3 Compact spaces and complete spaces

In this chapter, we will focus on two important notions, compact spaces and complete spaces.

3.1 Compact set

We present the notion of compact metric spaces in this section. First, we define such spaces using open coverings, known as the Borel–Lebesgue property (Definition 3.1.3), and later on, we will see that this property is actually equivalent to the sequential characterization, known as the Bolzano–Weierstraß property (Definition 3.1.19). We will also see the properties of compact sets under continuous functions in Section 3.1.2. In particular, this generalizes the notion of a segment in $\mathbb{R}$, and we will establish a generalization of the intermediate value theorem in Proposition 3.1.12.

3.1.1 Borel–Lebesgue property

Borel–Lebesgue property is a property defined using the notion of *open coverings*.

Definition 3.1.1 : Given a subset $A \subseteq M$ and a collection $\mathcal{C} = (C_i)_{i\in I}$ of subsets. We say that $\mathcal{C}$ is a covering (覆蓋) of A, or $\mathcal{C}$ covers A, if

$$A \subseteq \bigcup_{i\in I} C_i.$$

Additionally, if all the C_i's are open subsets and satisfy the above condition, we say that $\mathcal{C}$ is an open covering (開覆蓋) of A.

Example 3.1.2 : In the metric space $(M, d) = (\mathbb{R}, |\cdot|)$, the collections

$$\mathcal{I}_1 = \{(a, b) : 0 < a < b < 1\},$$
$$\mathcal{I}_2 = \{(\tfrac{1}{n}, \tfrac{2}{n}) : n \geqslant 2\}$$

are both coverings of $(0, 1)$, where $\mathcal{I}_1$ is uncountable, but $\mathcal{I}_2$ is countable.

Below, we define the notion of a compact metric space and a compact set in a metric space.

Definition 3.1.3 (Borel–Lebesgue property) **:** Let (M, d) be a metric space.

(1) The metric space (M, d) is said to satisfy the *Borel–Lebesgue property* if from any open covering of M, we can extract a finite subcovering.

(2) We say that (M, d) is compact (緊緻) if it satisfies the Borel–Lebesgue property. In other words, M is compact if for any collection $(U_i)_{i \in I}$ of open sets of M such that $M \subseteq \bigcup_{i \in I} U_i$ ($\Leftrightarrow M = \bigcup_{i \in I} U_i$), we can find a finite subfamily $J \subseteq I$ such that $M \subseteq \bigcup_{i \in J} U_i$.

(3) A subset $K \subseteq M$ is said to be a *compact set* if the induced metric space (K, d) is compact. In other words, K is a compact set if for any collection $(U_i)_{i \in I}$ of open sets of M such that $K \subseteq \bigcup_{i \in I} U_i$, we can find a finite subfamily $J \subseteq I$ such that $K \subseteq \bigcup_{i \in J} U_i$.

Example 3.1.4 :

(1) Any finite metric space is compact.

(2) $(\mathbb{R}, |\cdot|)$ is not compact because from the open covering $\mathbb{R} = \cup_{n \geqslant 1}(-n, n)$, we cannot extract a finite subcovering.

(3) In $(\mathbb{R}, |\cdot|)$, the subset $(0, 1)$ is not compact because from the covering $\cup_{n \geqslant 1}(\frac{1}{n}, 1 - \frac{1}{n})$, we cannot extract a finite subcovering.

Remark 3.1.5 : There is a dual version of the Borel–Lebesgue property by taking complementary sets. A metric space (M, d) is compact if and only if for any family $(F_i)_{i \in I}$ of closed sets such that $\cap F_i = \varnothing$, there exists a finite subfamily $J \subseteq I$ such that $\cap_{j \in J} F_j = \varnothing$. In particular, in a metric space (M, d), we may look at the two following properties.

(i) (M, d) is compact.

(ii) For any non-increasing sequence $(F_n)_{n \geqslant 1}$ of nonempty closed sets, the intersection $\cap_{n \geqslant 1} F_n$ is nonempty.

Note that (ii) can be compared to the Cantor's intersection theorem (Theorem 2.2.7), which involves non-increasing sequences of bounded, closed, and nonempty subsets in $\mathbb{R}^n$.

We clearly have (i) $\Rightarrow$ (ii). If additionally, the metric space has the property that from any open subcovering, we may extract a *countable* subcovering (known as the Lindelöf covering property, see Theorem 3.1.28), then (ii) $\Rightarrow$ (i). In particular, if M is a subspace of the Euclidean space $\mathbb{R}^n$, we have (ii) $\Rightarrow$ (i).

Proposition 3.1.6 : *Let $K \subseteq M$ be a compact set. Then, K is closed and bounded.*

Proof : We first show that K is bounded. Take $x \in K$, then $(B(x,n))_{n\geqslant 1}$ is an open covering of K. By compactness, we can find a finite subcovering, so K is bounded.

Next, we prove that K is closed. By contradiction, assume that K is not closed. We can find an accumulation point y of K such that $y \notin K$. For each $x \in K$, define $r_x = \frac{1}{2}d(x,y)$. Then, the collection $(B(x,r_x))_{x\in K}$ is an open covering of K, and the compactness of K gives us a finite subcovering, that is

$$K \subseteq \bigcup_{k=1}^{n} B(x_k, r_{x_k}),$$

for some $x_1, \ldots, x_n \in K$. Take $r = \min(r_{x_1}, \ldots, r_{x_n})$ and $x \in B(y,r)$, then we can see that

$$d(x,x_k) \geqslant d(y,x_k) - d(x,y) > 2r_{x_k} - r \geqslant r_{x_k},$$

for all $1 \leqslant k \leqslant n$. This means that x is not in any of the open balls $B(x_k, r_{x_k})$. Thus, we obtain that $K \cap B(y,r) = \varnothing$. This contradicts the fact that y is an accumulation point of K. Then, we may conclude that K is closed. □

Remark 3.1.7 : Later in Remark 3.1.34, we will see that a closed and bounded set is not necessarily compact in general.

