A1 Classical primitives: computations and methods

For a function $f : I \to \mathbb{R}$ defined on a connected interval of I, let us denote by F one of its primitives. We recall that primitives are not unique, but defined up to an additive constant. In the following table, you may find the primitives of some elementary functions.

$f(x)$	$F(x)$
$x^\alpha, \alpha \neq -1$	$\frac{x^{\alpha+1}}{\alpha+1}$
x^{-1}	$\ln\lvert x\rvert$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$

$f(x)$	$F(x)$
$\frac{1}{\cos^2 x}$	$\tan x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\frac{1}{x^2+1}$	$\arctan x$

Classical methods to compute primitives include change of variables (Proposition 5.3.33), integration by parts (Proposition 5.2.11), by induction (Exercise A1.2), etc. We present some other common methods below, depending on the form that the integrand takes.

A1.1 Rational fractions

Let f be a rational fraction over $\mathbb{R}$, that is $f = \frac{P}{Q} \in \mathbb{R}(X)$, where $P, Q \in \mathbb{R}[X]$ and $Q \neq 0$. It can be viewed as a map $x \mapsto f(x)$, which is well defined for $x \in \mathbb{R}$ with $Q(x) \neq 0$. The polynomial Q can be factorized in $\mathbb{R}[X]$ as a product of irreducible polynomials, that is,

$$Q(x) = \text{cst} \cdot \prod_{i=1}^{m} (x^2 + b_i x + c_i)^{m_i} \prod_{j=1}^{n} (x - r_j)^{n_j},$$

where $m, n \in \mathbb{N}_0$ and

- for every $1 \leqslant i \leqslant m$, we have $b_i, c_i \in \mathbb{R}$ satisfying $b_i^2 - 4c_i < 0$ and $m_i \in \mathbb{N}$,
- for every $1 \leqslant j \leqslant n$, we have $r_j \in \mathbb{R}$ and $n_j \in \mathbb{N}$.

Therefore, we may decompose f into partial fractions,

$$\frac{P(x)}{Q(x)} = T(x) + \sum_{i=1}^{m} \frac{M_i(x)}{(x^2 + b_i x + c_i)^{m_i}} + \sum_{j=1}^{n} \frac{N_i(x)}{(x - r_j)^{n_j}},$$

where

- $T \in \mathbb{R}[X]$ is a polynomial,
- for every $1 \leqslant i \leqslant m$, $M_i \in \mathbb{R}[X]$ is a polynomial of degree $\deg(M_i) \leqslant 2m_i - 1$,
- for every $1 \leqslant j \leqslant n$, $N_i \in \mathbb{R}[X]$ is a polynomial of degree $\deg(N_j) \leqslant n_j - 1$.

This can be further rewritten as

$$\frac{P(x)}{Q(x)} = T(x) + \sum_{i=1}^{m} \sum_{r=1}^{m_i} \frac{M_{i,r}(x)}{(x^2 + b_i x + c_i)^{r}} + \sum_{j=1}^{n} \sum_{s=1}^{n_j} \frac{N_{i,s}}{(x - r_j)^{s}},$$

where

- $T \in \mathbb{R}[X]$ is a polynomial,
- for every $1 \leqslant i \leqslant m$ and $1 \leqslant r \leqslant m_i$, $M_{i,r} \in \mathbb{R}[X]$ is a polynomial of degree at most 1,
- for every $1 \leqslant j \leqslant n$ and $1 \leqslant s \leqslant n_j$, $N_{i,s} \in \mathbb{R}[X]$ is a polynomial of degree at most zero, that is a constant.

Therefore, it is enough to determine the primitives of rational fractions of the forms,

$$\frac{1}{(x-r)^n}, n \in \mathbb{N} \quad \text{and} \quad \frac{sx+t}{(x^2+bx+c)^m}, b^2 - 4c < 0, \quad m \in \mathbb{N}.$$

For the first type of rational fraction, we easily find

$$\int \frac{\mathrm{d}x}{(x-r)^n} = \begin{cases} \frac{1}{(1-n)(x-r)^{n-1}} + \text{cst} & \text{if } n \neq 1, \\ \ln|x-r| + \text{cst} & \text{if } n = 1. \end{cases}$$

For the second type, we rewrite in the following form

$$\frac{sx+t}{(x^2+bx+c)^m} = \frac{2\alpha(x-p)}{\left[(x-p)^2+q^2\right]^m} + \frac{\beta}{\left[(x-p)^2+q^2\right]^m}$$

The first term has primitives in the following form,

$$\int \frac{2\alpha(x-p)}{\left[(x-p)^2+q^2\right]^m}\,\mathrm{d}x = \begin{cases} \frac{\alpha}{(1-m)\left[(x-p)^2+q^2\right]^{m-1}} + \text{cst} & \text{if } m \geqslant 2, \\ \alpha \ln\left[(x-p)^2+q^2\right] + \text{cst} & \text{if } m = 1. \end{cases}$$

For the second term, we perform a change of variables $x-p = q\tan\theta$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then, we find

$$\int \frac{\beta}{\left[(x-p)^2+q^2\right]^m}\,\mathrm{d}x = \frac{\beta}{q^{2m-1}}\int \cos^{2m-2}\theta\,\mathrm{d}\theta,$$

which can be evaluated recursively (see Wallis' integral in Exercise A1.2), or Section A1.2.

