1}{(1-n)(x-r)^{n-1}} + \text{cst} & \text{若 } n \neq 1, \\ \ln|x-r| + \text{cst} & \text{若 } n = 1. \end{cases}$$

對於第二種類型的，我們把他改寫成下列形式：

$$\frac{sx+t}{(x^2+bx+c)^m} = \frac{2\alpha(x-p)}{\left[(x-p)^2+q^2\right]^m} + \frac{\beta}{\left[(x-p)^2+q^2\right]^m}$$

第一項的原函數可以寫做：

$$\int \frac{2\alpha(x-p)}{\left[(x-p)^2+q^2\right]^m}\,\mathrm{d}x = \begin{cases} \frac{\alpha}{(1-m)\left[(x-p)^2+q^2\right]^{m-1}} + \text{cst} & \text{若 } m \geqslant 2, \\ \alpha \ln\left[(x-p)^2+q^2\right] + \text{cst} & \text{若 } m = 1. \end{cases}$$

針對第二像，我們先使用變數變換 $x - p = q\tan\theta$ 其中 $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$。這會讓我們得到：

$$\int \frac{\beta}{\left[(x-p)^2+q^2\right]^m}\,\mathrm{d}x = \frac{\beta}{q^{2m-1}} \int \cos^{2m-2}\theta\,\mathrm{d}\theta,$$

這可以透過歸納法來求值（參見習題 A1.2 中的 Wallis 積分），或是第 A1.2 節。

**範例 A1.1.1 ：** 求 $\frac{x}{(x^2+x+1)^2}$ 的一個原函數。我們寫

$$\frac{x}{(x^2+x+1)^2} = \frac{x+\frac{1}{2}}{((x+\frac{1}{2})^2+\frac{3}{4})^2} - \frac{1}{2}\frac{1}{((x+\frac{1}{2})^2+\frac{3}{4})^2}$$

From what we discussed above, we have

$$\int \frac{x+\frac{1}{2}}{((x+\frac{1}{2})^2+\frac{3}{4})^2}\,\mathrm{d}x = -\frac{1}{2}\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} + \mathrm{cst} = -\frac{1}{2}\frac{1}{x^2+x+1} + \mathrm{cst},$$

and, by setting $x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, we have

$$\begin{aligned}\int \frac{1}{((x+\frac{1}{2})^2+\frac{3}{4})^2}\,\mathrm{d}x &= \Big(\frac{2}{\sqrt{3}}\Big)^3 \int \cos^2\theta\,\mathrm{d}\theta = \Big(\frac{2}{\sqrt{3}}\Big)^3 \int \frac{1+\cos(2\theta)}{2}\,\mathrm{d}\theta \\ &= \frac{4}{3\sqrt{3}}(\theta + \frac{1}{2}\sin(2\theta)) + \mathrm{cst} \\ &= \frac{4}{3\sqrt{3}}(\theta + \frac{\tan\theta}{1+\tan^2\theta}) + \mathrm{cst} \\ &= \frac{4}{3\sqrt{3}}\arctan\Big(\frac{2x+1}{\sqrt{3}}\Big) + \frac{2}{3}\frac{x+\frac{1}{2}}{x^2+x+1} + \mathrm{cst}.\end{aligned}$$

Therefore, we have

$$\int \frac{x}{(x^2+x+1)^2}\,\mathrm{d}x = -\frac{2}{3\sqrt{3}}\arctan\Big(\frac{2x+1}{\sqrt{3}}\Big) - \frac{1}{3}\frac{x+2}{x^2+x+1} + \mathrm{cst}.$$

## A1.2 Polynomials in sine and cosine functions

Let $m, n \in \mathbb{N}_0$, and we want to find primitives of $\int \sin^m x \cos^n x\,\mathrm{d}x$.