Proposition 3.1.8 : *Let (M,d) be a compact metric space, and $K \subseteq M$ be a closed set. Then, K is a compact subset.*

Proof : If K is empty, then K is clearly compact. Suppose that K is not empty, and given an open covering $\mathcal{C} = \{C_i : i \in I\}$ of K. Since K is closed, $M\backslash K$ is open. Therefore, $\mathcal{C} \cup \{M\backslash K\}$ is an open covering of M. Since M is compact, we can find a finite subset $I' \subseteq I$ such that

$$M \subseteq \Big(\bigcup_{i\in I'} C_i\Big) \cup (M\backslash K).$$

Therefore,

$$K \subseteq \bigcup_{i\in I'} C_i,$$

which is a finite subcovering from $\mathcal{C}$. This shows that K is compact. □

Proposition 3.1.9 : *Compact subsets of a metric space (M, d) satisfy the following properties.*

(1) *Any finite union of compact subsets is compact.*

(2) *Any intersection of compact subsets is compact.*

Proof : The proofs are straightforward by the Borel–Lebesgue property.

(1) Let $n \geqslant 1$ and $K_1, \ldots, K_n$ be compact subsets of (M, d) and $K := K_1 \cup \cdots \cup K_n$. Let $\mathcal{C} = \{C_i : i \in I\}$ be an open covering of K. Then, $\mathcal{C}$ is also an open covering of K_m for $1 \leqslant m \leqslant n$. For each $m = 1, \ldots, n$, let us extract a finite subcovering of K_m from $\mathcal{C}$, that we denote by $\{C_i : i \in I_m\}$, where I_m is a finite subset of I. Then, the set $I' := \cup_{m=1}^{n} I_m$ is finite as well, and $\{C_i : i \in I'\}$ is a finite subcovering of K. This shows that K is a compact subset of (M, d).

(2) Let $(K_i)_{i \in I}$ be a family of compact sets and $K := \cap_{i \in I} K_i$. Since K is an intersection of closed sets, K is also closed. We may regard K as a subset of any compact set K_i, and it follows from Proposition 3.1.8 that K is also compact. □

3.1.2 Application to continuous functions

Proposition 3.1.10 : *Let (M, d) and (M', d') be two metric spaces and $f : M \to M'$ be a function. If f is continuous and M is compact, then the image $f(M)$ is a compact subset of M'; in particular, $f(M)$ is closed and bounded.*

Remark 3.1.11 :

(1) We note that in Section 2.5.3, continuous functions are characterized by their preimage, or inverse image, of open and closed sets. This proposition establishes the property that, the image of any closed subset of M (so compact by Proposition 3.1.8) is also closed in M'. We note that this property does not hold in general, see Remark 2.5.15.

(2) We also note that if $f : M \to M'$ is continuous and M' is compact, the preimage $f^{-1}(M')$ is not necessarily compact. A constant function $f : \mathbb{R} \to \mathbb{R}$ gives us a counterexample.

Proof : Let $\mathcal{C} = \{C_i : i \in I\}$ be an open covering of $f(M)$. It follows from Proposition 2.5.11 that $f^{-1}(C_i)$ is an open set in M for $i \in I$. Therefore, $\{f^{-1}(C_i) : i \in I\}$ forms an open covering of M, and the compactness of M allows us to extract a finite subcovering, denoted by $\{f^{-1}(C_i) : i \in J\}$ for some finite subset $J \subseteq I$. Therefore,

$$f(M) \subseteq f\Big(\bigcup_{i\in J} f^{-1}(C_i)\Big) = \bigcup_{i\in J} f(f^{-1}(C_i)) \subseteq \bigcup_{i\in J} C_i$$

gives a finite subcovering from $\mathcal{C}$ of $f(M)$. □

Proposition 3.1.12 : *Let (M, d) be a compact metric space and $f : M \to \mathbb{R}$ be a continuous function. Then, f is bounded and attains its maximum and minimum, that is, there exists $a, b \in M$ such that*

$$f(a) = \inf_{x\in M} f(x) \quad \textit{and} \quad f(b) = \sup_{x\in M} f(x).$$

Proof : It follows from Proposition 3.1.10 that $f(M)$ is compact, thus also bounded and closed, in $\mathbb{R}$. Let $m = \inf_{x\in M} f(x)$. Then, m is an adherent point to $f(M)$. Since $f(M)$ is closed, we also have $m \in f(M)$, that is $m = f(a)$ for some $a \in M$. The proof is similar for the supremum / maximum. □

Remark 3.1.13 : We note that it is important to assume that M is compact. For a counterexample, if $(M, d) = (\mathbb{R}, |\cdot|)$ and take $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \arctan(x)$, then $f(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2})$, and it is clear that the supremum and the infimum are not attained.

We may apply Proposition 3.1.12 to compact sets of $\mathbb{R}$, giving us an improved version of the intermediate value theorem (Theorem 2.7.18).

