2 Topology on metric spaces and normed spaces

2.1 Elementary notions

In the first section, we start with metric spaces and normed spaces, on which we will define the notion of topology.

2.1.1 Metric spaces, normed spaces, and examples

Definition 2.1.1 : Given a set M. We say that a function $d : M \times M \to \mathbb{R}$ is a *distance or metric* (距離) on M if

(i) (Positive definiteness) $d(x, y) \geqslant 0$ with equality if and only if $x = y$.

(ii) (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in M$.

(iii) (Triangle inequality) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in M$.

We also say that (M, d) is a *metric space* (賦距空間) if d is a distance on M.

Example 2.1.2 : Below we give a few common examples of metric spaces.

(1) On $\mathbb{R}$, the function $d(x, y) = |x - y|$ is a distance.

(2) On $\mathbb{R}^n$, we may define the following *Euclidean distance* (歐氏距離) ,

$$d(x, y) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}, \quad x, y \in \mathbb{R}^n.$$

(3) On $\mathbb{R}^n$, the following functions are distances.

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|,$$
$$d_\infty(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

(4) For any nonempty set M, define

$$d(x,y)=\begin{cases}0 & \text{if } x=y,\\ 1 & \text{if } x\neq y.\end{cases}$$

This is called a *discrete metric* and (M,d) is called a *discrete metric space* (離散賦距空間) .

Definition 2.1.3 : Let V be a vector space over a field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. A map $\|\cdot\|: V\to\mathbb{R}_+$ is said to be a *norm* on V if

(i) (Positive definiteness) $\|x\|=0$ if and only if $x=0$.

(ii) (Homogeneity) For every $\lambda\in\mathbb{K}$ and $x\in V$, we have $\|\lambda x\|=|\lambda|\,\|x\|$.

(iii) (Triangle inequality) For any $x,y\in V$, we have $\|x+y\|\leqslant\|x\|+\|y\|$.

If $\|\cdot\|$ is a norm on V, then we say that $(V,\|\cdot\|)$ is a *normed vector space* (賦範向量空間) , or a *normed space* (賦範空間) .

Example 2.1.4 : Given a normed space $(V,\|\cdot\|)$, the map $d(x,y):=\|x-y\|$ defines a distance on V, making (V,d) a metric space. Therefore, whenever we want to consider a normed space as a metric space, we choose this distance by default.

Example 2.1.5 : Below are some classical norms that we consider on $\mathbb{R}^n$. For $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, define

$$\|x\|_1:=\sum_{i=1}^n|x_i|,\quad \|x\|_2:=\sqrt{\sum_{i=1}^n|x_i|^2},\quad \|x\|_\infty:=\sup_{1\leqslant i\leqslant n}|x_i|. \tag{2.1}$$

You may check that the properties (1)–(3) in Definition 2.1.3 are satisfied.

Example 2.1.6 : The following spaces of real sequences are also normed spaces,

$$\ell^1(\mathbb{R}) := \Big\{a = (a_n)_{n\geqslant 1} \in \mathbb{R}^{\mathbb{N}} : \|a\|_1 := \sum_{n\geqslant 1} |a_n| < \infty\Big\},$$
$$\ell^2(\mathbb{R}) := \Big\{a = (a_n)_{n\geqslant 1} \in \mathbb{R}^{\mathbb{N}} : \|a\|_2 := \sqrt{\sum_{n\geqslant 1} |a_n|^2} < \infty\Big\},$$
$$\ell^\infty(\mathbb{R}) := \Big\{a = (a_n)_{n\geqslant 1} \in \mathbb{R}^{\mathbb{N}} : \|a\|_\infty := \sup_{n\geqslant 1} |a_n| < \infty\Big\}.$$

Example 2.1.7 : Given a set X and a normed vector space $(V, \|\cdot\|)$. Write $\mathcal{B}(X, V)$ for the set of bounded functions from X to V, which can be checked to be a vector space. Then, we may equip $\mathcal{B}(X, V)$ with the following norm,

$$\|f\|_\infty := \sup_{x\in X} \|f(x)\|, \qquad f \in \mathcal{B}(X, V).$$

Example 2.1.8 : Let $a < b$ be two real numbers. Consider the space of continuous functions defined on $[a, b]$ with values in $\mathbb{R}$, denoted by $\mathcal{C}([a, b], \mathbb{R})$. It is not hard to check that $\mathcal{C}([a, b], \mathbb{R})$ is a vector space. We may equip the following vector subspaces with the corresponding norms,

$$L^1([a, b], \mathbb{R}) := \Big\{f \in \mathcal{C}([a, b], \mathbb{R}) : \|f\|_1 = \int_a^b |f(t)|\,\mathrm{d}t < \infty\Big\},$$
$$L^2([a, b], \mathbb{R}) := \Big\{f \in \mathcal{C}([a, b], \mathbb{R}) : \|f\|_2 = \sqrt{\int_a^b |f(t)|^2\,\mathrm{d}t} < \infty\Big\},$$
$$L^\infty([a, b], \mathbb{R}) := \Big\{f \in \mathcal{C}([a, b], \mathbb{R}) : \|f\|_\infty = \sup_{t\in[a,b]} |f(t)|\Big\}.$$

Example 2.1.9 : On the vector space $\mathbb{K}[X]$ of polynomials with coefficients in a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, that is

$$\mathbb{K}[X] = \Big\{\sum_{n=0}^{N} a_n X^n : a_n \in \mathbb{K}, 0 \leqslant n \leqslant N, N \geqslant 0\Big\}.$$

We may define the following norms on $\mathbb{K}[X]$.

(a) A polynomial P can be uniquely written as $P = \sum_{n=0}^{\infty} a_n X^n$, where only finitely many terms of $(a_n)_{n\geqslant 0}$ are nonzero. Then, we define

$$\|P\|_1 = \sum_{n\geqslant 0} |a_n|, \quad \|P\|_2 = \sqrt{\sum_{n\geqslant 0} |a_n|^2}, \quad \text{and } \|P\|_\infty = \max_{n\geqslant 0} |a_n|.$$

(b) We are given real numbers $a < b$ and see a polynomial P as a function $t \mapsto P(t)$ on $[a, b]$. Then, we define

$$\|P\|_1 = \int_a^b |P(t)|\,\mathrm{d}t, \quad \|P\|_2 = \sqrt{\int_a^b |P(t)|^2\,\mathrm{d}t}, \quad \text{and } \|P\|_\infty = \max_{t\in[a,b]} |P(t)|.$$

Definition 2.1.10 : A *Euclidean space* (歐氏空間) is a finite dimensional vector space V over $\mathbb{R}$, equipped with an *inner product* (內積) $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ satisfying

(i) (Positive definiteness) $\langle x, x\rangle \geqslant 0$ with equality if and only if $x = 0$.

(ii) (Symmetry) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$.

(iii) (Linearity) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $a, b \in \mathbb{R}$ and $x, y, z \in V$.

Example 2.1.11 : The vector space $\mathbb{R}^n$ with the following inner product

$$\langle x, y\rangle = \sum_{i=1}^n x_i y_i, \quad x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$$

is a Euclidean space.

Proposition 2.1.12 : *Given a Euclidean space* $(V, \langle\cdot,\cdot\rangle)$, *we may define*

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad \forall x \in \mathbb{R}^n. \tag{2.2}$$

Then, $\|\cdot\|$ *is a norm on* V*, which is the canonical norm on the Euclidean space* V*.*

Proof : We only need to check that the function defined in (2.2) satisfies the triangular inequality. It is a classical proof, see Exercise 2.5. □

In what follows, we will fix a metric space (M, d) and define several notions in this space. If you need a concrete space to help you visualize, think of (1) or (2) in Example 2.1.2, but please bear in mind that these notions can be made sense of in any abstract metric space (M, d). Also, some behaviors might be quite different in a general metric space, for instance, look at the balls (defined below, also see Example 2.1.30) in a discrete metric space such as (4) in Example 2.1.2.

Definition 2.1.13 : Given $x \in M$ and $r \geqslant 0$, we define

$$\begin{aligned} B(x,r) &= \{y \in M : d(x,y) < r\}, \\ \overline{B}(x,r) &= \{y \in M : d(x,y) \leqslant r\}, \\ S(x,r) &= \{y \in M : d(x,y) = r\}. \end{aligned}$$

We say that $B(x,r)$ is the *open ball* (開球) centered at x of radius r, $\overline{B}(x,r)$ is the *closed ball* (閉球) centered at x of radius r, and $S(x,r)$ is the *sphere* (球殼) centered at x of radius r. If the set M is equipped with different distances, we may write $B_d(x,r)$, $\overline{B}_d(x,r)$, or $S_d(x,r)$ to specify the balls are defined using the distance d.

Remark 2.1.14 : Note that we have $B(x,r) \cup S(x,r) = \overline{B}(x,r)$ for any $x \in M$ and $r \geqslant 0$. We also have $B(x,0) = \varnothing$ and $\overline{B}(x,0) = \{x\}$ for any $x \in M$.

Definition 2.1.15 : Given a nonempty subset $A \subseteq M$, we define its *diameter* (直徑) by

$$\delta(A) = \sup_{x,y \in A} d(x,y).$$

And we say that A is *bounded* (有界) if $A = \varnothing$ or $\delta(A) < +\infty$. Otherwise, A is *unbounded* (無界).

Definition 2.1.16 : Given two nonempty subsets A and B of M, we define the distance between A and B to be

$$d(A,B) = \inf_{\substack{x \in A \\ y \in B}} d(x,y).$$

We also define the distance between a point x and a subset $A \subseteq M$ to be

$$d(x,A) = d(\{x\},A) = \inf_{y \in A} d(x,y).$$

Remark 2.1.17 : The distance d, originally defined on the metric space (M, d), can be generalized to a map

$$d : (\mathcal{P}(M)\backslash\{\varnothing\})^2 \to \mathbb{R}$$

as we see in Definition 2.1.16. However, this map d does not define a distance on nonempty subsets $\mathcal{P}(M)\backslash\{\varnothing\}$ in the sense of Definition 2.1.1. For example, if we take $(M, d) = (\mathbb{R}, |\cdot|)$, then $d(A, B) = 0$ for $A = [0, 2]$ and $B = [1, 3]$ without having $A = B$. However, we may still call it a *distance* by abuse of language.

2.1.2 Open sets and closed sets

Below, let us fix a metric space (M, d) and define open sets and closed sets on this space. The *topology* of (M, d) is characterized by such sets.

Definition 2.1.18 : Given a subset $A \subseteq M$. We say that A is an *open set* (開集) or open in M if $A = \varnothing$ or

$$\forall x \in A, \exists r > 0 \quad \text{such that} \quad B(x, r) \subseteq A.$$

Example 2.1.19 : Below are a few examples of open sets.

(1) Open balls are open sets.

(2) Take $(M, d) = (\mathbb{R}, |\cdot|)$, then the intervals (a, b) with $-\infty \leqslant a < b \leqslant \infty$ are open sets.

(3) In a metric space (M, d), fix a subset $A \subseteq M$ and $r > 0$. Then, the set

$$A_r = \{y \in M : d(y, A) < r\}$$

is open for the following reason. Let $y \in A_r$, write $\varepsilon = \frac{1}{2}(r - d(y, A)) > 0$. Then, for $z \in B(y, \varepsilon)$, the triangle inequality gives

$$\forall x \in A, \quad d(z, x) \leqslant d(z, y) + d(y, x) < \varepsilon + d(y, x).$$

By taking the infimum over $x \in A$ in the above inequality, we find, for $z \in B(y, \varepsilon)$ that,

$$d(z, A) = \inf_{x\in A} d(z, x) \leqslant \varepsilon + \inf_{x\in A} d(y, x) = \varepsilon + d(y, A) = \tfrac{1}{2}(r + d(y, A)) < r.$$

That is, $B(y, \varepsilon) \subseteq A_r$.

Proposition 2.1.20 : *Open sets in (M, d) satisfy the following properties.*

(1) The empty set $\varnothing$ and the whole space M are both open sets.

(2) Any union of open sets is still an open set.

(3) Any finite intersection of open sets is still an open set.

Proof :

(1) The empty set $\varnothing$ is open by definition. The whole space M is open because for any point $x \in M$ and any $r > 0$, we have $B(x, r) \subseteq M$.

(2) Let $(A_i)_{i \in I}$ be a family of open sets in M and denote $A = \bigcup_{i \in I} A_i$. We want to show that A is also open. Given $x \in A$. By definition, we can choose $i \in I$ such that $x \in A_i$. Since A_i is open, we may take $r > 0$ such that $B(x, r) \subseteq A_i$. Therefore, we also have $B(x, r) \subseteq A$. In conclusion, we are able to find an open ball centered at any point of A that is entirely contained in A, we have shown that A is open.

(3) Let $(A_i)_{1 \leqslant i \leqslant n}$ be a finite family of open sets. Write $A = \bigcap_{i=1}^{n} A_i$, and we want to show that A is also an open set. Given $x \in A$. For every $i = 1, \ldots, n$, we have $x \in A_i$, since A_i is open, we can find $r_i > 0$ such that $B(x, r_i) \subseteq A_i$. Take $r := \min(r_1, \ldots, r_n) > 0$, then we can check that $B(x, r) \subseteq B(x, r_i) \subseteq A_i$, which means that $B(x, r) \subseteq A$. □

Remark 2.1.21 : It is important to note that *any intersection* of open sets is not necessarily an open set. For example, consider $I_n = (-\frac{1}{n}, \frac{1}{n})$, which is open in $\mathbb{R}$ for $n \geqslant 1$, but

$$I := \bigcap_{n \geqslant 1} I_n = \{0\}$$

is clearly not an open set (in $\mathbb{R}$).