- If $m = 2k+1$ is odd, then we write

$$\int \sin^m x \cos^n x\,\mathrm{d}x = \int (1-\cos^2 x)^k \cos^n x \sin x\,\mathrm{d}x = -\int (1-\cos^2 x)^k \cos^n x\,\mathrm{d}\cos x,$$

  and the change of variables $t = \cos x$ allows us to conclude.
- If $n = 2\ell+1$ is odd, then we write

$$\int \sin^m x \cos^n x\,\mathrm{d}x = \int \sin^m x(1-\sin^2 x)^\ell \cos x\,\mathrm{d}x = \int \sin^m x(1-\sin^2 x)^\ell\,\mathrm{d}\sin x,$$

- If both $m$ and $n$ are even, we use the following identities

$$\cos x = \frac{1}{2}(e^{\mathrm{i}\,x} + e^{-\mathrm{i}\,x}) \quad \text{and} \quad \sin x = \frac{1}{2\,\mathrm{i}}(e^{\mathrm{i}\,x} - e^{-\mathrm{i}\,x}),$$

  which simplifies the integrand into a polynomial in $e^x$, whose primitive is easy to compute.

Note that in the first two cases, it is also possible to apply the method in the last case, but the resulting polynomial would be less nicer to deal with than the direct method mentioned above.



從上面的討論，我們有

$$\int \frac{x+\frac{1}{2}}{((x+\frac{1}{2})^2+\frac{3}{4})^2}\,\mathrm{d}x = -\frac{1}{2}\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} + \mathrm{cst} = -\frac{1}{2}\frac{1}{x^2+x+1} + \mathrm{cst},$$

此外，設 $x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$，我們得到

$$\begin{aligned}\int \frac{1}{((x+\frac{1}{2})^2+\frac{3}{4})^2}\,\mathrm{d}x &= \Big(\frac{2}{\sqrt{3}}\Big)^3 \int \cos^2\theta\,\mathrm{d}\theta = \Big(\frac{2}{\sqrt{3}}\Big)^3 \int \frac{1+\cos(2\theta)}{2}\,\mathrm{d}\theta \\ &= \frac{4}{3\sqrt{3}}(\theta + \frac{1}{2}\sin(2\theta)) + \mathrm{cst} \\ &= \frac{4}{3\sqrt{3}}(\theta + \frac{\tan\theta}{1+\tan^2\theta}) + \mathrm{cst} \\ &= \frac{4}{3\sqrt{3}}\arctan\Big(\frac{2x+1}{\sqrt{3}}\Big) + \frac{2}{3}\frac{x+\frac{1}{2}}{x^2+x+1} + \mathrm{cst}.\end{aligned}$$

因此，我們有

$$\int \frac{x}{(x^2+x+1)^2}\,\mathrm{d}x = -\frac{2}{3\sqrt{3}}\arctan\Big(\frac{2x+1}{\sqrt{3}}\Big) - \frac{1}{3}\frac{x+2}{x^2+x+1} + \mathrm{cst}.$$

## 第二節 正弦和餘弦函數的多項式

令 $m, n \in \mathbb{N}_0$，我們想找 $\int \sin^m x \cos^n x\,\mathrm{d}x$ 的原函數。

- 如果 $m = 2k+1$ 是奇數，那麼我們這樣算：

$$\int \sin^m x \cos^n x\,\mathrm{d}x = \int (1-\cos^2 x)^k \cos^n x \sin x\,\mathrm{d}x = -\int (1-\cos^2 x)^k \cos^n x\,\mathrm{d}\cos x,$$

  然後我們可以透過變數變換 $t = \cos x$ 總結。
- 如果 $n = 2\ell+1$ 是奇數，那麼我們這樣算：

$$\int \sin^m x \cos^n x\,\mathrm{d}x = \int \sin^m x(1-\sin^2 x)^\ell \cos x\,\mathrm{d}x = \int \sin^m x(1-\sin^2 x)^\ell\,\mathrm{d}\sin x,$$