Example A1.1.1 : Find a primitive of $\frac{x}{(x^2+x+1)^2}$. We write

$$\frac{x}{(x^2+x+1)^2} = \frac{x+\frac{1}{2}}{((x+\frac{1}{2})^2+\frac{3}{4})^2} - \frac{1}{2}\frac{1}{((x+\frac{1}{2})^2+\frac{3}{4})^2}$$

From what we discussed above, we have

$$\int \frac{x+\frac{1}{2}}{((x+\frac{1}{2})^2+\frac{3}{4})^2}\,\mathrm{d}x = -\frac{1}{2}\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} + \text{cst} = -\frac{1}{2}\frac{1}{x^2+x+1} + \text{cst},$$

and, by setting $x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, we have

$$\begin{aligned} \int \frac{1}{((x+\frac{1}{2})^2+\frac{3}{4})^2}\,\mathrm{d}x &= \left(\frac{2}{\sqrt{3}}\right)^3 \int \cos^2\theta\,\mathrm{d}\theta = \left(\frac{2}{\sqrt{3}}\right)^3 \int \frac{1+\cos(2\theta)}{2}\,\mathrm{d}\theta \\ &= \frac{4}{3\sqrt{3}}(\theta + \frac{1}{2}\sin(2\theta)) + \text{cst} \\ &= \frac{4}{3\sqrt{3}}(\theta + \frac{\tan\theta}{1+\tan^2\theta}) + \text{cst} \\ &= \frac{4}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{2}{3}\frac{x+\frac{1}{2}}{x^2+x+1} + \text{cst}. \end{aligned}$$

Therefore, we have

$$\int \frac{x}{(x^2+x+1)^2}\,\mathrm{d}x = -\frac{2}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{1}{3}\frac{x+2}{x^2+x+1} + \text{cst}.$$

A1.2 Polynomials in sine and cosine functions

Let $m, n \in \mathbb{N}_0$, and we want to find primitives of $\int \sin^m x \cos^n x \, \mathrm{d}x$.

- If $m = 2k + 1$ is odd, then we write

$$\int \sin^m x \cos^n x \, \mathrm{d}x = \int (1 - \cos^2 x)^k \cos^n x \sin x \, \mathrm{d}x = -\int (1 - \cos^2 x)^k \cos^n x \, \mathrm{d} \cos x,$$

  and the change of variables $t = \cos x$ allows us to conclude.

- If $n = 2\ell + 1$ is odd, then we write

$$\int \sin^m x \cos^n x \, \mathrm{d}x = \int \sin^m x (1 - \sin^2 x)^\ell \cos x \, \mathrm{d}x = \int \sin^m x (1 - \sin^2 x)^\ell \, \mathrm{d} \sin x,$$

- If both m and n are even, we use the following identities

$$\cos x = \frac{1}{2}(e^{\mathrm{i}\, x} + e^{-\mathrm{i}\, x}) \quad \text{and} \quad \sin x = \frac{1}{2\,\mathrm{i}}(e^{\mathrm{i}\, x} - e^{-\mathrm{i}\, x}),$$

  which simplifies the integrand into a polynomial in e^x, whose primitive is easy to compute.

Note that in the first two cases, it is also possible to apply the method in the last case, but the resulting polynomial would be less nicer to deal with than the direct method mentioned above.

A1.3 Rational fractions in sine and cosine functions

Let $R \in \mathbb{R}(X, Y)$ be a rational fraction in two variables, and we want to find a primitive of $R(\sin x, \cos x)$. We apply the change of variables $t = \tan(\frac{x}{2})$, and note that

$$\sin x = \frac{2\tan(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})} \quad \text{and} \quad \cos x = \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}.$$

This allows us to rewrite

$$\int R(\sin x, \cos x) \, \mathrm{d}x = \int R\Big(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\Big) \frac{2\,\mathrm{d}t}{1+t^2},$$

whose integrand is a rational fraction in t, so we can apply the method from Section A1.1.