Corollary 3.1.14 : *Let $I \subseteq \mathbb{R}$ be a segment and $f : I \to \mathbb{R}$ be a continuous function. Then, $f(I)$ is also a segment.*

Proof : We note that a segment in $\mathbb{R}$ is compact, which is a direct consequence of Remark 3.1.5 and the fact that it satisfies the Cantor's intersection theorem. Therefore, $f(I)$ is compact in $\mathbb{R}$, and it follows from Proposition 3.1.6 that $f(I)$ is bounded and closed. Additionally, we see from Theorem 2.7.18 that $f(I)$ is an interval. We conclude by saying that a bounded and closed interval in $\mathbb{R}$ is a segment. □

Corollary 3.1.15 : *Let $(M,d) \to (M',d')$ be a continuous and bijective function. If (M,d) is compact, then f^{-1} is continuous, and f is a homeomorphism.*

Proof : Suppose that M is compact. To show that $g := f^{-1} : M' \to M$ is continuous, we may use the characterization from Proposition 2.5.11. Let A be a closed subset of M. It follows from Proposition 3.1.8 that A is compact. Then, Proposition 3.1.10 tells us that $g^{-1}(A) = f(A)$ is also compact, thus closed in M'. □

Remark 3.1.16 : It is important to assume that (M,d) is compact. For a counterexample (Exercise 2.47), consider $f : [0,1) \to \mathbb{U}$ defined by

$$f(x) = e^{2\pi \mathrm{i} x}, \quad x \in [0,1).$$

The function f is clearly continuous and bijective. However, f^{-1} is not continuous at $f(0) = 1$, because we may take the sequence $x_n = 1 - \frac{1}{n}$ and $y_n = f(x_n)$. Then, $y_n \xrightarrow[n\to\infty]{} 1 = f(0)$, but $f^{-1}(y_n) = x_n$ does not converge in $[0,1)$.

Theorem 3.1.17 (Heine–Cantor theorem) **:** *Let $f : (M,d) \to (M',d')$ be a continuous function. Suppose that M is compact. Then, f is uniformly continuous.*

Proof : Let $\varepsilon > 0$. For every $x \in M$, since f is continuous at x, we may find $\delta_x > 0$ such that

$$y \in B(x, \delta_x) \quad \Rightarrow \quad f(y) \in B(f(x), \tfrac{\varepsilon}{2}). \tag{3.1}$$

Clearly, the set of open balls $\{B(x, \frac{\delta_x}{2}) : x \in M\}$ forms an open covering of M. (Note that here, we divide the radii by 2.) The compactness of M allows us to extract a finite subcovering, that is

$$M \subseteq \bigcup_{i=1}^{n} B(x_i, \tfrac{\delta_{x_i}}{2})$$

for some $n \geqslant 1$ and $x_1, \dots, x_n \in M$. Let $\delta = \frac{1}{2} \min_{1\leqslant i\leqslant n} \delta_{x_i}$. For $y, y' \in M$ with $d(y,y') < \delta$, we can find $1 \leqslant i \leqslant n$ such that $y \in B(x_i, \frac{\delta_{x_i}}{2})$. Then,

$$d(y', x_i) \leqslant d(y', y) + d(y, x_i) < \delta + \tfrac{\delta_{x_i}}{2} \leqslant \delta_{x_i}.$$

That is, $y, y' \in B(x_i, \delta_{x_i})$. Therefore, by (3.1), we find

$$d(f(y), f(y')) \leqslant d(f(y), f(x_i)) + d(f(x_i), f(y')) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

□

Remark 3.1.18 : In the first-year calculus, you should have seen Heine–Cantor theorem in $\mathbb{R}$, which states that a continuous function $f : I \to \mathbb{R}$ on a segment $I \subseteq \mathbb{R}$ is uniformly continuous.

3.1.3 Sequential characterization

Definition 3.1.19 (Bolzano–Weierstraß property) **:** A metric space (M, d) is said to satisfy the Bolzano–Weierstraß property (Bolzano–Weierstraß 性質) if from any sequence $(x_n)_{n\geqslant 1}$ of points of M, we can extract a convergent subsequence $(x_{\varphi(n)})_{n\geqslant 1}$ with limit in M.

Theorem 3.1.20 : *In a metric space (M, d), the Borel–Lebesgue property and the Bolzano–Weierstraß property are equivalent. In other words, the metric space (M, d) is compact if and only if every sequence in (M, d) has a convergent subsequence.*

Proof : Borel–Lebesgue ⇒ Bolzano–Weierstraß. Let us assume that K is compact, and given a sequence $(x_n)_{n\geqslant 1}$ with values in K. Let $A = \{x_n : n \geqslant 1\}$ be the range of the sequence. If A is finite, then we can easily find a convergent subsequence of $(x_n)_{n\geqslant 1}$. Suppose that A is infinite, and that $(a_n)_{n\geqslant 1}$ does not have any convergent subsequence. This means that for every $x \in K$, there exists a $\varepsilon_x > 0$ such that $A \cap B(x, \varepsilon_x) = \varnothing$ or $\{x\}$. Additionally, these open balls $\{B(x, \varepsilon_x) : x \in K\}$ form an open covering of K. Due to the compactness of K, we may find a finite subcovering, that is $x_1, \dots, x_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon_{x_i}).$$

Therefore,

$$A = A \cap K \subseteq \bigcup_{i=1}^{n} (A \cap B(x_i, \varepsilon_{x_i})).$$

However, the set A on the l.h.s. is infinite, but on the r.h.s., each term in the finite union is either empty or a singleton, so is still a finite set. This leads to a contradiction.

For the converse, we will first prove the following two lemmas. □

Definition 3.1.21 : A metric space (M, d) is said to be precompact (預緊緻), or relatively compact (相對緊緻), if for all $\varepsilon > 0$, there exists a finite covering of M by open balls of radius ε.