- 如果 $m$ 和 $n$ 都是偶數，我們使用下列關係式：

$$\cos x = \frac{1}{2}(e^{\mathrm{i}\,x} + e^{-\mathrm{i}\,x}) \quad 以及 \quad \sin x = \frac{1}{2\,\mathrm{i}}(e^{\mathrm{i}\,x} - e^{-\mathrm{i}\,x}),$$

  這讓我們可以把被積分函數化簡為 $e^x$ 的多項式，他的原函數是好計算的。

我們注意到，在最前面的兩點中，我們也可以使用第三點的方式來計算，但這樣的計算會比較繁瑣一點。

## A1.3 Rational fractions in sine and cosine functions

Let $R \in \mathbb{R}(X,Y)$ be a rational fraction in two variables, and we want to find a primitive of $R(\sin x, \cos x)$. We apply the change of variables $t = \tan(\frac{x}{2})$, and note that

$$\sin x = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} \quad \text{and} \quad \cos x = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}.$$

This allows us to rewrite

$$\int R(\sin x, \cos x)\,\mathrm{d}x = \int R\Big(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\Big)\frac{2\,\mathrm{d}t}{1+t^2},$$

whose integrand is a rational fraction in $t$, so we can apply the method from Section A1.1.

**Example A1.3.1 :** Find a primitive of $\frac{1}{\sin x}$. We consider the change of variables $t = \tan(\frac{x}{2})$, and we have

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{1+t^2}{2t}\frac{2\,\mathrm{d}t}{1+t^2} = \int \frac{\mathrm{d}t}{t} = \ln|t| + \mathrm{cst} = \ln\Big|\tan\Big(\frac{x}{2}\Big)\Big| + \mathrm{cst}.$$

## A1.4 Rational fractions in hyperbolic sine and hyperbolic cosine functions

Let $R \in \mathbb{R}(X,Y)$ be a rational fraction in two variables, and we want to find a primitive of $R(\sinh x, \cosh x)$. We may apply several different methods, depending on the integrand, among which the most common ones are the following.

- We apply the change of variables $t = \tanh(\frac{x}{2})$, and note that

$$\sinh x = \frac{2\tanh(\frac{x}{2})}{1-\tanh^2(\frac{x}{2})} \quad \text{and} \quad \cosh x = \frac{1+\tanh^2(\frac{x}{2})}{1-\tanh^2(\frac{x}{2})}.$$

  Then, we reduce to the case where the integrand is a rational fraction in $t$.
- We apply the change of variables $t = e^x$, and the integrand becomes a rational fraction in $t$.

## A1.5 Ration fractions in $e^x$

If $f$ is a rational fraction in $e^x$, that is $f(x) = R(e^x)$ where $R \in \mathbb{R}(X)$, then by the change of variables $t = e^x$, we find

$$\int f(x)\,\mathrm{d}x = \int R(e^x)\,\mathrm{d}x = \int \frac{R(t)}{t}\,\mathrm{d}t,$$

which is a rational fraction in $t$.



## 第三節 正弦和餘弦函數的有理函數

令 $R \in \mathbb{R}(X,Y)$ 為雙變數的有理函數，我們想找 $R(\sin x, \cos x)$ 的原函數。透過變數變換 $t = \tan(\frac{x}{2})$ 並注意到下面關係式：

$$\sin x = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} \quad 以及 \quad \cos x = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})},$$

我們可以得到：

$$\int R(\sin x, \cos x)\,\mathrm{d}x = \int R\Big(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\Big)\frac{2\,\mathrm{d}t}{1+t^2},$$