Remark 2.1.22 : Given a set X, we say that a collection of (some) subsets τ of X is a *topology* on X if the properties in Proposition 2.1.20 are satisfied, where we replace "open set" by "element in X". These properties are considered as *axioms* of a topology. The elements in τ are called *open sets*, and (X, τ) is called a *topological space*. This generalization is compatible with what has been discussed above, since in the case of a metric space M, the topology τ simply contains all the subsets A satsfying Definition 2.1.18. We may also note that, a set M equipped with two different distances d_1 and d_2 gives rise to different topological spaces. They may also define the *same topology*, in the sense that a subset $A \subseteq M$ is open in (M, d_1) if and only if it is open in (M, d_2). We will see some examples in Example 2.3.4 and have a longer discussion in Section 2.5.4.

Definition 2.1.23 : Given $A \subseteq M$. We say that A is a *closed set* (閉集) or closed in M if $A^c = M \backslash A$ is open.

Example 2.1.24 : Below are a few examples of closed sets.

(1) Closed balls are closed sets.

(2) In the metric space $(M, d) = (\mathbb{R}, |\cdot|)$, the intervals $[a, b]$ with $-\infty < a < b < \infty$ are closed sets. However, the intervals $[a, b)$ with $-\infty < a < b < \infty$ are neither open nor closed.

(3) In a metric space (M, d), fix a subset $A \subseteq M$ and $r > 0$. Then, the set

$$\overline{A}_r = \{y \in M : d(y, A) \leqslant r\}$$

is closed. Let $y \in M \backslash \overline{A}_r$ and write $\varepsilon = \frac{1}{2}(d(y, A) - r)$. Then, we may show that $B(y, \varepsilon) \subseteq M \backslash \overline{A}_r$.

Proposition 2.1.25 : *Closed sets in (M, d) satisfy the following properties.*

(1) The empty set $\varnothing$ and the whole space M are both closed sets.

(2) Any finite union of closed sets is still a closed set.

(3) Any intersection of closed sets is still a closed set.

Proof : We actually have the same proofs as in Proposition 2.1.20 by noting that the complementary of a closed set is an open set. □

Question 2.1.26: Is any union of closed sets still a closed set? If yes, please prove it; otherwise, please give a counterexample.

2.1.3 Closure, interior, boundary

In the metric space (M, d), not all the subsets are necessarily open or closed, see Example 2.1.24 (2). Given a subset $A \subseteq M$, we can define its closure (closed set), interior (open set), and boundary (difference between them).

We start with the definition of closure and discuss some of its properties.

Definition 2.1.27 : Given a subset A of M, we denote by $\mathrm{cl}(A)$, or $\overline{A}$, the *closure* (閉包) of A, which is the smallest closed set containing A. In other words,

$$\mathrm{cl}(A) = \overline{A} := \bigcap_{\substack{G \supseteq A \\ G \text{ is closed}}} G. \tag{2.3}$$

Proposition 2.1.28 : *A subset A is closed in M if and only if $\overline{A} = A$.*

Proof : We are given a subset A of M.

$\boxed{\Rightarrow}$ We first assume that A is closed. Using the definition given in Eq. (2.3), any subset G in the intersection on the r.h.s. contains A and we may also choose $G = A$. Therefore, it is clear that the intersection gives A.

$\boxed{\Leftarrow}$ We assume that $\overline{A} = A$. Since $\overline{A}$ is closed, A is also closed. □

Proposition 2.1.29 : *Let $A \subseteq M$ and $x \in M$. Then, the following properties are equivalent.*

(1) $x \in \overline{A}$.

(2) For all $\varepsilon > 0$, there exists $a \in A$ such that $d(a, x) < \varepsilon$; or alternatively, $A \cap B(x, \varepsilon) \neq \varnothing$.

(3) $d(x, A) = 0$.

In other words, we may also write the closure $\overline{A}$ as

$$\overline{A} = \{y \in M : d(y, A) = 0\}.$$

Proof : We prove that (1) ⇒ (2) ⇒ (3) ⇒ (1).

- (1) ⇒ (2). Let $x \in \overline{A}$. Given $\varepsilon > 0$, we want to find $a \in A$ such that $d(a, x) < \varepsilon$. Define

$$\overline{A}_\delta := \{y \in M : d(y, A) \leqslant \delta\}, \quad \forall \delta \geqslant 0.$$

Since $\overline{A}_\delta$ is a closed set for any $\delta \geqslant 0$, and it contains A, by the definition of $\overline{A}$, we deduce that $x \in \overline{A}_\delta$ for any $\delta \geqslant 0$. By taking $\delta = \frac{\varepsilon}{2}$, we know that $d(x, A) \leqslant \frac{\varepsilon}{2}$, that is, we may find $a \in A$ such that $d(a, x) < \varepsilon$.

- (2) ⇒ (3). Fix $\varepsilon > 0$. By (2), we can find $a \in A$ with $d(a, x) < \varepsilon$. Therefore, we have $d(x, A) \leqslant d(a, x) < \varepsilon$. Since $\varepsilon > 0$ can be taken to be arbitrarily small, we conclude that $d(x, A) = 0$.

- (3) ⇒ (1). By contradiction, suppose that $x \notin \overline{A}$. Since $(\overline{A})^c$ is open and contains x, we may find $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq (\overline{A})^c$. This means that $d(x, a) \geqslant \varepsilon$ for any $a \in A$, which contradicts with (3). □

Example 2.1.30 : Below are some examples of closure.

(1) In a normed space $(V, \|\cdot\|)$, the closure of the centered unit open ball is the centered unit closed ball, i.e.,

$$\overline{B(0,1)} = \overline{B}(0,1).$$

(2) If we consider $M = \{0, 1\}$ with the discrete metric $d(x, y) = \mathbb{1}_{x \neq y}$. Then, we have

$$\overline{B(x,1)} \subsetneq \overline{B}(x,1), \qquad \forall x \in M.$$

Actually, $B(x, 1) = \{x\}$ is open and closed at the same time, implying that $\overline{B(x,1)} = B(x, 1)$. However, the closed ball $\overline{B}(x, 1)$ is the whole space M. This is still valid as long as we consider a discrete metric space (M, d) given in Example 2.1.2 (4), where the set M contains more than 2 points.

(3) For $(M, d) = (\mathbb{R}, |\cdot|)$, the closure of an open interval (a, b) with $-\infty < a < b < \infty$ is $[a, b]$.

Definition 2.1.31 : A subset A of M is said to be *dense* (稠密) (in M) if $\overline{A} = M$.

Remark 2.1.32 : To check whether a subset A is dense in M, we may use the property (2) or (3) in Proposition 2.1.29.

Below is an interpretation of the density property in $\mathbb{R}$.

Lemma 2.1.33 : *For $(M,d) = (\mathbb{R}, |\cdot|)$, a subset A is dense if and only if $(a,b) \cap A \neq \varnothing$ for all $a < b$.*

Proof : Let us first assume that A is dense in $\mathbb{R}$, that is $\overline{A} = \mathbb{R}$. Let $a < b$, $x = \frac{1}{2}(a+b)$ and $\varepsilon = \frac{1}{2}(b-a)$. Then, $(a,b) \cap A = B(x,\varepsilon) \cap A$, which is nonempty by (2) of Proposition 2.1.29

Let A be a subset of $\mathbb{R}$ such that $A \cap (a,b)$ is nonempty for all $a < b$. Given $x \in \mathbb{R}$, we want to show that $x \in \overline{A}$. For any $\varepsilon > 0$, take $a = x - \varepsilon$ and $b = x + \varepsilon$, since $(a,b) \cap A = B(x,\varepsilon) \cap A$ is nonempty by assumption, by (2) of Proposition 2.1.29, we deduce that $x \in \overline{A}$. □

Example 2.1.34 : Both the set of rationals $\mathbb{Q}$ and the set of irrationals $\mathbb{R}\backslash\mathbb{Q}$ are dense in $\mathbb{R}$, i.e. $\overline{\mathbb{Q}} = \overline{\mathbb{R}\backslash\mathbb{Q}} = \mathbb{R}$.

Next, we define the notion of interior points and interior of a set. We will see that it is quite similar to the notion of closure (after taking the complement).

Definition 2.1.35 : Let $A \subseteq M$ and $x \in A$. We call x an *interior point* (內點) of A if there exists $\varepsilon > 0$ such that $x \in B(x,\varepsilon) \subseteq A$.

Definition 2.1.36 : Given a subset A of M, we denote by $\operatorname{int}(A)$, or $\mathring{A}$, the *interior* (開核) of A, which is the largest open set contained in A. In other words,

$$\operatorname{int}(A) = \mathring{A} := \bigcup_{\substack{G \subseteq A \\ G \text{ is open}}} G. \tag{2.4}$$

Proposition 2.1.37 : *Given a subset A of M. Then, $\operatorname{int}(A)$ contains exactly the interior points of A.*

Proof : Let $x \in A$ be an interior point of A. By Definition 2.1.35, we may find $\varepsilon > 0$ such that $x \in B(x,\varepsilon) \subseteq A$. It means that $B(x,\varepsilon)$ is an element in the union on the r.h.s. of Eq. (2.4). Therefore, $x \in B(x,\varepsilon) \subseteq \operatorname{int}(A)$.

Given $x \in \operatorname{int}(A)$, by definition, there exists an open set $G \subseteq A$ with $x \in G$. Since G is open, by Definition 2.1.18, there exists $\varepsilon > 0$ such that the open ball $B(x,\varepsilon)$ contains x. □

Proposition 2.1.38 : *A subset A is open in M if and only if $\mathring{A} = A$.*

Proof : The proof is similar to that of Proposition 2.1.28. □

Example 2.1.39 : Below are some examples of interior.

(1) In a normed space $(V, \|\cdot\|)$, the interior of the centered unit closed ball is the centered unit open ball, i.e.,

$$\operatorname{int}(\overline{B}(0,1)) = B(0,1).$$

However, in a general metric space, this equality might not hold anymore, see Example 2.1.30 (2) for a similar phenomenon.

(2) We do not necessarily have $\mathring{\overline{A}} = A$. For example, take $(M,d) = (\mathbb{R}, |\cdot|)$ and $A = (0,1) \cup (1,2)$. We find $\overline{A} = [0,2]$ and $\mathring{\overline{A}} = (0,2) \neq A$.

(3) For $(M,d) = (\mathbb{R}, |\cdot|)$, the interior of a closed interval (a,b) with $-\infty < a < b < \infty$ is (a,b).

(4) For $(M,d) = (\mathbb{R}, |\cdot|)$, the interior of $\mathbb{Q}$ or $\mathbb{R}\backslash\mathbb{Q}$ is $\varnothing$.

Proposition 2.1.40 : *Given a subset $A \subseteq M$, we have*

$$\operatorname{int}(A) = M \backslash \operatorname{cl}(M\backslash A) \quad \textit{and} \quad \operatorname{cl}(A) = M \backslash \operatorname{int}(M \backslash A).$$

Proof : By symmetry, it is only sufficient to show $\operatorname{int}(A) = M\backslash \operatorname{cl}(M\backslash A)$ for any subset $A \subseteq M$. Let

$A \subseteq M$. We are going to prove using directly Eq. (2.3) and Eq. (2.4). We write

$$\begin{aligned} M\backslash \operatorname{int}(A) &= M\backslash\Bigg(\bigcup_{\substack{G\subseteq A \\ G \text{ is open}}} G \Bigg) = \bigcap_{\substack{G\subseteq A \\ G \text{ is open}}} (M\backslash G) \\ &= \bigcap_{\substack{M\backslash G\supseteq M\backslash A \\ G \text{ is open}}} (M\backslash G) = \bigcap_{\substack{F\supseteq M\backslash A \\ F \text{ is closed}}} F = \operatorname{cl}(M\backslash A). \end{aligned}$$

□

Definition 2.1.41 : Given a subset A of M, we define the *boundary* (邊界) of A as $\partial A := \overline{A}\backslash \mathring{A}$.

Example 2.1.42 :

(1) For $(M, d) = (\mathbb{R}, |\cdot|)$ and $A = [0, 1)$, then $\partial A = \{0, 1\}$.

(2) For $(M, d) = (\mathbb{R}^2, |\cdot|)$ and $A = [0, 1) \times \{0\}$, then $\partial A = [0, 1] \times \{0\}$.

2.2 Adherent points and accumulation points

2.2.1 In general metric spaces

Definition 2.2.1 : Given a subset A of M and $x \in M$.

(1) We say that x is an *adherent point* (附著點) of A if for any $\varepsilon > 0$,

$$B(x, \varepsilon) \cap A \neq \varnothing.$$

We write $\operatorname{Adh}(A)$ for the set of adhrent points of A.

(2) We say that x is an *accumulation point* (匯聚點) of A if for any $\varepsilon > 0$,

$$B(x, \varepsilon) \cap A \neq \varnothing \quad \text{and} \quad B(x, \varepsilon) \cap A \neq \{x\}.$$

We write $\operatorname{Acc}(A)$ for the set of accumulation points of A.

(3) We say that x is an *isolated point* (孤立點) of A if there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) \cap A = \{x\}.$$

We write $\mathrm{Iso}(A)$ for the set of isolated points of A.

Remark 2.2.2 : From the definition above, we note that

(1) The set of adherent points is exactly the closure, that is $\mathrm{Adh}(A) = \overline{A}$, see Proposition 2.1.29.

(2) The set of adherent points can be written as the disjoin union of the two other sets, i.e., $\mathrm{Adh}(A) = \mathrm{Acc}(A) \sqcup \mathrm{Iso}(A)$;

(3) A is dense in M if and only if all the points in M are adherent points of A, or $\mathrm{Adh}(A) = M$.

Example 2.2.3 : In the metric space $(M, d) = (\mathbb{R}, |\cdot|)$, consider the set $A := \{\frac{1}{n}, n \in \mathbb{N}\}$. Then,

- 0 is an accumulation point of A;
- all the points $\frac{1}{n}$, where $n \geqslant 1$ is a positive integer, are isolated points of A;
- the points in $A \cup \{0\}$ are adherent points of A.