Lemma 3.1.22 : *Let (M, d) be a metric space satisfying the Bolzano–Weierstraß property. Then, it is also precompact.*

Proof : By contradiction, suppose that there exists $\varepsilon > 0$ such that we cannot find a finite covering by open balls of radius ε. Let us construct a sequence $(x_n)_{n\geqslant 1}$ in M by induction such that

- for every $n \neq m \geqslant 1$, $d(x_n, x_m) \geqslant \varepsilon$,
- for every $n \geqslant 1$, $M \neq \cup_{i=1}^{n} B(x_i, \varepsilon)$.

Let $x_1 \in M$, then by the assumption, $M \neq B(x_1, \varepsilon)$. Let $k \geqslant 1$ and suppose that we have constructed $x_1, \ldots, x_k$ such that $d(x_n, x_m) \geqslant \varepsilon$ for $1 \leqslant n \neq m \leqslant k$ and $M \neq \cup_{i=1}^{k} B(x_i, \varepsilon)$. Then, we can find $x_{k+1} \in M \backslash (\cup_{i=1}^{k} B(x_i, \varepsilon))$. This allows us to see that $d(x_{k+1}, x_i) \geqslant \varepsilon$ for $1 \leqslant i \leqslant k$, and again by assumption, we know that $M \neq \cup_{i=1}^{k+1} B(x_i, \varepsilon)$.

The sequence $(x_n)_{n\geqslant 1}$ constructed above is not Cauchy, and none of its subsequence is Cauchy either. Thus, by Proposition 2.4.6, it does not have a convergent subsequence. This contradicts the Bolzano–Weierstraß property. □

Lemma 3.1.23 : *Let (M, d) be a metric space satisfying the Bolzano–Weierstraß property. Consider an open covering $(C_i)_{i\in I}$ of M. Then, there exists $\varepsilon > 0$ such that*

$$\forall x \in M, \exists i \in I, \quad B(x, \varepsilon) \subseteq C_i.$$

Proof : By contradiction, suppose that for all $\varepsilon > 0$, we can find $x \in M$ such that

$$B(x, \varepsilon) \nsubseteq C_i, \quad \forall i \in I.$$

In particular, for every $n \geqslant 1$, let us choose $x_n \in M$ such that

$$B_n := B(x_n, \tfrac{1}{n}) \nsubseteq C_i, \quad \forall i \in I. \tag{3.2}$$

Using the Bolzano–Weierstraß property, we may find a subsequence $(x_{\varphi(n)})_{n\geqslant 1}$ that converges to some limit $\ell \in M$. By the assumption, we may find $i \in I$ such that $\ell \in C_i$, and $r > 0$ such that $B(\ell, 2r) \subseteq C_i$. The convergence of the subsequence $(x_{\varphi(n)})_{n\geqslant 1}$ further implies that there exists $N \geqslant 1$ such that

$$d(x_{\varphi(n)}, \ell) < r, \quad \forall n \geqslant N.$$

Therefore, for $n \geqslant N$ with $\varphi(n) > 1/r$, we have

$$\forall y \in B_{\varphi(n)}, \quad d(\ell, y) \leqslant d(\ell, x_{\varphi(n)}) + d(x_{\varphi(n)}, y) < r + 1/\varphi(n) < 2r.$$

It follows from the above line that $B_{\varphi(n)} \subseteq B(\ell, 2r) \subseteq C_i$, which contradicts Eq. (3.2). □

Now, let us finish the proof of Theorem 3.1.20.

Proof of Theorem 3.1.20: Suppose that the metric space (M, d) satisfies the Bolzano–Weierstraß property. Let $(C_i)_{i\in I}$ be an open covering of M. By Lemma 3.1.23, we may fix $\varepsilon > 0$ such that

$$\forall x \in M, \exists i \in I, \quad B(x, \varepsilon) \subseteq C_i. \tag{3.3}$$

By Lemma 3.1.22, we can find finitely many open balls of radius $\varepsilon > 0$ that cover M. Let $n \geqslant 1$ and $x_1, \ldots, x_n \in M$ such that

$$M = \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

By Eq. (3.3), for $1 \leqslant i \leqslant n$, we may find $j_i \in I$ such that $B(x_i, \varepsilon) \subseteq C_{j_i}$. This allows us to conclude that

$$M = \bigcup_{i=1}^{n} C_{j_i}.$$

□

To conclude this subsection, we sum up what we have shown in the following corollary, which gives us useful criterions to check whether a metric space is compact. Additionally, we will also see a few applications.

Corollary 3.1.24 : *The metric space (M, d) is compact if and only if one of the following properties is satisfied.*

(1) Every sequence of M has a subsequential limit in M.

(2) Every infinite subset of M has an accumulation point in M.

Proposition 3.1.25 : *Let (M, d) be a compact metric space and $(x_n)_{n\geqslant 1}$ be a sequence in M with only one subsequential limit x. Then, $(x_n)_{n\geqslant 1}$ converges to x.*

Proof : By contradiction, assume that the sequence does not converge to x. We can find $\varepsilon > 0$ such that for all $N \geqslant 1$, there exists $n \geqslant N$ with $d(x_n, x) \geqslant \varepsilon$. This gives us a subsequence $(x_{\varphi(n)})_{n\geqslant 1}$ such that $d(x_{\varphi(n)}, x) \geqslant \varepsilon$ for all $n \geqslant 1$. Since M is compact, we may extract a convergent subsequence from $(x_{\varphi(n)})_{n\geqslant 1}$, that is

$$\lim_{n\to\infty} x_{\varphi\circ\psi(n)} = y \in M.$$

We have $d(x, y) \geqslant \varepsilon$, so $x \neq y$, and y is also a subsequential limit of $(x_n)_{n\geqslant 1}$, which is a contradiction.
□

Below, we are going to consider a product of metric spaces indexed by I. We distinguish two settings: (i) I is finite, or $I = [N] = \{1, \ldots, N\}$ for some integer $N \geqslant 1$; (ii) I is countably infinite, or $I = \mathbb{N}$. We recall the notations and definitions we saw in Section 2.6 and Exercise 2.46.