當中的被積分函數是個 $t$ 的有理函數，因此我們可以使用第 A1.1 節當中提到的方法。

**範例 A1.3.1 ：** 找 $\frac{1}{\sin x}$ 的原函數。透過變數變換 $t = \tan(\frac{x}{2})$，我們得到

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{1+t^2}{2t}\frac{2\,\mathrm{d}t}{1+t^2} = \int \frac{\mathrm{d}t}{t} = \ln|t| + \mathrm{cst} = \ln\Big|\tan\Big(\frac{x}{2}\Big)\Big| + \mathrm{cst}.$$

## 第四節 雙曲正弦和雙區餘弦函數的有理函數

令 $R \in \mathbb{R}(X,Y)$ 為雙變數有理函數，我們想要找 $R(\sinh x, \cosh x)$ 的原函數。根據被積分函數的形式，我們可以使用不同方法。比較常使用的技巧如下：

- 透過變數變換 $t = \tanh(\frac{x}{2})$ 並注意到

$$\sinh x = \frac{2\tanh(\frac{x}{2})}{1-\tanh^2(\frac{x}{2})} \quad 以及 \quad \cosh x = \frac{1+\tanh^2(\frac{x}{2})}{1-\tanh^2(\frac{x}{2})}.$$

  這樣被積分函數會變成 $t$ 的有理函數。
- 我們使用變數變換 $t = e^x$，這樣被積分函數會變成 $t$ 的有理函數。

## 第五節 $e^x$ 的有理函數

如果 $f$ 是個 $e^x$ 的有理函數，也就是 $f(x) = R(e^x)$ 其中 $R \in \mathbb{R}(X)$，那麼使用變數變換 $t = e^x$，我們會有

$$\int f(x)\,\mathrm{d}x = \int R(e^x)\,\mathrm{d}x = \int \frac{R(t)}{t}\,\mathrm{d}t,$$

這會是個 $t$ 的有理函數。

**Example A1.5.1 :** Find a primitive of $\frac{1}{\cosh x}$. We consider the change of variables $t = e^x$, and find

$$\int \frac{\mathrm{d}x}{\cosh x} = \int \frac{2\,\mathrm{d}x}{e^x + e^{-x}} = \int \frac{2\,\mathrm{d}t}{t}\frac{1}{t+\frac{1}{t}} = \int \frac{2\,\mathrm{d}t}{1+t^2} = 2\arctan(e^x) + \text{cst}.$$

## A1.6 Abelian integrals of first type

Let $R \in \mathbb{R}(X,Y)$ be a rational fraction in two variables, and we want to find a primitive of

$$R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right), \quad n \in \mathbb{N}.$$

- If $ad - bc = 0$, then the $n$-th root does not depend on $x$, and the computation is easy.
- If $ad - bc \neq 0$, we consider the change of variables

$$t = \sqrt[n]{\frac{ax+b}{cx+d}} \quad \text{and} \quad x = g(t) = \frac{dt^n - b}{a - t^n c}.$$

This allows us to rewrite

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) \mathrm{d}x = \int R(g(t), t) g'(t)\,\mathrm{d}t,$$

whose integrand is a rational fraction in $t$.

**Example A1.6.1 :** We want to compute

$$\int \frac{\mathrm{d}x}{\sqrt{1+x} + \sqrt[3]{1+x}}.$$

We consider the change of variables $t = \sqrt[6]{1+x}$, and we have

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{1+x} + \sqrt[3]{1+x}} &= \int \frac{6t^5\,\mathrm{d}t}{t^3+t^2} = 6\int \left(t^2 - t + 1 - \frac{1}{1+t}\right)\mathrm{d}t \\ &= 2t^3 - 3t^2 + 6t - 6\ln|1+t| + \text{cst} \\ &= 2\sqrt{1+x} - 3\sqrt[3]{1+x} + 6\sqrt[6]{1+x} - 6\ln|1+\sqrt[6]{1+x}| + \text{cst}.\end{aligned}$$