Example A1.3.1 : Find a primitive of $\frac{1}{\sin x}$. We consider the change of variables $t = \tan(\frac{x}{2})$, and we have

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{1+t^2}{2t}\frac{2\,\mathrm{d}t}{1+t^2} = \int \frac{\mathrm{d}t}{t} = \ln|t| + \text{cst} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + \text{cst}.$$

A1.4 Rational fractions in hyperbolic sine and hyperbolic cosine functions

Let $R \in \mathbb{R}(X, Y)$ be a rational fraction in two variables, and we want to find a primitive of $R(\sinh x, \cosh x)$. We may apply several different methods, depending on the integrand, among which the most common ones are the following.

- We apply the change of variables $t = \tanh(\frac{x}{2})$, and note that

  $$\sinh x = \frac{2\tanh(\frac{x}{2})}{1-\tanh^2(\frac{x}{2})} \quad \text{and} \quad \cosh x = \frac{1+\tanh^2(\frac{x}{2})}{1-\tanh^2(\frac{x}{2})}.$$

  Then, we reduce to the case where the integrand is a rational fraction in t.

- We apply the change of variables $t = e^x$, and the integrand becomes a rational fraction in t.

A1.5 Ration fractions in e^x

If f is a rational fraction in e^x, that is $f(x) = R(e^x)$ where $R \in \mathbb{R}(X)$, then by the change of variables $t = e^x$, we find

$$\int f(x)\,\mathrm{d}x = \int R(e^x)\,\mathrm{d}x = \int \frac{R(t)}{t}\,\mathrm{d}t,$$

which is a rational fraction in t.

Example A1.5.1 : Find a primitive of $\frac{1}{\cosh x}$. We consider the change of variables $t = e^x$, and find

$$\int \frac{\mathrm{d}x}{\cosh x} = \int \frac{2\,\mathrm{d}x}{e^x + e^{-x}} = \int \frac{2\,\mathrm{d}t}{t}\frac{1}{t+\frac{1}{t}} = \int \frac{2\,\mathrm{d}t}{1+t^2} = 2\arctan(e^x) + \text{cst}.$$

A1.6 Abelian integrals of first type

Let $R \in \mathbb{R}(X, Y)$ be a rational fraction in two variables, and we want to find a primitive of

$$R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right), \quad n \in \mathbb{N}.$$

- If $ad - bc = 0$, then the n-th root does not depend on x, and the computation is easy.
- If $ad - bc \neq 0$, we consider the change of variables

  $$t = \sqrt[n]{\frac{ax+b}{cx+d}} \quad \text{and} \quad x = g(t) = \frac{dt^n - b}{a - t^n c}.$$

  This allows us to rewrite

  $$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) \mathrm{d}x = \int R(g(t), t) g'(t) \, \mathrm{d}t,$$

  whose integrand is a rational fraction in t.

Example A1.6.1 : We want to compute

$$\int \frac{\mathrm{d}x}{\sqrt{1+x} + \sqrt[3]{1+x}}.$$

We consider the change of variables $t = \sqrt[6]{1+x}$, and we have

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1+x} + \sqrt[3]{1+x}} &= \int \frac{6t^5 \, \mathrm{d}t}{t^3 + t^2} = 6 \int \left(t^2 - t + 1 - \frac{1}{1+t}\right) \mathrm{d}t \\
&= 2t^3 - 3t^2 + 6t - 6 \ln|1+t| + \mathrm{cst} \\
&= 2\sqrt{1+x} - 3\sqrt[3]{1+x} + 6\sqrt[6]{1+x} - 6 \ln|1 + \sqrt[6]{1+x}| + \mathrm{cst}.
\end{aligned}$$

A1.7 Abelian integrals of second type

Let $R \in \mathbb{R}(X, Y)$ be a rational fraction in two variables, and we want to find a primitive of $R(x, \sqrt{ax^2 + bx + c})$.

- If $a = 0$, this reduces to the case in Section A1.6.
- If $a \neq 0$, but $b^2 - 4ac = 0$, the square root gets factorzied and we have a rational fraction in x, and we proceed as in Section A1.1.

Therefore, we may assume that $a \neq 0$ and $b^2 - 4ac \neq 0$.

- First, we look at the case where $b^2 - 4ac > 0$.

- For $a < 0$, we rewrite

  $$\sqrt{ax^2+bx+c} = \sqrt{-a}\sqrt{q^2-(x-p)^2},$$

  for some $p, q \in \mathbb{R}$. Then, the change of variables $x - p = q\cos\theta$ allows us to obtain

  $$\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x = \int R(p+q\cos\theta, q\sqrt{-a}\sin\theta)\,\mathrm{d}\theta.$$

  Then it simplifies to the computation of a primitive of rational fractions in sine and cosine functions, see Section A1.3.

- For $a > 0$, we rewrite

  $$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{(x-p)^2-q^2},$$

  for some $p, q \in \mathbb{R}$. We consider the change of variables $x - p = q\varepsilon\cosh(t)$, where $\varepsilon \in \{1, -1\}$, chosen depending on the interval where we look for a primitive. This allows us to obtain

  $$\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x = \int R(p+q\cosh t, q\sqrt{a}\sinh t)\,\mathrm{d}t.$$

  Then it simplifies to the computation of a primitive of rational fractions in hyperbolic sine and hyperbolic cosine functions.