(i) Let $N \geqslant 1$ and $(M_1, d_1), \ldots, (M_N, d_N)$ be metric spaces. We consider the product space $M = \prod_{i\in I} M_i = M_1 \times \cdots \times M_N$ and the product distance as in Definition 2.6.1, defined by

$$d(x, y) = \max_{1\leqslant i\leqslant N} d_i(x_i, y_i), \quad x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in M.$$

(ii) Let $((M_n, d_n))_{n\geqslant 1}$ be a sequence of uniformly bounded metric spaces. We consider the product space $M = \prod_{i\in I} M_i = \prod_{n\geqslant 1} M_n$ and the product distance as in Exercise 2.46, defined by

$$d(x, y) = \sum_{n\geqslant 1} 2^{-n} d_n(x_n, y_n), \quad x = (x_n)_{n\geqslant 1}, y = (y_n)_{n\geqslant 1} \in M.$$

We note that in both settings, the convergence in the product space is equivalent to the convergence of each coordinate in the corresponding metric space. In other words, for any sequence $(x^{(k)})_{k\geqslant 1}$ in M, we have

$$\lim_{k\to\infty} x^{(k)} = x \quad \Leftrightarrow \quad \lim_{k\to\infty} x_i^{(k)} = x_i, \forall i \in I.$$

Proposition 3.1.26 : *Given metric spaces and define the corresponding product metric space as in (i) or (ii). Then, (M, d) is compact if and only if (M_n, d_n) is compact for all $n \in I$.*

Proof : We are going to show that M_i is compact for all $i \in I$ using the continuity of projection maps. More precisely, for a fixed $i \in I$, M_i can be seen as the image of M under the projection map Proj_i, which is continuous by Proposition 2.6.5. Then, it follows from Proposition 3.1.10 that M_i is also compact.

For the converse, we are going to use the characterization from Corollary 3.1.24. Given a sequence of points $(x^{(k)})_{k\geqslant 1}$ from (M, d), we are going to find a convergent subsequence of it. We just saw that, in both (i) and (ii) cases, the convergence of a sequence in M is equivalent to the convergence of all the coordinates, which simplifies the construction we are going to present below. We will construct extractions $(\varphi_n)_{n\in I}$ by induction, where $I = [N]$ for some $N \geqslant 1$ or $I = \mathbb{N}$.

First, note that since M_1 is compact, we may find a subsequence $(x^{(\varphi_1(k))})_{k\geqslant 1}$ such that $(x_1^{(\varphi_1(k))})$ converges in M_1. Then, we apply the compactness of M_2 to the sequence $(x^{(\varphi_1(k))})_{k\geqslant 1}$, which allows us to find a subsequence $(x^{(\varphi_1\circ\varphi_2(k))})_{k\geqslant 1}$ such that $(x_2^{(\varphi_1\circ\varphi_2(k))})_{k\geqslant 1}$ converges in M_2. By doing this, we can find extractions $(\varphi_n)_{n\in I}$ such that

$$\lim_{k\to\infty} x_n^{(\varphi_1\circ\cdots\circ\varphi_n(k))} = \ell_n \in M_n, \quad \forall n \in I. \tag{3.4}$$

(i) In the case that $I = [N]$ for some $N \geqslant 1$, we may define the extraction $\psi := \varphi_1 \circ \ldots \varphi_N$. For each $1 \leqslant n \leqslant N$, since the sequence $(x_n^{\psi(k)})_{k\geqslant 1}$ is a subsequence of the convergent sequence $(x_n^{(\varphi_1\circ\cdots\circ\varphi_n(k))})_{k\geqslant 1}$, it follows from Eq. (3.4) that it also converges to the same limit ℓ_n.

(ii) In the case that $I = \mathbb{N}$, we cannot copy the same prove as above, since it does not make sense to consider the composition of infinitely many functions. To get around of this, we are going to construct an extraction using the diagonal argument, that is

$$(x^{(\psi(n))})_{n\geqslant 1}, \quad \psi(n) := \varphi_1 \circ \cdots \circ \varphi_n(n).$$

For each $n \in \mathbb{N}$, we see that $(x_n^{(\psi(k))})_{k\geqslant n}$ is a subsequence of the convergent sequence $(x_n^{(\varphi_1\circ\cdots\circ\varphi_n(k))})_{k\geqslant n}$, so $(x_n^{(\psi(k))})_{k\geqslant 1}$ also converges to the same limit ℓ_n.

Therefore, we have established that

$$\lim_{n\to\infty} x^{(\psi(n))} = \ell := (\ell_n)_{n\in I} \in M.$$

□

3.1.4 Heine–Borel property in finite dimensional Euclidean spaces

We know from Proposition 3.1.6 that a compact set is closed and bounded. In this subsection, we will see that in the Euclidean space $\mathbb{R}^n$, closed and bounded sets are also compact, which is known as the Heine–Borel theorem. In particular, this allows us to have a simpler criterion to check whether a subset of $\mathbb{R}^n$ is compact, without using the Borel–Lebesgue property, which is the very first definition of compactness in Definition 3.1.3. However, keep in mind that this equivalence does not hold in a more general metric space, as we will see in Remark 3.1.34.