## A1.7 Abelian integrals of second type



**範例 A1.5.1 ：** 求 $\frac{1}{\cosh x}$ 的原函數。我們考慮變數變換 $t = e^x$，這會給出

$$\int \frac{\mathrm{d}x}{\cosh x} = \int \frac{2\,\mathrm{d}x}{e^x + e^{-x}} = \int \frac{2\,\mathrm{d}t}{t}\frac{1}{t+\frac{1}{t}} = \int \frac{2\,\mathrm{d}t}{1+t^2} = 2\arctan(e^x) + \text{cst}.$$

## 第六節 第一型 Abel 積分

令 $R \in \mathbb{R}(X,Y)$ 為雙變數有理函數，我們想要找

$$R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right), \quad n \in \mathbb{N}$$

的原函數。

- 如果 $ad - bc = 0$，那麼 $n$ 次方根不取決於 $x$，這情況下的計算是簡單的。
- 如果 $ad - bc \neq 0$，我們考慮下面這個變數變換：

$$t = \sqrt[n]{\frac{ax+b}{cx+d}} \quad \text{以及} \quad x = g(t) = \frac{dt^n - b}{a - t^n c}.$$

這讓我們可以改寫：

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) \mathrm{d}x = \int R(g(t), t) g'(t)\,\mathrm{d}t,$$

當中的被積分函數是 $t$ 的有理函數。

**範例 A1.6.1 ：** 我們想計算

$$\int \frac{\mathrm{d}x}{\sqrt{1+x} + \sqrt[3]{1+x}}.$$

我們考慮變數變換 $t = \sqrt[6]{1+x}$，這樣我們有

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{1+x} + \sqrt[3]{1+x}} &= \int \frac{6t^5\,\mathrm{d}t}{t^3+t^2} = 6\int \left(t^2 - t + 1 - \frac{1}{1+t}\right)\mathrm{d}t \\ &= 2t^3 - 3t^2 + 6t - 6\ln|1+t| + \text{cst} \\ &= 2\sqrt{1+x} - 3\sqrt[3]{1+x} + 6\sqrt[6]{1+x} - 6\ln|1+\sqrt[6]{1+x}| + \text{cst}.\end{aligned}$$

## 第七節 第二型 Abel 積分

Let $R \in \mathbb{R}(X,Y)$ be a rational fraction in two variables, and we want to find a primitive of $R(x, \sqrt{ax^2+bx+c})$.

- If $a = 0$, this reduces to the case in Section A1.6.
- If $a \neq 0$, but $b^2 - 4ac = 0$, the square root gets factorzied and we have a rational fraction in $x$, and we proceed as in Section A1.1.

Therefore, we may assume that $a \neq 0$ and $b^2 - 4ac \neq 0$.

- First, we look at the case where $b^2 - 4ac > 0$.
  - For $a < 0$, we rewrite
    $$\sqrt{ax^2+bx+c} = \sqrt{-a}\sqrt{q^2-(x-p)^2},$$
    for some $p, q \in \mathbb{R}$. Then, the change of variables $x - p = q\cos\theta$ allows us to obtain
    $$\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x = \int R(p+q\cos\theta, q\sqrt{-a}\sin\theta)\,\mathrm{d}\theta.$$
    Then it simplifies to the computation of a primitive of rational fractions in sine and cosine functions, see Section A1.3.
  - For $a > 0$, we rewrite
    $$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{(x-p)^2-q^2},$$
    for some $p, q \in \mathbb{R}$. We consider the change of variables $x - p = q\varepsilon\cosh(t)$, where $\varepsilon \in \{1, -1\}$, chosen depending on the interval where we look for a primitive. This allows us to obtain
    $$\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x = \int R(p+q\cosh t, q\sqrt{a}\sinh t)\,\mathrm{d}t.$$
    Then it simplifies to the computation of a primitive of rational fractions in hyperbolic sine and hyperbolic cosine functions.
- For the case where $b^2 - 4ac < 0$, we need to have $a > 0$. We rewrite
  $$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{(x-p)^2+q^2},$$
  Then, the change of variables $x - p = q\sinh\theta$ allows us to obtain
  $$\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x = \int R(p+q\sinh t, q\sqrt{a}\cosh t)\,\mathrm{d}t.$$
  Then it simplifies to the computation of a primitive of rational fractions in hyperbolic sine and hyperbolic cosine functions.