Proposition 2.2.4 : *Given a subset A of M and $x \in M$. The following properties are equivalent.*

(1) x is an accumulation point of A.

(2) For any $\varepsilon > 0$, $B(x, \varepsilon) \cap A$ contains infinitely many points.

Proof : By definition, it is clear that (2) ⇒ (1).

Assume that x is an accumulation point of A. Fix $\varepsilon > 0$ and let us construct a pairwise distinct sequence $(x_n)_{n \geqslant 1}$ of points in $B(x, \varepsilon) \cap A$ by induction.

Take $\varepsilon_1 = \varepsilon$, by definition we can find $x_1 \in B(x, \varepsilon_1) \cap A$ with $x_1 \neq x$. Let $n \geqslant 1$ and assume that pairwise distinct $x_1, \ldots, x_n$ and $\varepsilon_1 > \cdots > \varepsilon_n$ have been constructed, and satisfying

$$\varepsilon_1 > d(x, x_1) = \varepsilon_2 > \cdots > d(x, x_n) =: \varepsilon_{n+1}.$$

Again by definition, we can find $x_{n+1} \in B(x, \varepsilon_{n+1}) \cap A$ with $x_{n+1} \neq x$. Moreover, we know that $d(x, x_{n+1}) < \varepsilon_{n+1} = d(x, x_n)$, so x_{n+1} is distinct from all the previous $x_1, \ldots, x_n$. □

2.2.2 In Euclidean spaces $\mathbb{R}^n$

Let us consider Euclideans spaces $\mathbb{R}^n$ for some positive integer $n \geqslant 1$. Recall that the canonical norm is defined via the associated inner product (Proposition 2.1.12), which also leads to the canonical metric on $\mathbb{R}^n$ (Example 2.1.4).

Theorem 2.2.5 (Bolzano–Weierstraß theorem) **:** *Let $A \subseteq \mathbb{R}^n$ be a bounded set. If A contains infinitely many points, then there exists at least one point in $\mathbb{R}^n$ which is an accumulation point of A.*

Remark 2.2.6 : The choice of a Euclidean space is important here. For example, if we consider the discrete metric space as in Example 2.1.2 (4), then in $\mathbb{R}$, the subset of rationals $\mathbb{Q} \subseteq \overline{B}(0,1)$ is bounded and infinite. However, $\mathbb{Q}$ does not have any accumulation point in $\mathbb{R}$. In fact, for $x \in \mathbb{R}$ and $\varepsilon \in (0,1)$, we have $B(x,\varepsilon) \cap \mathbb{Q} = \{x\}$ or $\varnothing$, depending on whether $x \in \mathbb{Q}$ or $x \in \mathbb{R} \backslash \mathbb{Q}$.

Proof : Since A is bounded, there exists $M > 0$ such that $A \subseteq [-M, M]^n$. We are going to construct sequences $(a_k^{(i)})_{k\geqslant 1}$ and $(b_k^{(i)})_{k\geqslant 1}$ for $1 \leqslant i \leqslant n$ such that

(a) for all $1 \leqslant i \leqslant n$, $(a_k^{(i)})_{k\geqslant 1}$ is non-decreasing, $(b_k^{(i)})_{k\geqslant 1}$ is non-increasing, and the difference $b_k^{(i)} - a_k^{(i)}$ tends to 0 when $k \to \infty$,

(b) for every $k \geqslant 1$, the intersection $A \cap B_k$ contains infinitely many points, where

$$B_k := I_k^{(1)} \times \cdots \times I_k^{(n)}, \quad I_k^{(i)} := [a_k^{(i)}, b_k^{(i)}], \quad 1 \leqslant i \leqslant n.$$

We proceed by induction on k.

Let $a_1^{(i)} = -M$ and $b_1^{(i)} = M$ for all $1 \leqslant i \leqslant n$. Let $k \geqslant 1$ and suppose that $(a_\ell^{(i)})_{1\leqslant \ell\leqslant k}$ and $(b_\ell^{(i)})_{1\leqslant \ell\leqslant k}$ have been constructed, are non-decreasing and non-increasing respectively, and satisfy (b).

We may divide each of $I_k^{(i)}$ into two segments of equal length $I_k^{(i)} := I_{k,1}^{(i)} \cup I_{k,2}^{(i)}$, that is

$$I_{k,1}^{(i)} = [a_k^{(i)}, c_k^{(i)}], \quad I_{k,2}^{(i)} = [c_k^{(i)}, b_k^{(i)}], \quad c_k^{(i)} := \tfrac{1}{2}(a_k^{(i)} + b_k^{(i)}),$$

leading to 2^n subsets of B_k whose union is B_k itself,

$$B_{k,r}^{(i)} = I_{k,r_1}^{(i)} \times \cdots \times I_{k,r_n}^{(i)}, \quad r = (r_1, \ldots, r_n) \in \{1,2\}^n.$$

Since

$$A \cap B_k = \bigcup_{r \in \{1,2\}^n} (A \cap B_{k,r}^{(i)})$$

is an infinite set, at least one of the $A \cap B_{k,r}^{(i)}$ needs to be infinite as well. Let r be such that $A \cap B_{k,r}^{(i)}$ is infinite. Then, for $1 \leqslant i \leqslant n$, we let

$$(a_{k+1}^{(i)}, b_{k+1}^{(i)}) = \begin{cases} (a_k^{(i)}, c_k^{(i)}) & \text{if } r_i = 1, \\ (c_k^{(i)}, b_k^{(i)}) & \text{if } r_i = 2. \end{cases}$$

Then, it is not hard to check that $a_k^{(i)} \leqslant a_{k+1}^{(i)}$, $b_k^{(i)} \geqslant b_{k+1}^{(i)}$, and $b_{k+1}^{(i)} - a_{k+1}^{(i)} = \frac{1}{2}(b_k^{(i)} - a_k^{(i)})$.

Now that we have constructed the sequences $(a_k^{(i)})_{k \geqslant 1}$ and $(b_k^{(i)})_{k \geqslant 1}$ for $1 \leqslant i \leqslant n$ as above, we know that $(a_k^{(i)})_{k \geqslant 1}$ and $(b_k^{(i)})_{k \geqslant 1}$ both converge and have the same limit, denoted by x_i. We want to show that $x := (x_1, \dots, x_n)$ is an accumulation point of A. To see this, we are going to fix $\varepsilon > 0$, and want to show that $A \cap B(x, \varepsilon)$ contains infinitely many points. By the above construction, it is not hard to see that $x \in B_k$ for all $k \geqslant 1$. For large enough $k \geqslant 1$, we may see that $B_k \subseteq B(x, \varepsilon)$, therefore, $A \cap B(x, \varepsilon)$ also contains infinitely many points. □

Theorem 2.2.7 (Cantor intersection theorem) **:** *Given a sequence of nonempty closed sets $(A_k)_{k \geqslant 1}$ in $\mathbb{R}^n$. Suppose that*

- $A_{k+1} \subseteq A_k$ *for all* $k \geqslant 1$,
- A_1 *is bounded.*

Then, the intersection $A = \bigcap_{k \geqslant 1} A_k$ is closed and nonempty.

Remark 2.2.8 : It is important to assume that A_k's are closed sets and that A_1 is bounded.

- If A_k's are not closed, take $A_k = (0, \frac{1}{k})$ for instance, then $\bigcap_{k \geqslant 1} A_k = \varnothing$.
- If A_1 is not bounded, take $A_k = [k, \infty)$ for instance, then $\bigcap_{k \geqslant 1} A_k = \varnothing$.

Proof : First, it is easy to see that A is closed being an intersection of closed sets, see Proposition 2.1.25. Then, we need to show that A is nonempty using Bolzano–Weierstraß theorem.

If there exists an $k \geqslant 1$ such that A_k is finite, then it is clear that the sequence $(A_k)_{k \geqslant 1}$ needs to stablize to a nonempty set, and the intersection A is nonempty. Therefore, we may assume that A_k is infinite for all $k \geqslant 1$.

For each $k \geqslant 1$, we may find $x_k \in A_k$ such that the sequence $(x_k)_{k\geqslant 1}$ is pariwise distinct. We also note that, due to the fact that $(A_k)_{k\geqslant 1}$ is non-increasing, we have $x_k \in A_m$ for any $k \geqslant m \geqslant 1$. Since $X = \{x_k : k \geqslant 1\}$ is a bounded set containing infinitely many points, by Bolzano–Weierstraß theorem, it has an accumulation point $x \in \mathbb{R}^n$. We need to check that x is indeed in A, or equivalently, x is in A_m for all $m \geqslant 1$.

Given $m \geqslant 1$ and $\varepsilon > 0$. From Proposition 2.2.4, it follows that $B(x, \varepsilon)$ contains infinitely many points of X. Apart from a finite number of them (those with index $k < m$), all the other points are also in A_m. Therefore, $A_m \cap B(x, \varepsilon)$ is also infinite, which means that x is also an accumulation point of A_m. Since A_m is closed, we find $x \in A_m$. □

Example 2.2.9 : We define a sequence of subsets of $\mathbb{R}$ by induction,

$$C_0 = [0,1], \quad C_{n+1} = \tfrac{1}{3}C_n \cup (\tfrac{1}{3}C_n + \tfrac{2}{3}), \quad \forall n \geqslant 0.$$

Let $\mathcal{C} := \cap_{n\geqslant 0} C_n$. The set $\mathcal{C}$ is called *Cantor set*, and has the following properties.

(1) $\mathcal{C}$ is a nonempty closed set.

(2) $\mathcal{C}$ is equinumerous to $\{0,1\}^{\mathbb{N}}$, so uncountable.

(3) The "length" of $\mathcal{C}$ is zero.

2.3 Subspace topology

Given a metric space (M, d) and a subset $S \subseteq M$, we want to equip S with a distance so that it can become a metric space. The most natural way is consider the restricted distance $d_{S\times S}$, which is the distance d restricted on $S \times S$, sometimes also denoted by d by abuse of notations. Then, (S, d) is a metric space, and its topology is called *induced topology* (誘導拓撲) , *trace topology* (跡拓撲) , *subspace topology* (子空間拓撲) , or *relative topology* (相對拓撲) .

Proposition 2.3.1 : *Let S be a subset of M.*

(1) The open sets of S are exactly the sets $A \cap S$ where A is an open set of M.

(2) The closed sets of S are exactly the sets $A \cap S$ where A is a closed set of M.

Proof : A closed set is the complement of an open set, so it is enough to check (1). An open set is described by open balls (Definition 2.1.18), so we only need to check (1) for open balls. This is trivial, because we have

$$B_S(x,\varepsilon) = B_M(x,\varepsilon) \cap S, \quad \forall x \in M, \varepsilon > 0. \qquad \square$$

Example 2.3.2 : In the metric space $((0,1], |\cdot|)$,

- $(0,x)$ and $(x,1]$ are open sets for $x \in (0,1)$,
- $(0,x]$ and $[x,1]$ are closed sets for $x \in (0,1)$.

Remark 2.3.3 : We see from Example 2.3.2 that when we talk about closed or open sets, it is important to mention the ambient space.

Example 2.3.4 : On the space $(0,1]$, we may consider the topology induced by the metric space $(\mathbb{R}, |\cdot|)$ as mentioned in Example 2.3.2. Alternatively, we may also define a distance d on $(0,1]$, given by

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \qquad \forall x, y \in (0,1].$$

We may check in Exercise 2.23 that these two metric spaces *define the same open sets*. In other words, an open set of $((0,1], |\cdot|)$ is also an open set of $((0,1], d)$, and vice versa.

2.4 Limits

2.4.1 Definition and properties

In this section, we are given a sequence $(a_n)_{n\geqslant 1}$ with values in a metric space (M,d). When we want to talk about a subsequence of $(a_n)_{n\geqslant 1}$, we may write

- either $(a_{n_k})_{k\geqslant 1}$ for a strictly increasing sequence $(n_k)_{k\geqslant 1}$ and $n_1 \geqslant 1$,
- or $(a_{\varphi(n)})_{n\geqslant 1}$ for a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, called *extraction* (萃取函數) .

Definition 2.4.1 :

- Let $\ell \in M$. We say that $(a_n)_{n\geqslant 1}$ converges to ℓ, and write

$$a_n \xrightarrow[n\to\infty]{} \ell \qquad \text{or} \qquad \lim_{n\to\infty} a_n = \ell,$$

  if for any $\varepsilon > 0$, there exists $N \geqslant 1$ such that $d(a_n, \ell) < \varepsilon$ for all $n \geqslant N$.

- We say that $(a_n)_{n\geqslant 1}$ converges if there exists $\ell \in M$ such that $(a_n)_{n\geqslant 1}$ converges to ℓ.

- If $(a_n)_{n\geqslant 1}$ does not converge, we say that $(a_n)_{n\geqslant 1}$ diverges.

Remark 2.4.2 :

(1) For $(M, d) = (\mathbb{R}, |\cdot|)$, we recover the classical (if you have seen) definition of the limit of a sequence in $\mathbb{R}$.

(2) The convergence $a_n \xrightarrow[n\to\infty]{} \ell$ in a metric space (M, d) can also be interpreted in an equivalent way as the convergence $d(a_n, \ell) \xrightarrow[n\to\infty]{} 0$ in $(\mathbb{R}, |\cdot|)$.

(3) The notion of convergence is a *topological notion*, in the sense that it only depends on the topology (we recall its definition in Remark 2.1.22) that the space is equipped with. See Exercise 2.24.

Example 2.4.3 :

(1) For $(M, d) = (\mathbb{R}, |\cdot|)$, the sequence defined by $a_n = (-1)^n$, $n \geqslant 1$, does not converge. However, the subsequences $(a_{2n})_{n\geqslant 1}$ and $(a_{2n+1})_{n\geqslant 1}$ converge respectively to 1 and -1.