Let us consider the following countable collection of open balls in $\mathbb{R}^n$,

$$\mathcal{G} = \{B(x,r) : x = (x_1, \dots, x_n) \in \mathbb{Q}^n, r \in \mathbb{Q}\}.$$

Lemma 3.1.27 : *Let $x \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$ be an open set containing x. Then, there exists $G \in \mathcal{G}$ such that $x \in G \subseteq A$.*

Proof : To find such an open ball $G \in \mathcal{G}$ satisfying $x \in G \subseteq A$, we need to find a point y with rational coordinates that is close enough to x, and take $G = B(y, \varepsilon)$ for some small enough rational $\varepsilon > 0$.

Since A is an open set, we may find $\varepsilon > 0$ such that $x \in B(x, \varepsilon) \subseteq A$. Then, we take $y \in \mathbb{Q}^n$ such that $d(x,y) < \frac{\varepsilon}{4}$. This is possible because $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Let $r \in \mathbb{Q} \cap [\frac{\varepsilon}{4}, \frac{\varepsilon}{2}]$, which guarantees that $x \in B(y,r)$. We are going to check that $B(y,r) \subseteq A$. Given $z \in B(y,r)$. It follows from the triangular inequality that

$$d(x,z) \leqslant d(x,y) + d(y,z) < \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{2} = \tfrac{3}{4}\varepsilon.$$

Therefore, $B(y,r) \subseteq B(x,\varepsilon) \subseteq A$. □

Theorem 3.1.28 (Lindelöf covering theorem) **:** *Let $A \subseteq \mathbb{R}^n$ and $\mathcal{C}$ be an open covering of A. Then, there is a countable subfamily of $\mathcal{C}$ that also covers A.*

Remark 3.1.29 : This theorem is interesting only when the open covering $\mathcal{C}$ is uncountable, since otherwise, the statement is trivial. Additionally, here we do not require any additional condition for the subset A.

Proof : We write the elements of the open covering $\mathcal{C}$ as $\mathcal{C} := \{C_i : i \in I\}$ for some index set I. The collection $\mathcal{G}$ contains countably many open balls, we may enumerate its elements as $\mathcal{G} = \{G_1, G_2, \dots\}$.

For each $x \in A$, we may fix $i = i(x) \in I$ such that $x \in C_i$, and by Lemma 3.1.27, there exists at least one $G \in \mathcal{G}$ such that $x \in G \subseteq C_i$, so the map $f : A \to \mathbb{N}$ given by

$$\forall x \in A, \qquad f(x) := \min\{j \geqslant 1 : x \in G_j \subseteq C_i\}$$

is well defined. Then, let

$$J := \{f(x) : x \in A\},$$

which is countable, and it follows from the above construction that

$$A = \bigcup_{j \in J} G_j.$$

To conclude, for each $j \in J$, we may choose an arbitrary $x_j \in f^{-1}(\{j\})$, and the corresponding element $C_{i(x_j)}$ in the open covering $\mathcal{C}$. Therefore, we have obtained a countable subfamily of $\mathcal{C}$ that covers A, that is

$$A \subseteq \bigcup_{j \in J} C_{i(x_j)}.$$

□

Theorem 3.1.30 (Heine–Borel theorem) **:** *Let $K \subseteq \mathbb{R}^n$. If K is closed and bounded, then K is compact.*

Remark 3.1.31 : We note that by Lindelöf covering theorem (Theorem 3.1.28), we may extract a countable subcovering from any open covering of K. This theorem gives us sufficient conditions so that a *finite* (open) subcovering exists. We note that these conditions are also necessary, as mentioned in Proposition 3.1.6.

Proof : Given an open covering $\mathcal{C}$ of K. Lindelöf covering theorem gives us a countable subcovering of $\mathcal{C}$, denote by

$$\{C_i : i \geqslant 1\} \subseteq \mathcal{C}$$

that also covers K. For $n \geqslant 1$, let

$$S_n := \bigcup_{i=1}^{n} C_i$$

which is an open set. We want to show that there exists $n \geqslant 1$ such that $K \subseteq S_n$.

Let us consider another sequence of subsets defined by

$$A_n := K \backslash S_n = K \cap S_n^c, \quad \forall n \geqslant 1.$$

We can easily see that for all $n \geqslant 1$, the set A_n is closed, the inclusion $A_{n+1} \subseteq A_n$ holds, and $A_1 \subseteq K$ is bounded. By contradiction, suppose that there does not exist $n \geqslant 1$ with $K \subseteq S_n$, which means that all the sets A_n are nonempty. We may apply Cantor intersection theorem (Theorem 2.2.7), telling us that the intersection $A = \cap_{n \geqslant 1} A_n$ is nonempty. This gives $x \in A$, that is $x \in K$ and $x \notin S_n$ for all $n \geqslant 1$, which is impossible because $K \subseteq \cup_{n \geqslant 1} C_n$. □

Remark 3.1.32 : In the beginning of this subsection, we assumed that $\mathbb{R}^n$ is equipped with the Euclidean norm $\|\cdot\|_2$. However, for any other given equivalent norm $\|\cdot\|$, the normed spaces $(\mathbb{R}^n, \|\cdot\|_2)$ and $(\mathbb{R}^n, \|\cdot\|)$ are also topologically equivalent (Remark 2.5.26). This means that the notion of open sets, closed sets, and bounded sets stays the same, the notion of compactness also stays unchanged, since it only depends on the notion of open sets (Definition 3.1.3). Therefore, the Heine–Borel theorem is still valid if we equip $\mathbb{R}^n$ with other equivalent norms, such as $\|\cdot\|_1$ or $\|\cdot\|_\infty$ (Example 2.5.25).