**Example A1.7.1 :** We want to find a primitive of $\frac{1}{\sqrt{1-x^2}}$ and $\sqrt{1-x^2}$ on $(-1,1)$. From above, we consider the change of variables $x = \cos\theta$, so we have
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int \frac{-\sin\theta}{\sin\theta}\,\mathrm{d}\theta = -\int \mathrm{d}\theta = -\theta + \mathrm{cst},$$



令 $R \in \mathbb{R}(X,Y)$ 為雙變數有理函數，我們想要找 $R(x, \sqrt{ax^2+bx+c})$ 的原函數。

- 如果 $a = 0$，我們回到第 A1.6 節的情況。
- 如果 $a \neq 0$，但 $b^2 - 4ac = 0$，平方根可以被化簡，我們會直接得到 $x$ 的有理函數，接著我們使用第 A1.1 節中的方法即可。

因此，我們可以假設 $a \neq 0$ 以及 $b^2 - 4ac \neq 0$。

- 首先，我們先看 $b^2 - 4ac > 0$ 的情況。
  - 對於 $a < 0$，我們改寫
    $$\sqrt{ax^2+bx+c} = \sqrt{-a}\sqrt{q^2-(x-p)^2},$$
    其中 $p, q \in \mathbb{R}$。這樣的話，變數變換 $x - p = q\cos\theta$ 讓我們得到
    $$\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x = \int R(p+q\cos\theta, q\sqrt{-a}\sin\theta)\,\mathrm{d}\theta.$$
    這可以化簡為由正弦和餘弦函數構成的有理函數原函數的計算，見第 A1.3 節。
  - 對於 $a > 0$，我們改寫
    $$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{(x-p)^2-q^2},$$
    其中 $p, q \in \mathbb{R}$。我們考慮變數變換 $x - p = q\varepsilon\cosh(t)$，其中根據我們想要找原函數的區間，我們取 $\varepsilon \in \{1, -1\}$。這讓我們得到
    $$\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x = \int R(p+q\cosh t, q\sqrt{a}\sinh t)\,\mathrm{d}t.$$
    這可以化簡為由雙曲函數構成的有理函數原函數的計算，見第 A1.4 節。
- 當 $b^2 - 4ac < 0$ 時，我們一定要有 $a > 0$。我們改寫
  $$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{(x-p)^2+q^2},$$
  這樣一來，變數變換 $x - p = q\sinh\theta$ 讓我們得到
  $$\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x = \int R(p+q\sinh t, q\sqrt{a}\cosh t)\,\mathrm{d}t.$$
  這可以化簡為由雙曲函數構成的有理函數原函數的計算，見第 A1.4 節。

**範例 A1.7.1 ：** 我們想要找 $\frac{1}{\sqrt{1-x^2}}$ 和 $\sqrt{1-x^2}$ 在 $(-1,1)$ 上的原函數。根據上面的討論，我們考慮變數變換 $x = \cos\theta$，所以會得到
$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int \frac{-\sin\theta}{\sin\theta}\,\mathrm{d}\theta = -\int \mathrm{d}\theta = -\theta + \mathrm{cst},$$

and

$$\int \sqrt{1-x^2} = \int (-\sin^2\theta)\,\mathrm{d}\theta = \int \frac{\cos(2\theta)-1}{2}\,\mathrm{d}\theta = \frac{\sin(2\theta)}{4} - \frac{\theta}{2} + \mathrm{cst}$$
$$= \frac{\sin\theta\cos\theta}{2} - \frac{\theta}{2} + \mathrm{cst} = \frac{x\sqrt{1-x^2}}{2} - \frac{\arccos x}{2} + \mathrm{cst}.$$