(2) The sequence $(a_n = \frac{1}{n})_{n\geqslant 1}$ converges to 0 in $[0, 1]$ but diverges in $(0, 1]$.

(3) If we consider a discrete metric space, see Example 2.1.2 (4), then any convergent sequence $(a_n)_{n\geqslant 1}$ is eventually constant , i.e., there exists $N \geqslant 1$ such that $a_n = a_N$ for all $n \geqslant N$.

Lemma 2.4.4 : *The sequence $(a_n)_{n\geqslant 1}$ can converge to at most one point $\ell \in M$.*

Proof : By contradiction, suppose that $(a_n)_{n\geqslant 1}$ converges to ℓ_1 and ℓ_2 with $\ell_1 \neq \ell_2$. Given $\varepsilon > 0$, we may find $N_1, N_2 \geqslant 1$ such that

$$d(a_n, \ell_1) < \varepsilon, \qquad \forall n \geqslant N_1,$$
$$d(a_n, \ell_2) < \varepsilon, \qquad \forall n \geqslant N_2.$$

Therefore, we can take $n \geqslant \max(N_1, N_2)$ and apply the triangle inequality to deduce that

$$d(\ell_1, \ell_2) \leqslant d(a_n, \ell_1) + d(a_n, \ell_2) < 2\varepsilon.$$

Since ε can be arbitrarily small, for $\varepsilon < \frac{1}{2}d(\ell_1, \ell_2)$, we find a contradiction. □

2.4.2 Cauchy sequences and complete spaces

Definition 2.4.5 : A sequence $(a_n)_{n\geqslant 1}$ is said to be a *Cauchy sequence* (柯西序列) if for any $\varepsilon > 0$, there exists $N \geqslant 1$ such that

$$d(a_n, a_m) < \varepsilon, \qquad \forall n, m \geqslant N. \tag{2.5}$$

Proposition 2.4.6 : *If $(a_n)_{n\geqslant 1}$ is a convergent sequence in (M, d), then it is a Cauchy sequence.*

Remark 2.4.7 : We note that a Cauchy sequence does not converge necessarily. For example, in the metric space $(M, d) = ((0, 1], |\cdot|)$, the sequence $(a_n = \frac{1}{n})_{n\geqslant 1}$ is Cauchy, but does not converge.

Proof : Suppose that $(a_n)_{n\geqslant 1}$ is a convergent sequence with limit ℓ. Given $\varepsilon > 0$. By the definition of convergence, we may find $N \geqslant 1$ such that for any $n \geqslant N$, we have $d(a_n, \ell) < \frac{\varepsilon}{2}$. Therefore, for any $n, m \geqslant N$, we have

$$d(a_n, a_m) \leqslant d(a_n, \ell) + d(a_m, \ell) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$ □

Proposition 2.4.8 : *A Cauchy sequence is always bounded.*

Proof : Let $(a_n)_{n\geqslant 1}$ be a Cauchy sequence with values in a metric space (M, d). Fix $\varepsilon > 0$ and $N \geqslant 1$ such that Eq. (2.5) holds. The set $\{a_1, \dots, a_N\}$ is finite, so bounded. The set $\{a_n : n \geqslant N\}$ is also bounded because of the Cauchy condition

$$d(a_N, a_n) < \varepsilon, \qquad \forall n \geqslant N.$$

□

Remark 2.4.9 : We note that the notion of Cauchy sequence is not a topological notion. It cannot be defined by open sets, and depends on the distance that the metric space is equipped with. We may come back to the example mentioned in Example 2.3.4. The sequence $(a_n = \frac{1}{n})_{n\geqslant 1}$ is Cauchy in $((0,1], |\cdot|)$, but is not Cauchy in $((0,1], d)$, although they define the same notion of open sets. To see this, we have for any fixed $N \geqslant 1$ and $n, m \geqslant N$,

$$|a_n - a_m| \leqslant \frac{1}{N} \quad \text{but} \quad d(a_n, a_m) = |n - m|.$$

Definition 2.4.10 :

- A metric space (M, d) is said to be *complete* (完備) if every Cauchy sequence in (M, d) converges to a limit in M.
- A complete normed vector space $(V, \|\cdot\|)$ is called a *Banach space* (Banach 空間) .

Example 2.4.11 :

(1) Euclidean spaces $\mathbb{R}^n$ with $n \geqslant 1$ are complete.

(2) $\mathbb{Q}$ is not complete. We may consider an irrational point $x \in \mathbb{R}\backslash\mathbb{Q}$ and a sequence of rational numbers $(x_n)_{n\geqslant 1}$ converging to x in $\mathbb{R}$. This sequence is a Cauchy sequence in $\mathbb{Q}$ but does not converge in $\mathbb{Q}$.

Leter in Section 3.2, we will have a more thorough discussion about complete spaces.

2.4.3 Limits and adhernet points

In this subsection, we are going to give sequential characterizations of some topological notions, especially the notion of adherent points and closed sets, which can be better understood using sequences.

Below, we are given a sequence $(a_n)_{n\geqslant 1}$. For $p \geqslant 1$, let us write $A_p := \{a_n : n \geqslant p\}$ for the *range* (值域) of the sequence $(a_n)_{n\geqslant p}$ and $A := A_1$. We may also define

$$\mathcal{L} := \{\ell \in M : \text{there exists } \varphi : \mathbb{N} \to \mathbb{N} \text{ that is strictly increasing such that } a_{\varphi(n)} \xrightarrow[n\to\infty]{} \ell\}$$

to be the set of all the subsequential limits.

Proposition 2.4.12 : *Let $\ell \in M$ and suppose that $(a_n)_{n\geqslant 1}$ converges to ℓ. Then,*

(1) A is bounded,

(2) ℓ is an adherent point of A, that is $\ell \in \overline{A}$.

Proof : (1) is a direct consequence of Proposition 2.4.6 and Proposition 2.4.8.

To show (2), let us fix $\varepsilon > 0$. By the definition of convergence, we can find $N \geqslant 1$ such that $d(a_n, \ell) < \varepsilon$ for $n \geqslant N$. We deduce that $B(\ell, \varepsilon) \supseteq A_N = \{a_n : n \geqslant N\}$, where A_N is not empty. Since this property holds for any arbitrarily $\varepsilon > 0$, we deduce that ℓ is an adherent point of A. □

Proposition 2.4.13 : *Let $A \subseteq M$ be a subset and $\ell \in M$.*

(1) If ℓ is an adherent point of A, then one can find a sequence $(a_n)_{n\geqslant 1}$ with values in A that converges to ℓ.

(2) If ℓ is an accumulation point of A, then one can find a sequence $(a_n)_{n\geqslant 1}$ with values in $A\backslash\{\ell\}$ that converges to ℓ.

Proof : The construction is similar in both cases, let us start with (1). Let $\ell \in M$ be an adherent point of A. For every $n \geqslant 1$, since $B(\ell, \frac{1}{n}) \cap A$ is not empty, we may find $a_n \in A$ such that $d(\ell, a_n) < \frac{1}{n}$. We can easily see that the sequence $(a_n)_{n\geqslant 1}$ converges to ℓ. For (2), we may take $a_n \in B(\ell, \frac{1}{n}) \cap (A\backslash\{\ell\})$, which is nonempty for all $n \geqslant 1$. □

It is also convenient to use limits to describe closure and closed sets, which can be seen as a consequence of the above propositions.

Corollary 2.4.14 : *Let $A \subseteq M$ be a subset and $x \in M$. Then, $x \in \overline{A}$ if and only if there exists a sequence of points in A that converges to x.*

Proof : It is a direct consequence of Proposition 2.4.12 and Proposition 2.4.13. □

Corollary 2.4.15 : *Let $A \subseteq M$ be a subset. Then, A is closed if and only if every convergent sequence (in M) of points of A converges to a limit in A.*

Proof : It is a direct consequence of Corollary 2.4.14. □

The following proposition tells us when a sequence converges.

Proposition 2.4.16 : *Let $\ell \in M$. The sequence $(a_n)_{n\geqslant 1}$ converges to ℓ if and only if every subsequence $(a_{\varphi(n)})_{n\geqslant 1}$ converges to ℓ.*

Proof : We first assume that $(a_n)_{n\geqslant 1}$ converges to ℓ. Let $(a_{\varphi(n)})_{n\geqslant 1}$ be a subsequence of $(a_n)_{n\geqslant 1}$. Fix $\varepsilon > 0$. By the definition of convergence, there exists $N \geqslant 1$ such that $d(a_n, \ell) < \varepsilon$ for $n \geqslant N$. Since φ is strictly increasing, we also have $\varphi(n) \geqslant N$ for $n \geqslant N$. Therefore, $d(a_{\varphi(n)}, \ell) < \varepsilon$ for $n \geqslant N$.

If every subsequence of $(a_n)_{n\geqslant 1}$ converges to ℓ, then the original sequence also converges to ℓ, since $\varphi(n) = n$ is also an extraction. □

Before closing this subsection, we see a more general proposition which describes the structure of $\mathcal{L}$, the set of all the subsequential limits of $(a_n)_{n\geqslant 1}$.

Proposition 2.4.17 : *Let $\ell \in M$. The following properties are equivalent.*

(1) $\ell \in \mathcal{L}$.

(2) $\ell \in \overline{A_p}$ for all $p \geqslant 1$.

(3) ℓ is either an accumulation point of A, or ℓ appears infinitely many times in the sequence $(a_n)_{n\geqslant 1}$.

In particular, this implies that the set of the subsequential limits of $(a_n)_{n\geqslant 1}$ may also be rewritten as $\mathcal{L} = \cap_{p\geqslant 1}\overline{A_p}$, which is closed.

Proof : We are going to show that (1) ⇒ (2) ⇒ (3) ⇒ (1).

- (1) ⇒ (2). Suppose that $\ell \in \mathcal{L}$, that is there exists an extraction $\varphi : \mathbb{N} \to \mathbb{N}$ such that $a_{\varphi(n)} \xrightarrow[n\to\infty]{} \ell$. Therefore, it follows from Proposition 2.4.12 that

$$\ell \in \overline{\{a_{\varphi(n)} : n \geqslant 1\}} \subseteq \overline{A_{\varphi(1)}}.$$

  For any non-negative integer $p \geqslant 1$, the map $\varphi_p : \mathbb{N} \to \mathbb{N}, n \mapsto \varphi(n+p)$ is still an extraction, and the convergence $a_{\varphi_p(n)} \xrightarrow[n\to\infty]{} \ell$ still holds. Therefore, we deduce that $\ell \in \overline{A_{\varphi(p)}}$ for $p \geqslant 1$. Since the sequence of subsets $(A_p)_{p\geqslant 1}$ is non-increasing (for the inclusion), we deduce that

$$\bigcap_{p\geqslant 1} \overline{A_p} = \bigcap_{p\geqslant 1} \overline{A_{\varphi(p)}}.$$

- (2) ⇒ (3). Suppose that $\ell \in \overline{A_p}$ for all $p \geqslant 1$ and that ℓ does not appear infinitely many times in $(a_n)_{n\geqslant 1}$. Let $p \geqslant 1$ such that $a_n \neq \ell$ for all $n \geqslant p$. Since $\ell \in \overline{A_p}$ and $\ell \notin A_p$, we know that ℓ is an accumulation point of A_p, so also an accumulation point of A.

- (3) ⇒ (1). If ℓ appears infinitely many times in $(a_n)_{n\geqslant 1}$, it is easy to construct a subsequence with limit ℓ. Now, suppose that ℓ is an accumulation point of A. It follows from Proposition 2.4.13 that we may find $f : \mathbb{N} \to \mathbb{N}$ (not necessarily an extraction) such that $a_{f(n)} \xrightarrow[n\to\infty]{} \ell$ and $a_{f(n)} \in A\backslash\{\ell\}$ for all $n \geqslant 1$. The map f cannot be bounded, since otherwise $(a_{f(n)})_{n\geqslant 1}$ would only take finitely many different values, the sequence $(a_{f(n)})_{n\geqslant 1}$, being convergent, would be eventually constant (constant for large n), and would not be able to converge to ℓ. Thus, we may find an subsequence of $(f(n))_{n\geqslant 1}$ that is strictly increasing, denoted $(f \circ \varphi(n))_{n\geqslant 1}$. Then, $\psi := f \circ \varphi : \mathbb{N} \to \mathbb{N}$ is an extraction and $a_{\psi(n)} \xrightarrow[n\to\infty]{} \ell$. □

2.4.4 In a normed space

In this subsection, we are given a normed vector space $(V, \|\cdot\|)$ over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Proposition 2.4.18 : *Let $(x_n)_{n\geqslant 1}$ and $(y_n)_{n\geqslant 1}$ be two sequences in V. Suppose that*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y.$$

Then,

(1) $x_n + y_n \xrightarrow[n\to\infty]{} x + y$,

(2) $\lambda x_n \xrightarrow[n\to\infty]{} \lambda x$ *for any* $\lambda \in \mathbb{K}$,

(3) $\|x_n\| \xrightarrow[n\to\infty]{} \|x\|$.

Proof :

(1) Let us fix $\varepsilon > 0$ and take $N \geqslant 1$ such that for $n \geqslant N$, we have

$$\|x_n - x\| < \varepsilon \quad \text{and} \quad \|y_n - y\| < \varepsilon.$$

For $n \geqslant N$, we have

$$\|(x_n + y_n) - (x + y)\| \leqslant \|x_n - x\| + \|y_n - y\| < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have shown that $x_n + y_n \xrightarrow[n\to\infty]{} x + y$.