- For the case where $b^2 - 4ac < 0$, we need to have $a > 0$. We rewrite

  $$\sqrt{ax^2+bx+c} = \sqrt{a}\sqrt{(x-p)^2+q^2},$$

  Then, the change of variables $x - p = q\sinh\theta$ allows us to obtain

  $$\int R(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x = \int R(p+q\sinh t, q\sqrt{a}\cosh t)\,\mathrm{d}t.$$

  Then it simplifies to the computation of a primitive of rational fractions in hyperbolic sine and hyperbolic cosine functions.

Example A1.7.1 : We want to find a primitive of $\frac{1}{\sqrt{1-x^2}}$ and $\sqrt{1-x^2}$ on $(-1, 1)$. From above, we consider the change of variables $x = \cos\theta$, so we have

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int \frac{-\sin\theta}{\sin\theta}\,\mathrm{d}\theta = -\int \mathrm{d}\theta = -\theta + \mathrm{cst},$$

and

$$\int \sqrt{1-x^2} = \int (-\sin^2\theta)\,\mathrm{d}\theta = \int \frac{\cos(2\theta)-1}{2}\,\mathrm{d}\theta = \frac{\sin(2\theta)}{4} - \frac{\theta}{2} + \text{cst}$$
$$= \frac{\sin\theta\cos\theta}{2} - \frac{\theta}{2} + \text{cst} = \frac{x\sqrt{1-x^2}}{2} - \frac{\arccos x}{2} + \text{cst}.$$

Example A1.7.2 : We want to find a primitive of $\frac{1}{\sqrt{x^2-1}}$. The function $x \mapsto \frac{1}{\sqrt{x^2-1}}$ is defined on $(-\infty,-1) \cup (1,+\infty)$, and on each of the connected components, we can have a different primitive. From above, we consider the change of variables $x = \varepsilon \cosh t$, where $\varepsilon = 1$ if we look at an interval $I \subseteq (1,+\infty)$; $\varepsilon = -1$ if we look at an interval $I \subseteq (-\infty,-1)$. Note that in both cases, $t > 0$. This allows us to write

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \int \frac{\varepsilon \sinh t}{\sinh t}\,\mathrm{d}t = \varepsilon t + \text{cst} = \varepsilon \operatorname{arccosh}(\varepsilon x) + \text{cst}$$
$$= \ln|x + \sqrt{x^2-1}| + \text{cst}.$$

Example A1.7.3 : We want to find a primitive of $\frac{1}{\sqrt{x^2+1}}$. From above, we consider the change of variables $x = \sinh t$, so we have

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \int \frac{\cosh t\,\mathrm{d}t}{\cosh t} = t + \text{cst} = \operatorname{arcsinh} x + \text{cst}.$$

Below, you see a more complete table on primitives. Note that computing primitives is a hard task, and for many functions, we are not able to find a closed form for their primitives. In such cases, mean-value theorems (Section 5.3.5) or comparison of integrals (Section 7.1.4) can be useful to estimate the values of integrals.

$f(x)$	$F(x)$
a^x, $a > 0, a \neq 1$	$\frac{a^x}{\ln a}$
$\tan x$	$-\ln\lvert\cos x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\frac{1}{\sin^2 x}$	$-\cot x$
$\tanh x$	$\ln(\cosh x)$
$\coth x$	$\ln(\sinh x)$
$\frac{1}{\cosh^2 x}$	$\tanh x$
$\frac{1}{\sinh^2 x}$	$-\coth x$

$f(x)$	$F(x)$
$\frac{1}{\sqrt{1-x^2}}$, $a > 0$	$\arcsin x$
$\frac{1}{\sqrt{a^2-x^2}}$, $a > 0$	$\arcsin \frac{x}{a}$
$\frac{1}{\sqrt{x^2+1}}$	$\ln(x + \sqrt{1+x^2})$
$\frac{1}{\sqrt{x^2+a^2}}$, $a \neq 0$	$\ln(x + \sqrt{a^2+x^2})$
$\frac{1}{\sqrt{x^2-1}}$	$\ln\lvert x + \sqrt{x^2-1}\rvert$
$\frac{1}{\sqrt{x^2-a^2}}$, $a \neq 0$	$\ln\lvert x + \sqrt{x^2-a^2}\rvert$
$\frac{1}{x^2+a^2}$, $a \neq 0$	$\frac{1}{a}\arctan\frac{x}{a}$
$\frac{1}{a^2-x^2}$, $a \neq 0$	$\frac{1}{2a}\ln\lvert\frac{x+a}{x-a}\rvert$