**Example A1.7.2 :** We want to find a primitive of $\frac{1}{\sqrt{x^2-1}}$. The function $x \mapsto \frac{1}{\sqrt{x^2-1}}$ is defined on $(-\infty,-1)\cup(1,+\infty)$, and on each of the connected components, we can have a different primitive. From above, we consider the change of variables $x = \varepsilon\cosh t$, where $\varepsilon = 1$ if we look at an interval $I \subseteq (1,+\infty)$; $\varepsilon = -1$ if we look at an interval $I \subseteq (-\infty,-1)$. Note that in both cases, $t > 0$. This allows us to write

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \int \frac{\varepsilon\sinh t}{\sinh t}\,\mathrm{d}t = \varepsilon t + \mathrm{cst} = \varepsilon\,\mathrm{arccosh}(\varepsilon x) + \mathrm{cst}$$
$$= \ln|x+\sqrt{x^2-1}| + \mathrm{cst}.$$

**Example A1.7.3 :** We want to find a primitive of $\frac{1}{\sqrt{x^2+1}}$. From above, we consider the change of variables $x = \sinh t$, so we have

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \int \frac{\cosh t\,\mathrm{d}t}{\cosh t} = t + \mathrm{cst} = \mathrm{arcsinh}\,x + \mathrm{cst}.$$

Below, you see a more complete table on primitives. Note that computing primitives is a hard task, and for many functions, we are not able to find a closed form for their primitives. In such cases, mean-value theorems (Section 5.3.5) or comparison of integrals (Section 7.1.4) can be useful to estimate the values of integrals.

| $f(x)$ | $F(x)$ |
|---|---|
| $a^x$, $a>0$, $a\neq 1$ | $\frac{a^x}{\ln a}$ |
| $\tan x$ | $-\ln\lvert\cos x\rvert$ |
| $\cot x$ | $\ln\lvert\sin x\rvert$ |
| $\frac{1}{\sin^2 x}$ | $-\cot x$ |
| $\tanh x$ | $\ln(\cosh x)$ |
| $\coth x$ | $\ln(\sinh x)$ |
| $\frac{1}{\cosh^2 x}$ | $\tanh x$ |
| $\frac{1}{\sinh^2 x}$ | $-\coth x$ |

| $f(x)$ | $F(x)$ |
|---|---|
| $\frac{1}{\sqrt{1-x^2}}$, $a>0$ | $\arcsin x$ |
| $\frac{1}{\sqrt{a^2-x^2}}$, $a>0$ | $\arcsin\frac{x}{a}$ |
| $\frac{1}{\sqrt{x^2+1}}$ | $\ln(x+\sqrt{1+x^2})$ |
| $\frac{1}{\sqrt{x^2+a^2}}$, $a\neq 0$ | $\ln(x+\sqrt{a^2+x^2})$ |
| $\frac{1}{\sqrt{x^2-1}}$ | $\ln\lvert x+\sqrt{x^2-1}\rvert$ |
| $\frac{1}{\sqrt{x^2-a^2}}$, $a\neq 0$ | $\ln\lvert x+\sqrt{x^2-a^2}\rvert$ |
| $\frac{1}{x^2+a^2}$, $a\neq 0$ | $\frac{1}{a}\arctan\frac{x}{a}$ |
| $\frac{1}{a^2-x^2}$, $a\neq 0$ | $\frac{1}{2a}\ln\lvert\frac{x+a}{x-a}\rvert$ |



和

$$\int \sqrt{1-x^2} = \int (-\sin^2\theta)\,\mathrm{d}\theta = \int \frac{\cos(2\theta)-1}{2}\,\mathrm{d}\theta = \frac{\sin(2\theta)}{4} - \frac{\theta}{2} + \mathrm{cst}$$
$$= \frac{\sin\theta\cos\theta}{2} - \frac{\theta}{2} + \mathrm{cst} = \frac{x\sqrt{1-x^2}}{2} - \frac{\arccos x}{2} + \mathrm{cst}.$$