(2) We write directly

$$\|\lambda x_n - \lambda x\| = |\lambda|\,\|x_n - x\| \xrightarrow[n\to\infty]{} 0.$$

(3) The triangular inequality gives

$$|\,\|x_n\| - \|x\|\,| \leqslant \|x_n - x\| \xrightarrow[n\to\infty]{} 0.$$

□

2.4.5 Limit of a function

We consider two metric spaces (M, d) and (M', d'). Let $A \subseteq M$ be a subset of M, and let $f : A \to M'$ be a function from A to M'.

Definition 2.4.19 : Let a be an accumulation point of A and $b \in M'$. We say that when x tends to a, $f(x)$ tends to b, and write

$$\lim_{x \to a} f(x) = b,$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x \in A \backslash \{a\}, \quad d(x,a) < \delta \quad \Rightarrow \quad d'(f(x), b) < \varepsilon. \tag{2.6}$$

Proposition 2.4.20 : *Let a be an accumulation point of A and $b \in M'$. Then, the following properties are equivalent.*

(1) When x tends to a, $f(x)$ tends to b, that is

$$\lim_{x \to a} f(x) = b.$$

(2) For any sequence $(x_n)_{n \geqslant 1}$ with values in $A \backslash \{a\}$ converging to a, we have

$$\lim_{n \to \infty} f(x_n) = b.$$

Proof : Let us assume that (1) holds, that is $f(x) \to b$ when $x \to a$. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that Eq. (2.6) holds. Fix a sequence $(x_n)_{n \geqslant 1}$ with values in $A \backslash \{a\}$ converging to a. We may find $N \geqslant 1$ such that for $n \geqslant N$, we have $d(x_n, a) < \delta$. Therefore, for $n \geqslant N$, we also have $d'(f(x_n), b) < \varepsilon$. This shows that $f(x_n) \xrightarrow[n \to \infty]{} b$.

For the converse, let us proceed by contradiction. We assume that (2) holds but not (1). If (1) does not hold, we may find $\varepsilon > 0$ such that for every $n \geqslant 1$, there is $x_n \in A$ such that

$$0 < d(x_n, a) < \tfrac{1}{n} \quad \text{and} \quad d'(f(x_n), b) \geqslant \varepsilon.$$

It is clear that $(x_n)_{n \geqslant 1}$ converges to a, but $(f(x_n))_{n \geqslant 1}$ does not converge to b since there is always a positive distance at least ε between $f(x_n)$ and b. This contradicts (2). □

Proposition 2.4.21 : *Consider a normed vector space $(V, \|\cdot\|)$ over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $f, g : A \to V$*

be two functions, and a be an accumulation point of A. Assume that

$$\lim_{x \to a} f(x) = b, \quad \lim_{x \to a} g(x) = c.$$

Then,

(1) $\lim_{x \to a}(f(x) + g(x)) = b + c$,

(2) $\lim_{x \to a} \lambda f(x) = \lambda b$ *for every* $\lambda \in \mathbb{K}$,

(3) $\lim_{x \to a} \|f(x)\| = \|b\|$.

Proof : It is a direct consequence by applying Proposition 2.4.18 and Proposition 2.4.20. □

2.4.6 On the real line

Below, we are given a sequence $(a_n)_{n \geqslant 1}$ taking values in the metric space $(M, d) = (\mathbb{R}, |\cdot|)$. In Proposition 2.4.17, we saw how to characterize the subsequential limits of the sequence. We are going to see other notions of limits.

Definition 2.4.22 : We define

$$\overline{\lim_{n \to \infty}}\, a_n = \limsup_{n \to \infty} a_n := \inf_{n \geqslant 1} \sup_{k \geqslant n} a_k,$$
$$\underline{\lim_{n \to \infty}}\, a_n = \liminf_{n \to \infty} a_n := \sup_{n \geqslant 1} \inf_{k \geqslant n} a_k,$$

called the *upper limit* (上極限) and the *lower limit* (下極限) of $(a_n)_{n \geqslant 1}$.

Remark 2.4.23 : We note that, we may rewrite $\limsup_{n \to \infty} a_n$ as a non-increasing limit,

$$\limsup_{n \to \infty} a_n := \lim_{n \to \infty} \downarrow \sup_{k \geqslant n} a_k,$$

because the sequence $(\sup_{k \geqslant n} a_k)_{n \geqslant 1}$ is non-increasing. Similarly, $\liminf_{n \to \infty} a_n$ can be rewritten as a non-decreasing limit,

$$\liminf_{n \to \infty} a_n := \lim_{n \to \infty} \uparrow \inf_{k \geqslant n} a_k.$$

Example 2.4.24 :

(1) The sequence defined by $a_n = (-1)^n$ has upper limit 1 and lower limit -1.

(2) The sequence defined by $a_n = \sin(n)$ has upper limit 1 and lower limit -1.

Lemma 2.4.25 : *If $(a_{\varphi(n)})_{n\geqslant 1}$ is a convergent subsequence of $(a_n)_{n\geqslant 1}$, then its limit ℓ is an adherent point of $\{a_n : n \geqslant 1\}$ and satisfies*

$$\liminf_{n\to\infty} a_n \leqslant \ell := \lim_{n\to\infty} a_{\varphi(n)} \leqslant \limsup_{n\to\infty} a_n.$$

Proof : Let $(a_{\varphi(n)})_{n\geqslant 1}$ be a convergent subsequence of $(a_n)_{n\geqslant 1}$. It follows from Proposition 2.4.17 that its limit ℓ is an adherent point of the range $\{a_n : n \geqslant 1\}$.

Next, for any $n \geqslant 1$, we clearly have

$$\inf_{k\geqslant\varphi(n)} a_k \leqslant a_{\varphi(n)} \leqslant \sup_{k\geqslant\varphi(n)} a_k. \tag{2.7}$$

By taking a monotonic limit for the left inequality in Eq. (2.7), we find

$$\liminf_{n\to\infty} a_n = \sup_{n\geqslant 1} \inf_{k\geqslant\varphi(n)} a_k = \lim_{n\to\infty} \inf_{k\geqslant\varphi(n)} a_k \leqslant \lim_{n\to\infty} a_{\varphi(n)} = \ell.$$

If we do the same thing for the right inequality in Eq. (2.7), we find the other inequality. □

Lemma 2.4.26 : *There exist subsequences $(a_{\varphi(n)})_{n\geqslant 1}$ and $(a_{\psi(n)})_{n\geqslant 1}$ such that*

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_{\varphi(n)}, \qquad \text{and} \qquad \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_{\psi(n)}.$$

Proof : We are going to construct an extraction φ for the lower limit by induction. Let $\ell := \liminf_{n\to\infty} a_n$. Define

$$\varphi(1) := \inf\{n \geqslant 1 : \ell - 1 \leqslant a_n \leqslant \ell + 1\},$$
$$\forall n \geqslant 1, \qquad \varphi(n+1) := \inf\{n > \varphi(n) : \ell - \tfrac{1}{n} \leqslant a_n \leqslant \ell + \tfrac{1}{n}\}.$$

It is not hard to check that $\varphi(n)$ is well defined for all $n \geqslant 1$ and that φ is strictly increasing. Addi-

tionally, we easily see that $\lim a_{\varphi(n)} = \ell$. The construction works in a similar way for the upper limit.
□

Remark 2.4.27 : The above two lemmas justify the names of *upper limit* and *lower limit* given to $\limsup$ and $\liminf$.

Proposition 2.4.28 : *A sequence $(a_n)_{n\geqslant 1}$ in $\mathbb{R}$ converges if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n < \infty$.*

Proof : It is a direct consequence of Proposition 2.4.16 and the above lemmas (Lemma 2.4.26 and Lemma 2.4.25). □

Remark 2.4.29 : The limit of a real sequence needs not exist in general. However, its upper limit (resp. lower limit) always exist in $(-\infty, +\infty]$ (resp. in $[-\infty, +\infty)$). In order to write $\lim$, or to show that the limit exists, this proposition suggests that one may show that the upper limit and the lower limit are equal.

2.5 Continuity

2.5.1 Definition and properties

Below, we are given two metric spaces (M, d) and (M', d'). When we talk about balls in different metric spaces, we may add a subscript to avoid confusion. For example, $B_M(x, \varepsilon)$ or $B_d(x, \varepsilon)$ denotes the open ball centered at $x \in M$ with radius $\varepsilon > 0$ in (M, d).

Definition 2.5.1 : Given a function $f : (M, d) \to (M', d')$. We say that f is *continuous at* $x \in M$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall y \in M, \qquad d(x, y) < \delta \quad \Longrightarrow \quad d'(f(x), f(y)) < \varepsilon, \tag{2.8}$$

or equivalently,

$$f(B_M(x, \delta)) \subseteq B_{M'}(f(x), \varepsilon).$$

We say that f is *continuous* if it is continuous at all $x \in M$.

Example 2.5.2 :

(1) If we take $(M, d) = (M', d') = (\mathbb{R}, |\cdot|)$, then we recover the definition of continuity that we saw in the first-year calculus.

(2) The identity map $\mathrm{Id} : (M, d) \to (M, d), x \mapsto x$ is continuous.

(3) Fix $a \in M$. Then, the map $(M, d) \to (\mathbb{R}, |\cdot|), x \mapsto d(x, a)$ is continuous.

Remark 2.5.3 : If $a \in M$ is an accumulation point, then the continuity of f at a is equivalent to

$$\lim_{x \to a} f(x) = f(a).$$

If $a \in M$ is an isolated point, then any function $f : M \to M'$ is continuous at a, because for sufficiently small $\delta > 0$, the open ball $B(a, \delta)$ is reduced to the singleton $\{a\}$.

Proposition 2.5.4 : *Consider three metric spaces (M_1, d_1), (M_2, d_2), and (M_3, d_3). Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be two functions. Fix $x \in M_1$. If f is continuous at x and g is continuous at $f(x)$, then the composition $g \circ f : M_1 \to M_3$ is continuous at x.*

Proof : The proof is quite direct if we use Definition 2.5.1. Given $\varepsilon > 0$. Since g is continuous at $y := f(x)$, we may find $\eta > 0$ such that

$$g(B_{M_2}(y, \eta)) \subseteq B_{M_3}(g(y), \varepsilon).$$

Since f is continuous at x, we may find $\delta > 0$ such that

$$f(B_{M_1}(x, \delta)) \subseteq B_{M_2}(f(x), \eta) = B_{M_2}(y, \eta).$$

Putting the two above inclusions together, we find

$$(g \circ f)(B_{M_1}(x, \delta)) \subseteq g(B_{M_2}(y, \eta)) \subseteq B_{M_3}((g \circ f)(x), \varepsilon).$$

This leads to the continuity of $g \circ f$ at x. □

2.5.2 Sequential characterization

Proposition 2.5.5 : *Given a function $f : (M, d) \to (M', d')$ and $a \in M$. Then, the following properties are equivalent.*

(1) f is continuous at a.

(2) For every sequence $(x_n)_{n\geqslant 1}$ with values in M that converges to a, the sequence $(f(x_n))_{n\geqslant 1}$ with values in M' also converges to $f(a)$. In other words,

$$\lim_{n\to\infty} x_n = a \quad \Rightarrow \quad \lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big) = f(a).$$

Proof : The proof is similar to that of Proposition 2.4.20. □

Example 2.5.6 : The function $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0,

$$f(x) = \begin{cases} x\sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We can see this by taking any convergent sequence $(x_n)_{n\geqslant 1}$ with limit 0, then

$$|f(x_n)| = |x_n \sin(1/x_n)| \leqslant |x_n| \xrightarrow[n\to\infty]{} 0.$$

Proposition 2.5.7 : *Consider a normed vector space $(V, \|\cdot\|)$ over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $a \in M$ and $f, g : M \to V$ be two functions that are continuous at a. Then,*

(1) $x \mapsto f(x) + g(x)$ is continuous at a,

(2) $x \mapsto \lambda f(x)$ is continuous at a,

(3) $x \mapsto \|f(x)\|$ is continuous at a.

Proof : It is a direct consequence by applying Proposition 2.5.5 and Proposition 2.4.18. □

Example 2.5.8 : Let $n \geqslant 1$ and $P \in \mathbb{R}[X_1, \ldots, X_n]$ be a multivariate polynomial. Take $(M, d) = (\mathbb{R}^n, \|\cdot\|_1)$ and $(M', d') = (\mathbb{R}, |\cdot|)$. Then, the map $(a_1, \ldots, a_n) \mapsto P(a_1, \ldots, a_n)$ is continuous. This can be seen by using Proposition 2.5.5 and the following two facts.

(a) For any sequence $(a^k = (a_1^k, \ldots, a_n^k))_{k \geqslant 1}$ with values in $(\mathbb{R}^n, \|\cdot\|_1)$, we have

$$\lim_{k \to \infty} a^k = a = (a_1, \ldots, a_n) \quad \Leftrightarrow \quad \lim_{k \to \infty} a_i^k = a_i, \quad \forall i = 1, \ldots, n.$$

(b) For any real-valued sequences $(x_n)_{n \geqslant 1}$ and $(y_n)_{n \geqslant 1}$, we have

$$\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y \quad \Rightarrow \quad \lim_{n \to \infty} x_n y_n = xy$$

2.5.3 Characterization using preimage

Definition 2.5.9 : Given a function $f : (M, d) \to (M', d')$ and a subset $A \subseteq M'$. We recall the definition viewed in Definition 1.1.7 of preimage or inverse image (像原) of A under f,

$$f^{-1}(A) := \{x \in M : f(x) \in A\}.$$

Remark 2.5.10 : We recall the following properties for the preimage.

(1) If f is bijective, then the preimage of A under f is exactly the image of A under f^{-1}.

(2) If $A \subseteq B \subseteq M'$, then $f^{-1}(A) \subseteq f^{-1}(B) \subseteq M$.

(3) For $A \subseteq M$, we have $A \subseteq f^{-1}(f(A))$.

(4) For $A \subseteq M'$, we have $f(f^{-1}(A)) \subseteq A$.