Remark 3.1.33 : More generally speaking, the proof of Lindelöf covering theorem, Cantor intersection theorem, and Heine–Borel theorem can be generalized without many modifications to any finite dimensional normed vector space. In fact, we will see in Theorem 3.2.22 that all the norms in a finite dimensional normed vector space are equivalent. Since the results hold in $\mathbb{R}^n$ equipped with the Euclidean norm, it follows from Remark 3.1.32 that they also hold for any other equivalent norm. Then, a finite dimensional normed vector space is isomorphic to $\mathbb{R}^n$ with a properly chosen norm. In conclusion, in finite dimensional normed vector spaces, closed and bounded sets are compact.

Remark 3.1.34 : In a general metric space, the Heine–Borel theorem fails to hold. Take the normed space $\ell^\infty(\mathbb{R})$ for example, and consider the unit closed ball $B = \overline{B}(0,1)$, which is clearly closed and bounded. However, it is not compact, because it does not satisfy the Bolzano–Weierstraß property. To see this, let us look at the sequence $(a^{(i)})_{i \geqslant 1}$ of points in B, defined by

$$a^{(i)} = (a_n^{(i)})_{n \geqslant 1}, \quad a_n^{(i)} = \delta_{n,i} = \mathbb{1}_{n=i}, \quad \forall n \geqslant 1.$$

This sequence does not have any subsequential limit. This can be seen by the fact that

$$\left\|a^{(i)} - a^{(j)}\right\| = 1, \quad \forall i \neq j \geqslant 1.$$

To sum up this subsection, in a finite dimensional normed vector space, such as $\mathbb{R}^n$, the above Heine–Borel theorem gives us a useful criterion to check whether a subset is compact.

Corollary 3.1.35 : *Let $K \subseteq \mathbb{R}^n$. Then, K is compact if and only if K is closed and bounded.*

3.2 Complete spaces

In Section 2.4.2, we saw that any convergent sequence is Cauchy, but some Cauchy sequences need not converge. Intuitively, a Cauchy sequence is a sequence whose terms are uniformly close one to each other for large enough indices, so if such a sequence cannot converge, it means that there are some "holes" in the space. Our goal is to study some general properties of complete metric spaces (Section 3.2.1), discuss an important application (Section 3.2.2), and to conclude this section, we will discuss some important results about complete normed vector spaces, also known as Banach spaces (Section 3.2.3).

3.2.1 Definition and properties

We recall the definition of copmlete spaces and Banach spaces from Definition 2.4.10. It says that a metric space is complete if every Cauchy sequence converges, and a Banach space is simply a complete normed vector space.

Proposition 3.2.1 : *Let (M, d) be a metric space, and $S \subseteq M$ be a subset.*

(1) If (S, d) is complete, then S is closed.

(2) If (M, d) is complete and S is closed, then S is also complete.

Proof : The proofs follow directly by the definition of complete spaces, and the sequential characterization of closed sets given in Corollary 2.4.15.

(1) Suppose that (S, d) is complete. To show that S is closed, by the sequential characterization from Corollary 2.4.15, we need to check that given any convergent sequence $(x_n)_{n\geqslant 1}$ of points in S,

its limit should also be in S. If the sequence $(x_n)_{n\geqslant 1}$ is convergent in M, then it is a Cauchy sequence in M, so also a Cauchy sequence in S. The completeness of S allows us to conclude that the limit is in S.

(2) Suppose that (M,d) is complete and S is closed. Given a Cauchy sequence $(x_n)_{n\geqslant 1}$ in S. Since it is also a Cauchy sequence in the complete space M, it converges to a limit $\ell \in M$. Then, it follows from Corollary 2.4.14 that $\ell \in \overline{S} = S$, so the convergence also occurs in S. □

Proposition 3.2.2 : *Let (K,d) be a compact metric space. Then, K is also complete.*

Proof : Let $(x_n)_{n\geqslant 1}$ be a Cauchy sequence in K. The sequential characterization of compact sets Corollary 3.1.24 allows us to find an extraction φ such that $x_{\varphi(n)} \xrightarrow[n\to\infty]{} \ell$ for some $\ell \in K$. We shall show that the original sequence $(x_n)_{n\geqslant 1}$ also converges to ℓ.

Let $\varepsilon > 0$. Since $(x_n)_{n\geqslant 1}$ is Cauchy, we may find $N_1 \geqslant 1$ such that

$$d(x_n, x_m) < \tfrac{\varepsilon}{2}, \quad \forall n, m \geqslant N_1.$$

Since the subsequence $(x_{\varphi(n)})_{n\geqslant 1}$ converges, we may find $N_2 \geqslant 1$ such that

$$d(x_{\varphi(n)}, \ell) < \tfrac{\varepsilon}{2}, \quad \forall n \geqslant N_2.$$

Then, for $n \geqslant \max(N_1, N_2)$, we have

$$d(x_n, \ell) \leqslant d(x_n, x_{\varphi(n)}) + d(x_{\varphi(n)}, \ell) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

This shows that $x_n \xrightarrow[n\to\infty]{} \ell$. □

Proposition 3.2.3 : *Given a sequence of metric spaces $(M_1,d_1),\dots,(M_n,d_n)$ and consider the product metric space (M,d) given by $M = M_1 \times \cdots \times M_n$ and the product distance defined in Definition 2.6.1. Then, (M,d) is complete if and only if (M_i,d_i) is complete for all $1 \leqslant i \leqslant n$.*

Proof : See Exercise 3.22. □

Proposition 3.2.4 : *Let (M, d) be a complete space. Consider a sequence of closed sets $(A_n)_{n\geqslant 1}$ satisfying*

- $A_{n+1} \subseteq A_n$ *for all* $n \geqslant 1$,
- *the diameter goes to zero:* $\delta(A_n) \xrightarrow[n\to\infty]{} 0$.