**範例 A1.7.2 ：** 我們想要找 $\frac{1}{\sqrt{x^2-1}}$ 的原函數。函數 $x \mapsto \frac{1}{\sqrt{x^2-1}}$ 是定義在 $(-\infty,-1)\cup(1,+\infty)$ 上的，在每個連通元件上，我們可以有不同的原函數。從上面的方法，我們可以考慮變數變換 $x = \varepsilon\cosh t$，如果我們考慮區間 $I \subseteq (1,+\infty)$，則我們取 $\varepsilon = 1$；如果我們考慮區間 $I \subseteq (-\infty,-1)$，則我們取 $\varepsilon = -1$。注意到在這兩種情況下，我們都有 $t > 0$。這讓我們得到

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \int \frac{\varepsilon\sinh t}{\sinh t}\,\mathrm{d}t = \varepsilon t + \mathrm{cst} = \varepsilon\,\mathrm{arccosh}(\varepsilon x) + \mathrm{cst}$$
$$= \ln|x+\sqrt{x^2-1}| + \mathrm{cst}.$$

**範例 A1.7.3 ：** 我們想要找 $\frac{1}{\sqrt{x^2+1}}$ 的原函數。從上面的討論，我們可以考慮變數變換 $x = \sinh t$，進而得到

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \int \frac{\cosh t\,\mathrm{d}t}{\cosh t} = t + \mathrm{cst} = \mathrm{arcsinh}\,x + \mathrm{cst}.$$

下面有個比較完整的原函數表格。不要忘記，計算原函數不是容易的事情，而且對很多函數來說，我們無法把原函數用封閉形式來描述。在這種情況下，我們會使用均值定理（第 5.3.5 小節）或是積分的比較（第 7.1.4 小節）來估計我們想要求的積分值。

| $f(x)$ | $F(x)$ |
|---|---|
| $a^x$, $a>0$, $a\neq 1$ | $\frac{a^x}{\ln a}$ |
| $\tan x$ | $-\ln\lvert\cos x\rvert$ |
| $\cot x$ | $\ln\lvert\sin x\rvert$ |
| $\frac{1}{\sin^2 x}$ | $-\cot x$ |
| $\tanh x$ | $\ln(\cosh x)$ |
| $\coth x$ | $\ln(\sinh x)$ |
| $\frac{1}{\cosh^2 x}$ | $\tanh x$ |
| $\frac{1}{\sinh^2 x}$ | $-\coth x$ |

| $f(x)$ | $F(x)$ |
|---|---|
| $\frac{1}{\sqrt{1-x^2}}$, $a>0$ | $\arcsin x$ |
| $\frac{1}{\sqrt{a^2-x^2}}$, $a>0$ | $\arcsin\frac{x}{a}$ |
| $\frac{1}{\sqrt{x^2+1}}$ | $\ln(x+\sqrt{1+x^2})$ |
| $\frac{1}{\sqrt{x^2+a^2}}$, $a\neq 0$ | $\ln(x+\sqrt{a^2+x^2})$ |
| $\frac{1}{\sqrt{x^2-1}}$ | $\ln\lvert x+\sqrt{x^2-1}\rvert$ |
| $\frac{1}{\sqrt{x^2-a^2}}$, $a\neq 0$ | $\ln\lvert x+\sqrt{x^2-a^2}\rvert$ |
| $\frac{1}{x^2+a^2}$, $a\neq 0$ | $\frac{1}{a}\arctan\frac{x}{a}$ |
| $\frac{1}{a^2-x^2}$, $a\neq 0$ | $\frac{1}{2a}\ln\lvert\frac{x+a}{x-a}\rvert$ |