Proposition 2.5.11 : *Let $f : (M, d) \to (M', d')$ be a function. The following properties are equivalent.*

(1) f is continuous on M.

(2) The preimage of any open set of M' is open in M.

(3) The preimage of any closed set of M' is closed in M.

Proof : We are going to prove that (1) ⇔ (2) ⇔ (3).

- (1) ⇒ (2). Let $A' \subseteq M'$ be an open set and denote $A = f^{-1}(A')$. Given $x \in A$, we want to show that x is an interior point of A. Let $y = f(x) \in A'$. Since y is an interior point of A', we may find $\varepsilon > 0$ such that $B_{M'}(y, \varepsilon) \subseteq A'$. Using the continuity of f at x, we may find $\delta > 0$ such that $f(B_M(x, \delta)) \subseteq B_{M'}(y, \varepsilon) \subseteq A'$. Therefore, $x \in B_M(x, \delta) \subseteq f^{-1}(A')$.

- (2) ⇒ (1). Given $x \in M$ and $\varepsilon > 0$, it follows from (2) that $A = f^{-1}(B_{M'}(f(x), \varepsilon))$ is open. Since $x \in A$, we may find $\delta > 0$ such that $B_M(x, \delta) \subseteq A$. This implies that $f(B_M(x, \delta)) \subseteq f(A) = B_{M'}(f(x), \varepsilon)$, giving the continuity of f at x.

- (2) ⇒ (3). Let A' be a closed set in M', then $B' := M' \backslash A'$ is an open set. We know that

$$f^{-1}(A') = f^{-1}(M' \backslash B') = M \backslash f^{-1}(B').$$

  By (2), the set $f^{-1}(B')$ is open, so $f^{-1}(A')$ is closed.

- (3) ⇒ (2). The proof is similar. □

Remark 2.5.12 : In practice, to check that a function $f : (M, d) \to (M', d')$ is continuous, we only need to check the following modified condition:

(2') The preimage of any open ball of M' is open in M.

Example 2.5.13 : We identify the space $\mathcal{M}_n(\mathbb{R})$ of $n \times n$ real matrices as $\mathbb{R}^{n^2}$, and equip it with the usual norm $\|\cdot\|_1$. The determinant function $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous. Since $\mathbb{R}^* := \mathbb{R} \backslash \{0\}$ is open in $\mathbb{R}$, the set of invertible matrices

$$\mathcal{GL}_n(\mathbb{R}) := \{M \in \mathcal{M}_n(\mathbb{R}) : \det(M) \neq 0\} = \det^{-1}(\mathbb{R}^*)$$

is also open in $\mathcal{M}_n(\mathbb{R})$.

Definition 2.5.14 : Let $f : (M, d) \to (M', d')$ be a function. We say that f is

- an *open map* (開函數) if $f(A)$ is open in M' for any open set $A \subseteq M$;
- a *closed map* (閉函數) if $f(A)$ is closed in M' for any closed set $A \subseteq M$.

Remark 2.5.15 : Note that in Proposition 2.5.11, it is important to look at the *preimage*.

- A continuous function is not necessarily an open map. For example, a constant function from $\mathbb{R}$ to $\mathbb{R}$ maps the open set $\mathbb{R}$ to a point which is not open.
- A continuous function is not necessarily a closed map. For example, the function $\mathbb{R} \to \mathbb{R}, x \mapsto \tan(x)$ maps the closed set $\mathbb{R}$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$, which is not closed in $\mathbb{R}$.

2.5.4 Isomorphisms

We are going to introduce two notions of *isomorphisms* (同構) : *isometric isomorphism* and topological isomorphism (homeomorphism) (拓撲同構、同胚). Below, consider two metric spaces (M, d) and (M', d').

Definition 2.5.16 :

- A bijective function $f : (M, d) \to (M', d')$ is called an *isometry* (等距變換) if

$$d'(f(x), f(y)) = d(x, y), \quad \forall x, y \in M.$$

- If there exists an isometry between (M, d) and (M', d'), then we say that the metric spaces (M, d) and (M', d') are *isometric* or *isometrically isomorphic*.

Example 2.5.17 : Let us fix an integer $n \geqslant 1$. We denote by $\mathcal{M}_n(\mathbb{R}) = \mathcal{M}_{n\times n}(\mathbb{R})$ the vector space of n by n real matrices. We may equip $\mathcal{M}_n(\mathbb{R})$ with the norm $\|\cdot\|_{\mathcal{M},1}$ defined by

$$\forall M = (m_{i,j})_{1\leqslant i,j\leqslant n}, \quad \|M\|_{\mathcal{M},1} = \sum_{i=1}^{n}\sum_{j=1}^{n} |m_{i,j}|,$$

and consider the distance $d_{\mathcal{M},1}$ induced by the norm $\|\cdot\|_{\mathcal{M},1}$. Then, $(\mathcal{M}_n(\mathbb{R}), d_{\mathcal{M},1})$ and $(\mathbb{R}^{n^2}, d_1)$ are isometric. For example, the map

$$M = (m_{i,j})_{1\leqslant i,j\leqslant n} \mapsto (m_{1,1}, \dots, m_{1,n}, m_{2,1}, \dots, m_{2,n}, \dots, m_{n,1}, \dots, m_{n,n}),$$

is an isometry.

Definition 2.5.18 :

- Let $f : (M, d) \to (M', d')$ be a function. Suppose that f is bijective, so that f^{-1} is well defined. We say that f is a *homeomorphism* (同胚) , or *topological isomorphism* (拓撲同構) , if both f and f^{-1} are continuous.
- If there exists an homeomorphism f between (M, d) and (M', d'), then we say that the metric spaces (M, d) and (M', d') are homeomorphic or topologically isomorphic.

Remark 2.5.19 : An isometry is also an homeomorphism.

Example 2.5.20 : Let us consider $M = \mathbb{R}^2$ with different distances d_1 induced by $\|\cdot\|_1$, d_2 induced by $\|\cdot\|_2$, and the discrete distance d_{discrete}.

(1) The identity map $\mathrm{Id} : (\mathbb{R}^2, d_1) \to (\mathbb{R}^2, d_2)$ is a homeomorphism because we have

$$B_{d_1}(x, r) \subseteq B_{d_2}(x, r) \subseteq B_{d_1}(x, \sqrt{2}r). \tag{2.9}$$

(2) The identity map $\mathrm{Id} : (\mathbb{R}^2, d_{\text{discrete}}) \to (\mathbb{R}^2, d_1)$ is not a homeomorphism. This map is bijective and continuous, but its inverse f^{-1} is clearly not continuous.

Definition 2.5.21 : Let d and d' be two distances on M. We say that the two distances are *topologically equivalent* (拓撲等價) if they define the same topology, in the sense that a set in (M, d) is open if and only if it is also open in (M, d').

Example 2.5.22 : In $\mathbb{R}^2$, the distances d_1 and d_2 are topologically equivalent, as seen in Eq. (2.9).

Proposition 2.5.23 : *Let d and d' be two distances on M. The distances d and d' are topologically equivalent if and only if the identity map $\mathrm{Id} : (M, d) \to (M, d')$ is a homeomorphism.*

Proof : First, let us assume that the distances d and d' are topologically equivalent. It is clear that the identity map $\mathrm{Id} : (M, d) \to (M, d')$ is bijective. To show its continuity, consider an open set $A \subseteq (M, d')$. Then,

$$\mathrm{Id}^{-1}(A) = A \subseteq (M, d)$$

is still an open set due to the assumption. Hence, Id is continuous. Similarly, we can also show that Id^{-1} is continuous.

For the converse, we assume that the identity map $\mathrm{Id} : (M, d) \to (M, d')$ is a homeomorphism. By its continuity, any open set $A \subseteq (M, d')$ is still open in (M, d), and vice versa. It is exactly the definition of two distances which are topologically equivalent. □

Definition 2.5.24 :

- Given a vector space V and two norms N_1 and N_2 on V. They are said to be *equivalent* if there exist $b > a > 0$ such that

$$a\, N_1(x) \leqslant N_2(x) \leqslant b\, N_1(x), \qquad \forall x \in V.$$

- Given a space M and two distances d_1 and d_2 on M. They are said to be *equivalent* if there exist $b > a > 0$ such that

$$a\, d_1(x, y) \leqslant d_2(x, y) \leqslant b\, d_1(x, y), \qquad \forall x, y \in M.$$

Example 2.5.25 : In $\mathbb{R}^n$, the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are equivalent. In fact, we have

$$\|x\|_\infty \leqslant \|x\|_1 \leqslant \|x\|_2 \leqslant \sqrt{n}\, \|x\|_\infty, \quad \forall x \in \mathbb{R}^n.$$

Remark 2.5.26 :

(1) Two equivalent norms induce two distances that are also equivalent.

(2) Two equivalent distances define two metric spaces that are topologically equivalent. This can be seen using inclusion relations between balls defined by different distances Example 2.5.20 (1).

(3) Later in Theorem 3.2.22, we will see that on a finite dimensional vector space, all the norms are equivalent.

2.5.5 Uniform continuity

Definition 2.5.27 : Let $f : (M, d) \to (M', d')$ be a function. We say that f is *uniformly continuous* (均勻連續) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, y \in M, \qquad d(x, y) < \delta \implies d'(f(x), f(y)) < \varepsilon. \tag{2.10}$$

Example 2.5.28 : The function $f : \mathbb{R}_{>0} \to \mathbb{R}, x \mapsto \frac{1}{x}$ is continuous. It is not uniformly continuous on $(0, 1]$, but is uniformly continuous on $[1, \infty)$.

Remark 2.5.29 :

(1) An uniformly continuous function is continuous, but the inverse does not hold in general, as we just saw in Example 2.5.28.

(2) In the definition of uniform continuity, the choice of δ does not depend on x and y, that is why it is called *uniform*. You may compare (2.8) and (2.10) to see the difference.

(3) Uniform continuity is not a topological notion, in the sense that it cannot be defined only using the open sets. See Exercise 2.41.

(4) Given a uniformly continuous function $f : (M_1, d_1) \to (M_2, d_2)$ and distances d_1' and d_2' such that d_1 and d_1' are equivalent, d_2 and d_2' are equivalent. Then, it is not hard to see that the function $f : (M_1, d_1') \to (M_2, d_2')$ is also uniformly continuous.

Definition 2.5.30 : Let $f : (M, d) \to (M', d')$ be a function. Given $K > 0$. We say that f is *K-Lipschitz continuous* if

$$d'(f(x), f(y)) \leqslant K\, d(x, y), \qquad \forall x, y \in M.$$

We also say that f is *Lipschitz continuous* if there exists $K > 0$ such that f is K-Lipschitz continuous.

Corollary 2.5.31 : *Any Lipschitz continuous function is also uniformly continuous.*

Proof : It is a direct consequence by taking $\delta = \varepsilon/K$ in (2.10) if the function $f : (M, d) \to (M, d')$ is K-Lipschitz. □

Definition 2.5.32 : Given a space M and two distances d and d' on M. They are said to be *uniformly equivalent* (均勻等價) if the identity map $\mathrm{Id} : (M, d) \to (M, d')$ and its inverse are uniformly continuous.

Remark 2.5.33 : Two equivalent distances are uniformly equivalent, and two uniformly equivalent distances are topologically equivalent.

2.6 Product of metric spaces

Given n metric spaces $(M_1, d_1), \dots, (M_n, d_n)$. We define the product space $M = M_1 \times \cdots \times M_n$ and want to equip it with a distance. There are several ways to achieve this using the distances $d_1, \dots, d_n$. The canonical way is as follows.

Definition 2.6.1 : We may equip the product space M with the product distance d defined as follows,

$$d(x, y) = \max_{1 \leqslant i \leqslant n} d_i(x_i, y_i), \tag{2.11}$$

for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in M$.

Remark 2.6.2 : The open ball centered at $x = (x_1, \dots, x_n)$ with radius r under the distance (2.11) is given by

$$B_d(x, r) = B_{d_1}(x_1, r) \times \cdots \times B_{d_n}(x_n, r).$$

Remark 2.6.3 : We may also define other distances on the product space M. Let

$$D_1(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i) \quad \text{and} \quad D_2(x, y) = \sqrt{\sum_{i=1}^{n} d_i(x_i, y_i)^2},$$

which are also distances on M. They are equivalent to the product distance d defined in (2.11), because

$$d(x,y) \leqslant D_2(x,y) \leqslant D_1(x,y) \leqslant n\, d(x,y), \quad \forall x,y \in E.$$

Therefore, it does not really matter which of these three distances we choose on the product space M.

Definition 2.6.4 : For $1 \leqslant i \leqslant n$, we may define the *projection* on the i-th coordinate of the product space M,

$$\begin{array}{rrcl} \mathrm{Proj}_i : & M = M_1 \times \cdots \times M_n & \to & M_i \\ & x = (x_1, \ldots, x_n) & \mapsto & x_i. \end{array}$$

Proposition 2.6.5 : *The projection* Proj_i *is continuous and open (Definition 2.5.14) for all* $1 \leqslant i \leqslant n$.

Proof : Fix $1 \leqslant i \leqslant n$.

- First, let us check that Proj_i is continuous. Following Remark 2.5.12, we only need to check the preimage of an open ball under Proj_i is open. Let $y \in M_i$ and $\varepsilon > 0$. It is not hard to check that

$$\mathrm{Proj}_i^{-1}(B_{M_i}(y,\varepsilon)) = M_1 \times \cdots \times M_{i-1} \times B_{M_i}(y,\varepsilon) \times M_{i+1} \times \cdots \times M_n.$$

  The r.h.s. is clearly an open set.

- Then, let us check that Proj_i is an open map. Given an open set $A \subseteq M$ and $y \in \mathrm{Proj}_i(A)$. Then, there exists $x \in A$ with $x_i = y$. Since A is open, there exists $r > 0$ such that $B_d(x,r) \subseteq A$. We know that the open ball in the product space can be written as the product of open balls (Remark 2.6.2), we deduce that $\mathrm{Proj}_i(B_d(x,r)) = B_{d_i}(x_i,r)$. Therefore, $y = x_i = \mathrm{Proj}_i(x) \in B_{d_i}(x_i,r) = \mathrm{Proj}_i(B_d(x,r)) \subseteq \mathrm{Proj}_i(A)$, implying that y is an interior point of $\mathrm{Proj}_i(A)$. □

Proposition 2.6.6 : *Let* (M', d') *be a metric space,* $a \in M'$*, and* $f : M' \to M$ *be a function. Then,* f *is continuous at* a *if and only if* $f_i := \mathrm{Proj}_i \circ f$ *is continuous at* a *for all* $1 \leqslant i \leqslant n$.

Proof : If f is continuous at a, it is not hard to see that f_i is continuous at a for all $1 \leqslant i \leqslant n$ by composition (Proposition 2.5.4). Conversely, suppose that f is a function such that f_i is continuous at a for all $1 \leqslant i \leqslant n$, we are goinig to show that f is also continuous at a. Let $\varepsilon > 0$. For each $1 \leqslant i \leqslant n$, we can find $\delta_i > 0$ such that for $x \in M$,

$$d'(x, a) < \delta_i \quad \Rightarrow \quad d_i(f_i(x), f_i(a)) < \varepsilon.$$

Since the product space $M = M_1 \times \cdot \times M_n$ is equipped with the metric defined in (2.11), by letting $\delta = \min_{1 \leqslant i \leqslant n} \delta_i$, for $x \in M$, we have,

$$d'(x, a) < \delta \quad \Rightarrow \quad d(f(x), f(a)) = \max_{1 \leqslant i \leqslant n} d_i(f_i(x), f_i(a)) < \varepsilon.$$

This shows the continuity of f at a. □

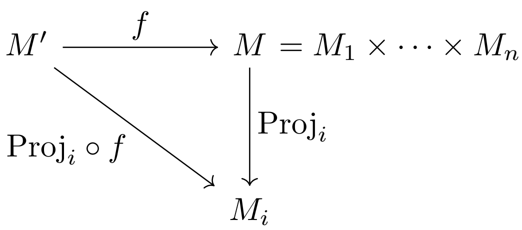

Figure 2.1: This diagram illustrates the relation between the function $f : M' \to M$, the projection $\text{Proj}_i : M \to M_i$, and thir composition.

Proposition 2.6.7 : *Let (M', d') be a metric space, $f : M \to M'$ be a function, and $a = (a_1, \ldots, a_n) \in M$. For $1 \leqslant i \leqslant n$, let us define the* partial function

$$\begin{array}{rccl} f^i: & M_i & \to & M' \\ & x & \mapsto & f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n). \end{array}$$

If f is continuous at a, then f^i is continuous at a_i for all $1 \leqslant i \leqslant n$.

Remark 2.6.8 : Note that the converse of Proposition 2.6.7 does not hold. For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(0, 0) = 0,$$
$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad \forall (x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}.$$

Take $a = (0, 0)$, then $f^1 \equiv 0$ and $f^2 \equiv 0$ are continuous functions, but

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2} \longrightarrow \frac{1}{2}, \quad \text{when } x \to 0.$$

Proof : For $x \in M_i$, let us write $a_x^{(i)} = (a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$. If $d_i(x, a_i) < \delta$, then it is clear that $d(a_x^{(i)}, a) < \delta$. Hence, if $x \in B_{d_i}(a_i, \delta)$, then $a_x^{(i)} \in B_d(a, \delta)$. This tells us that the continuity of f at a implies the continuity of f^i at a_i. □

2.7 Connectedness and arcwise connectedness

We are given a metric space (M, d), and we are going to study its connectedness properties below.

2.7.1 Connected spaces

Let us start with the definition of connected spaces.

Definition 2.7.1 (and properties) **:** We say that (M, d) is *connected* (連通) if one of the three following equivalent properties are satisfied.

(a) There is no partition of M into two disjoint nonempty open sets.

(b) There is no partition of M into two disjoint nonempty closed sets.

(c) The only subsets of M that are open and closed are $\varnothing$ and M.

Otherwise, we say that (M, d) is *disconnected* (不連通) . Similarly, in a metric space (M, d), a subset $A \subseteq M$ is said to be connected if the induced metric space (A, d) is connected.

Remark 2.7.2 : To check the property (a), one may assume that there exist open sets $A, B \subseteq M$ with $A \cap B = \varnothing$ and $A \cup B = M$, and show that either $A = \varnothing$ or $B = \varnothing$.

Proof : We are going to show that (a) ⇒ (b) ⇒ (c) ⇒ (a).

- (a) ⇒ (b). Suppose that there exist two closed sets A_1 and A_2 such that $M = A_1 \cup A_2$ and

$A_1 \cap A_2 = \varnothing$. Then, $B_1 = M \backslash A_1$ and $B_2 = M \backslash A_2$ are open sets. Moreoever, they satisfy $M = B_1 \cup B_2$ and $B_1 \cap B_2 = \varnothing$. By (a), we know that either $B_1 = \varnothing$ or $B_2 = \varnothing$, and it follows that $A_2 = \varnothing$ or $A_1 = \varnothing$.

- (b) $\Rightarrow$ (c). Let $A \subseteq M$ be open and closed. Then, $B := M \backslash A$ is also open and closed. Moreover, we have $M = A \cup B$ and $A \cap B = \varnothing$. Then, the assumption (b) implies that either $A = \varnothing$ or $B = \varnothing$, or equivalently, $A = \varnothing$ or M.
- (c) $\Rightarrow$ (a). Let A_1 and A_2 be two disjoint open sets such that $M = A_1 \cup A_2$. Then, A_1 can be rewritten as $A_1 = M \backslash A_2$, so it is also a closed set. By (c), we know that $A_1 = \varnothing$ or M. □

Remark 2.7.3 : The notion of connectedness is a topological notion, that is, it only depends on the notion of open sets (in the metric space), without the knowledge on the exact distance we are considering.

Example 2.7.4 :

(1) The metric space $\mathbb{R}^* = \mathbb{R} \backslash \{0\}$, induced by the Euclidean metric $(\mathbb{R}, |\cdot|)$, is not connected. Actually, we have $\mathbb{R}* = (-\infty, 0) \cup (0, \infty)$ which is a disjoint union of open sets.

(2) In any nonempty metric space, a singleton set $\{x\}$ is connected for every $x \in M$.

(3) Intervals of $\mathbb{R}$ are connected. We will prove this in Proposition 2.7.17.

(4) The set $\mathbb{Q}$ of rational numbers is disconnected.

2.7.2 Properties of connected spaces

Proposition 2.7.5 : *Let $f : (M, d) \to (M', d')$ be a continuous function. Suppose that M is connected. Then, $f(M)$ is also connected.*

Proof : Let A be an open and closed subset of $f(M)$. Thus, there exists an open subset $B_1 \subseteq M'$ and a closed subset $B_2 \subseteq M'$ such that

$$A = B_1 \cap f(M) = B_2 \cap f(M).$$

It follows from above that $f^{-1}(A) = f^{-1}(B_1) = f^{-1}(B_2)$, and the continuity of f implies that $f^{-1}(A)$ is open and closed in M. Since M is connected, we know that $f^{-1}(A) = \varnothing$ or M, that is $A = \varnothing$ or $f(M)$. □

Let us consider a discrete space with only two points $D = \{0, 1\}$ equipped with the discrete distance δ. Then, the metric space (D, δ) is disconnected because $D = \{0\} \cup \{1\}$ which is a disjoint union of closed (also open) sets. This discrete metric space will be useful for the characterization of connectedness.

Corollary 2.7.6 : *Let (M, d) be a metric space. Then, M is connected if and only if every continuous function $f : M \to D$ is constant.*

Proof : First, let us assume that M is connected. Given a continuous function $f : M \to D$, by Proposition 2.7.5, we know that the $f(M)$ is connected in D. Since D is disconnected, the image $f(M)$ cannot be the whole space, so $f(M) = \{0\}$ or $\{1\}$, that is, f is constant.

Suppose that every continuous function $f : M \to D$ is constant, and we want to show that M is connected. By contradiction, suppose that M is disconnected. Then, we can find two disjoint nonempty open subsets A and B such that $M = A \cup B$. Define $f : M \to D$ as follows,

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B. \end{cases}$$

The function f is clearly continuous because $\{0\}$ and $\{1\}$ are open sets in D, and their preimages $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$ are also open. However, f is not a constant function. □

Corollary 2.7.7 : *Let (M, d) be a metric space, and $A \subseteq M$ be a connected subset. Let S be a subset satisfying $A \subseteq S \subseteq \overline{A}$. Then, S is also connected.*

Proof : Let $f : S \to D = \{0, 1\}$ be a continuous function. Its restriction $f_{|A}$ on A is also continuous, thus constant, since A is connected. Assume for instance that $f_{|A} \equiv 0$. Let $x \in S$. By the continuity of f, there exists $\varepsilon > 0$ such that

$$y \in B(x, \varepsilon) \cap S \quad \Rightarrow \quad \delta(f(y), f(x)) < \tfrac{1}{2}.$$

This means that $f(y) = f(x)$ for $y \in B(x,\varepsilon) \cap S$. Additionally, since $S \subseteq \overline{A}$, we have $B(x,\varepsilon) \cap A \neq \varnothing$. We may choose $x' \in B(x,\varepsilon) \cap A$, then $f(x') = 0$, giving $f(y) = 0$ for $y \in B(x,\varepsilon) \cap S$. Therefore, $f \equiv 0$, so the result follows from Proposition 2.7.5. ☐

Proposition 2.7.8 : *Let (M,d) be a metric space and $(C_i)_{i\in I}$ be a family of connected subsets of M. Suppose that there exists $i_0 \in I$ such that*

$$C_i \cap C_{i_0} \neq \varnothing, \quad \forall i \in I.$$

Then, $C = \cup_{i\in I} C_i$ is connected.

Proof : Let $f : C = \cup_{i\in I} C_i \to D = \{0,1\}$ be a continuous function. For every $i \in I$, since C_i is connected, $f_{|C_i}$ is constant. In particular, we may assume that $f_{|C_{i_0}} \equiv 0$. Let $x \in C$ and $i \in I$ such that $x \in C_i$. Since $C_i \cap C_{i_0} \neq \varnothing$, we may find $x_0 \in C_i \cap C_{i_0}$. Due to the fact that $f_{|C_i}$ is constant, it follows that $f(x) = f(x_0) = 0$. Therefore, f is constant on C, and we conclude by Corollary 2.7.6. ☐

Remark 2.7.9 : In particular, if $(C_i)_{i\in I}$ is a family of connected subsets such that $\cap_{i\in I} C_i \neq \varnothing$, then $C = \cup_{i\in I} C_i$ is also connected.

Question 2.7.10: Let $(C_i)_{i\in I}$ be a countable family of connected subsets, i.e., $I = \{1,\ldots,p\}$ for some $p \geqslant 1$ or $I = \mathbb{N}$. Suppose that for every $i \in I$, $i \neq 1$, we have $C_{i-1} \cap C_i \neq \varnothing$. Show that $C = \cup_{i\in I} C_i$ is connected by rewriting the proof of Proposition 2.7.8.

Proposition 2.7.11 : *Given a sequence of metric spaces $(M_1,d_1),\ldots,(M_n,d_n)$ and consider the product metric space (M,d) given by $M = M_1 \times \cdots \times M_n$ and the product distance defined in Eq. (2.11). Then, (M,d) is connected if and only if (M_i,d_i) is connected for all $1 \leqslant i \leqslant n$.*

Proof : First, let us assume that M is connected. Fix $i \in \{1,\ldots,n\}$ and let $f : M_i \to D = \{0,1\}$ be a continuous function. Since the projection $\mathrm{Proj}_i : M \to M_i$ is continuous, the composition $f \circ \mathrm{Proj}_i : M \to D$ is also continuous. From the connectedness of M, we deduce that $f \circ \mathrm{Proj}_i$ is constant. Since $\mathrm{Proj}_i(M) = M_i$, it follows that f is also constant, that is M_i is connected.

Let us assume that (M_i,d_i) is connected for $1 \leqslant i \leqslant n$. Consider a continuous function $f : M \to D$.

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in M$. We want to show that $f(x) = f(y)$. First, it follows from Proposition 2.6.7 that the following map is continuous,

$$\begin{array}{rccc} f^1: & M_1 & \to & D \\ & z_1 & \mapsto & f(z_1, x_2, \ldots, x_n). \end{array}$$

The connectedness of M_1 implies that f^1 is constant, that is $f(x_1, x_2, \ldots, x_n) = f(y_1, x_2, \ldots, x_n)$. Then, we may look at the partial function at each of the following coordinates to conclude that $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$. Hence, the continuous function f is constant, and M is connected by Corollary 2.7.6. □

2.7.3 Connected components

Let (M, d) be a metric space. In this subsection, we are going to study the *connected components* of M, whose precise definition will be given below. Intuitively speaking, we want to decompose M into disjoint pieces of connected subspaces, and to achieve this, we will define an equivalence relation on M.

Definition 2.7.12 : We define the following binary relation $\mathcal{R}$ on (M, d),

$$x\mathcal{R}y \quad \Leftrightarrow \quad \text{there exists a connected subset } C \subseteq M \text{ such that } x, y \in C. \tag{2.12}$$

Proposition 2.7.13 : *The binary relation $\mathcal{R}$ defined in Eq. (2.12) is an equivalence relation.*

Proof : It is straightforward to check.

- (Reflexivity) For every $x \in M$, we have $x\mathcal{R}x$ since $\{x\}$ is connected.
- (Symmetry) If x, y are such that $x\mathcal{R}y$, then it follows from Eq. (2.12) that $y\mathcal{R}x$.
- (Transitivity) Let $x, y, z \in M$ such that $x\mathcal{R}y$ and $y\mathcal{R}z$. This means that there exist two connected subsets C and C' such that $x, y \in C$ and $y, z \in C'$. Since $C \cap C' \neq \varnothing$, it follows from Proposition 2.7.8 that $C \cup C'$ is also connected. We have $x, z \in C \cup C'$, so $x\mathcal{R}z$. □

Remark 2.7.14 : Proposition 2.7.13 allows us to define equivalence classes $M/\mathcal{R}$. For each $x \in M$, let us denote by $[x]$ its equivalence class. It is not hard to see that $[x]$ is given by the union of all the connected subsets containing x, which is again connected by Proposition 2.7.8. The subset $[x]$ is called a *connected component* (連通元件) of M. The connected components of M form a *partition* of M, that is a collection of disjoint subsets whose union is M. And we can see that M is connected if and only if it has only one connected component.

Corollary 2.7.15 : *The connected components of a metric space (M, d) are closed subsets. Moreover, if M only has finitely many connected components, then they are also open subsets.*

Proof : Let $x \in M$ and consider its connected component $[x]$. Since $[x] \subseteq \overline{[x]}$, it follows from Corollary 2.7.7 that $\overline{[x]}$ is also connected. We see that $\overline{[x]}$ also contains x, so $[x] = \overline{[x]}$, that is $[x]$ is a closed subset.

Suppose that M has only finitely many connected components, that is,

$$M = \bigcup_{i=1}^{N} \overline{[x_i]}, \quad N \geqslant 1, x_1, \ldots, x_N \in M.$$

Then, for any $1 \leqslant i \leqslant N$, we have

$$\overline{[x_i]} = M \setminus \bigcup_{\substack{1 \leqslant j \leqslant N \\ j \neq i}} \overline{[x_j]},$$

which is open, being the complement of a finite union of closed sets. □

Remark 2.7.16 : We give an example below of a subspace of $(\mathbb{R}, |\cdot|)$ which has one connected component that is not an open subset. Let

$$C = \Big(\bigcup_{n \geqslant 1} C_n \Big) \cup \{0\}, \quad C_n = [2^{-2n-1}, 2^{-2n}].$$

We first note that, all the C_n's and $\{0\}$ are connected components of C. It is also not hard to see that for each $n \geqslant 1$, the subset C_n is open and closed (in C) at the same time, because

$$\begin{aligned} C_n &= [2^{-2n-1}, 2^{-2n}] \cap C \\ &= (r \cdot 2^{-2n-1}, r^{-1} \cdot 2^{-2n}) \cap C, \quad \text{for some } r \in (\tfrac{1}{2}, 1). \end{aligned}$$

However, $\{0\}$ is a closed subset but not an open subset. To see this, suppose that it is open, that is we may find $\varepsilon > 0$ such that $B(0,\varepsilon) \cap C = \{0\}$. But for any $\varepsilon > 0$, the intersection $B(0,\varepsilon) \cap C$ contains not only 0 but also the subsets C_n's for sufficiently large n (as long as $n \geqslant \frac{1}{2}\log_2(1/\varepsilon)$).

2.7.4 Open sets and connected components in $\mathbb{R}$

We are going to look at the metric space $(\mathbb{R}, |\cdot|)$. Let us recall that $I \subseteq \mathbb{R}$ is an interval if for any $a, b \in I$, then

$$x \in (a,b) \quad \Rightarrow \quad x \in I. \tag{2.13}$$

There are four types of them,

$$\begin{aligned}
(a,b), &\quad -\infty \leqslant a \leqslant b \leqslant +\infty,\\
[a,b), &\quad -\infty < a \leqslant b \leqslant +\infty,\\
(a,b], &\quad -\infty \leqslant a \leqslant b < +\infty,\\
[a,b], &\quad -\infty < a \leqslant b < +\infty.
\end{aligned}$$

We note that the last type of intervals are also called *segments*.

Proposition 2.7.17 : *A subset I of $\mathbb{R}$ is connected if and only if it is an interval of $\mathbb{R}$.*

Proof : Let us assume that $I \subseteq \mathbb{R}$ is connected. By contradiction, if I is not an interval, it means that we may find $a, b \in I$ and $x \in (a,b)$ with $x \notin I$. In this case, we have $I \subseteq (-\infty, x) \cup (x, +\infty)$, so I is not connected.

For the converse, given an interval $I \subseteq \mathbb{R}$, we want to show that it is connected. If I is a singleton, it is clear. Let $I = (a,b)$ with $-\infty \leqslant a < b \leqslant +\infty$ and a continuous function $f : I \to D = \{0,1\}$. Suppose that f is not constant, that is there exists $x, y \in I$ such that

$$a < x < y < b \quad \text{and} \quad f(x) \neq f(y),$$

and, without loss of generality, we may assume $f(x) = 0$ and $f(y) = 1$. Consider the set

$$\Gamma = \{z \in I : z \geqslant x \text{ such that } f(t) = 0 \text{ for all } t \in [x,z]\}$$

The set Γ is nonempty because $x \in \Gamma$. Moreover, Γ is bounded from above by y. Let $c = \sup\Gamma \leqslant y$.

By the continuity of f, we have $f(c) = 0$. Additionally, the continuity of f at c implies that

$$\exists \varepsilon \in (0, b-y), \forall t \in [c, c+\varepsilon], \quad \delta(f(t), f(c)) < \tfrac{1}{2}.$$

This means that $f(t) = 0$ for $t \in [c, c+\varepsilon] \subseteq (a, b) = I$, so $c + \varepsilon \in \Gamma$. This contradicts the fact that c is the supremum of Γ. Therefore, f needs to be constant, and I is connected.

For a general interval I which is not a singleton, nor an open interval, we may write $J = \mathrm{int}(I)$ so that $J \subseteq I \subseteq \mathrm{cl}(J)$. Since J is of the form (a, b) with $-\infty \leqslant a < b \leqslant +\infty$, which has been discussed above, we know that J is connected. Then, it follows from Corollary 2.7.7 that I is also connected. ☐

The following theorem is the first application of Proposition 2.7.17.

Theorem 2.7.18 (Intermediate value theorem) **:** *Let I be an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a continuous function. Then, $f(I)$ is also an interval.*

Proof : Proposition 2.7.17 tells us that I is connected, then by applying Proposition 2.7.5, we also know that $f(I)$ is connected. Then, again by Proposition 2.7.17, we deduce that $f(I)$ is an interval. ☐

Remark 2.7.19 : Another way to interprete or apply the above theorem is as follows. If $f(a) \leqslant f(b)$ with $a < b$, then for any $\gamma \in [f(a), f(b)]$, we can find $c \in [a, b]$ such that $f(c) = \gamma$.

Another application of Proposition 2.7.17 is the following description on the structure of the open sets in $\mathbb{R}$. Below, let us fix a nonempty open subset $A \subseteq \mathbb{R}$.

Definition 2.7.20 : Let I be an open interval. We say that I is a *component interval* of A if

- $I \subseteq A$, and
- there is no open interval $J \neq I$ with $I \subseteq J \subseteq A$.

Theorem 2.7.21 (Representation theorem for open sets in $\mathbb{R}$) **:** *The subset A is the union of a countable collection of disjoint component intervals of A.*

Proof : It follows from Remark 2.7.14 that we may write down the connected components of A as

$$A/\mathcal{R} = \{[x_j] : j \in J\}, \tag{2.14}$$

where J is some index set, and $[x_j]$ denotes the equivalent class of $\mathcal{R}$, or connected component of A, represented by some $x_j \in A$. From Proposition 2.7.17, we know that each of $[x_j]$ is an interval of $\mathbb{R}$. We need to check that these intervals are component intervals in the sense of Definition 2.7.20.

Fix $j \in J$, let us denote $I_j = [x_j]$, $a_j = \inf I_j$, and $b_j = \sup I_j$, so that $(a_j, b_j) \subseteq I_j$. First, we want to show that I_j is an open interval, that is $I_j = (a_j, b_j)$. We want to show that $a_j \notin I_j$.

- If $a_j = -\infty$, then it is clear that $a_j \notin I_j$.
- If $a_j > -\infty$ with $a_j \in I_j$, then since $a_j \in A$, which is an open set, we may find $\varepsilon > 0$ such that $I'_j := (a_j - \varepsilon, a_j + \varepsilon) \subseteq A$. Since I'_j and I_j are both connected, and $I_j \cap I'_j \neq \varnothing$, it follows from Proposition 2.7.8 that $I_j \cup I'_j$ is still connected. This contradicts the fact that I_j is an equivalence class for the relation $\mathcal{R}$.

Therefore, $a_j \notin I_j$. Similarly, we may also show that $b_j \notin I_j$, that is $I_j = (a_j, b_j)$.

To show that I_j is maximal in the sense that, there is no open interval K such that $I_j \subsetneq K \subseteq A$, we use again the fact that $\mathcal{R}$ is an equivalence relation.

To conclude, it remains to show that J is countable. The set $\mathbb{Q}$ of rationals is countable and can be enumerated $\mathbb{Q} = \{q_1, q_2, \dots\}$. We may define a function $F : J \to \mathbb{N}$ as follows,

$$F(j) = \min\{n \geqslant 1 : q_n \in [x_j]\}, \qquad \forall j \in J.$$

The fact that F is an injection follows directly from the partition structure given by the equivalence relation. This allows us to conclude that (2.14) is a countable collection of component intervals. $\square$

2.7.5 Arcwise connectedness

Let us fix a metric space (M, d).

Definition 2.7.22 : Let $\gamma : [0,1] \to (M, d)$ be a continuous function with $a = \gamma(0)$ and $b = \gamma(1)$.

- We say that γ is a *path* from a to b.
- If $a \neq b$, the image $\gamma([0,1])$ is called an *arc* joining a and b.

- Suppose that (M, d) is a normed space, in the sense of Example 2.1.4. If γ writes as $\gamma(t) = tb + (1-t)a$ with value in M for all $t \in [0,1]$, then we say that $\gamma([0,1])$ is a *line segment* joining a and b, denoted by $[a, b]$.

Definition 2.7.23 : We say that M is arcwise connected (弧連通) if for any $a \neq b \in M$, there is an arc joining a and b.

Theorem 2.7.24 : *If M is arcwise connected, then M is also connected.*

Proof : Let $f : M \to D = \{0,1\}$ be a continuous function. Let $a, b \in M$ and $\gamma : [0,1] \to M$ be a continuous function such that $\gamma(0) = a$ and $\gamma(1) = b$. Then, the composition $f \circ \gamma : [0,1] \to D$ is continuous, so constant, because $[0,1]$ is connected. This means that $f(a) = (f \circ \gamma)(0) = (f \circ \gamma)(1) = f(b)$, so f is also constant. Thus, we can conclude that M is connected by Corollary 2.7.6. □

Example 2.7.25 :

(1) In the Euclidean space $\mathbb{R}^n$, any convex set A is arcwise connected. The reason is that, for any $x, y \in A$, the line segment $[x, y]$ is also in A, which is the definition of a convex set.

(2) Let $A \subseteq \mathbb{R}^2$ be defined as follows,

$$A := \{(0,0)\} \cup \{(x, \sin(1/x)) : x \in (0,1]\}.$$

This is a classical example of a space which is connected but not arcwise connected. We will prove this in Exercise 2.52.

Remark 2.7.26 :

(1) The above Theorem 2.7.24 is useful to show the connectedness of a metric space, because the arcwise connectedness is easier to visualize and to manipulate.

(2) Arcwise connectedness is also a topological notion. The reason is that, to define the notion of arcwise connectedness in Definition 2.7.23, we make use of continuous functions, which are characterized entirely by open sets, see Proposition 2.5.11.

(3) The converse of Theorem 2.7.24 does not hold. Example 2.7.25 (2) gives an example of metric space

that is connected but not arcwise connected.

Theorem 2.7.27 : *Let $(V, \|\cdot\|)$ be a normed vector space and A be an open set of V. Then, A is connected if and only if A is arcwise connected.*

Remark 2.7.28 : We note that it is important to assume that A is open. For example, the set A defined in Example 2.7.25 (2) is a subset in $\mathbb{R}^2$, and it is connected without being arcwise connected. Clearly, in this case, the subset A is not open.

Proof : If A is arcwise connected, we have already shown in Theorem 2.7.24 that A is connected. Now, suppose that A is connected. We fix $x_0 \in A$ and let

$$\Gamma = \{x \in A : \text{there is a path joining } x_0 \text{ and } x\}.$$

Our goal is to get $\Gamma = A$ by showing that Γ is open and closed in A at the same time.

- Γ is open. Let $x \in \Gamma$. Since x is also in the open set A, there exists $r > 0$ such that $B(x, r) \subseteq A$. Fix $y \in B(x, r)$, $y \neq x_0$, the line segment $[x, y]$ is also in A. Therefore, if γ_0 is a path from x_0 to x, and let γ_1 denote the line segment from x to y, then

$$\gamma(t) = \begin{cases} \gamma_0(2t), & t \in [0, \frac{1}{2}], \\ \gamma_1(2t-1), & t \in [\frac{1}{2}, 1] \end{cases} \tag{2.15}$$

  gives a path from x_0 to y.

- Γ is closed. To achieve this, let us be given $x \in \overline{\Gamma} \cap A$ and show that x is also in Γ. By the definition of open set and closure, we can find $r > 0$ such that $B(x, r) \subseteq A$ and $B(x, r) \cap \Gamma \neq \varnothing$. Choose $y \in B(x, r) \cap \Gamma$, then the line segment $[y, x]$ is contained in A, the same construction as Eq. (2.15) shows that x also needs to be in Γ. □