Then, there exists $x \in M$ such that $A := \cap_{n\geqslant 1} A_n = \{x\}$.

Proof : For every $n \geqslant 1$, choose $a_n \in A_n$. Since $\delta(A_n) \xrightarrow[n\to\infty]{} 0$, the sequence $(a_n)_{n\geqslant 1}$ is Cauchy. For any $p \geqslant 1$, the subset A_p is closed, it follows from Proposition 3.2.1 that A_p is complete. Therefore, $(a_n)_{n\geqslant p}$ converges in A_p to ℓ_p. The limit ℓ_p is actually the same for all $p \geqslant 1$ by the uniqueness of the limit, so let us denote the common limit by ℓ. Since $\ell \in A_p$ for any $p \geqslant 1$, we also have $\ell \in A := \cap_{n\geqslant 1} A_n$.

To conclude, the fact that $\delta(A_n) \xrightarrow[n\to\infty]{} 0$ implies that A can contain at most one element. □

Remark 3.2.5 : It is important to assume that $\delta(A_n) \xrightarrow[n\to\infty]{} 0$. For example, the sequence of closed sets $(A_n = [n, \infty))_{n\geqslant 1}$ satisfies all the other assumptions, but $\cap_{n\geqslant 1} A_n = \varnothing$.

3.2.2 Fixed point theorem

Definition 3.2.6 : Let (M, d) be a complete metric space. A function $f : M \to M$ is said to be a contraction map (壓縮映射) if there exists $k \in [0, 1)$ such that

$$d(f(x), f(y)) \leqslant k\, d(x, y), \qquad \forall x, y \in M. \tag{3.5}$$

Theorem 3.2.7 (Fixed point theorem) **:** *Let (M, d) be a complete metric space and $f : M \to M$ be a contraction. Then, f has a unique fixed point, that is there exists a unique $x \in M$ such that $f(x) = x$.*

Proof : Let $k \in [0, 1)$ be a constant such that Eq. (3.5) is satisfied. Fix $x_0 \in M$ and define $x_{n+1} = f(x_n)$ for $n \geqslant 1$. By induction, we find $d(x_{n+1}, x_n) \leqslant k^n d(x_1, x_0)$ for $n \geqslant 1$. Therefore, for any $m > n \geqslant 1$, we have

$$d(x_m, x_n) \leqslant d(x_m + x_{m-1}) + \cdots + d(x_{n+1}, x_n) \leqslant (k^{m-1} + \cdots + k^n) d(x_1, x_0) \leqslant \frac{k^n}{1-k} d(x_1, x_0).$$

This implies that the sequence $(x_n)_{n\geqslant 0}$ is Cauchy. The completeness of (M,d) implies that $(x_n)_{n\geqslant 0}$ converges to some limit that we denote by x.

Since f is Lipschitz continuous, it is also continuous (Corollary 2.5.31), and by the sequential characterization of continuity, we find

$$f(x) = f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

Therefore, x is a fixed point of f.

To show the uniqueness, we proceed as follows. Suppose $x, y \in M$ such that $f(x) = x$ and $f(y) = y$. Then,

$$0 \leqslant d(x,y) = d(f(x), f(y)) \leqslant k\, d(x,y).$$

Since $k < 1$, this is possible only if $d(x,y) = 0$, that is $x = y$. □

Remark 3.2.8 : As a byproduct of the proof, any iterative sequence defined by $x_{n+1} = f(x_n)$ for $n \geqslant 1$, where $x_0 \in M$ is fixed, converges to the unique fixed point of f.

Remark 3.2.9 : Note that the theorem does not hold if the assumption in Eq. (3.5) is replaced by

$$d(f(x), f(y)) < d(x,y), \qquad \forall x \neq y \in M.$$

For example, in the metric space $(M,d) = ([0,\infty), |\cdot|)$, the function $f(x) = x + e^{-x}$ does not have any fixed point. However, if we assume that (M,d) is compact, then the assumption in Eq. (3.5) is enough to obtain the fixed-point theorem, see Exercise 3.16.

3.2.3 Normed vector spaces

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 3.2.10 :

- We write $\mathcal{L}(V,W)$ for the set of linear maps from V to W.
- We write $\mathcal{L}_c(V,W)$ for the set of continuous linear maps from V to W. We may equip $\mathcal{L}_c(V,W)$

with the following norm, called operator norm (算子範數),

$$\forall f \in \mathcal{L}_c(V, W), \quad |||f||| = \sup_{\|x\|_V \leqslant 1} \|f(x)\|_W \in [0, \infty]. \tag{3.6}$$

By default, Eq. (3.6) is the norm we consider on $\mathcal{L}(V, W)$. We note that if $\dim V \geqslant 1$, that is $V \neq \{0\}$, we also have

$$\forall f \in \mathcal{L}_c(V, W), \quad |||f||| = \sup_{x \neq 0} \frac{\|f(x)\|_W}{\|x\|_V} = \sup_{\|x\|_V = 1} \|f(x)\|_W. \tag{3.7}$$

- The elements in the set $V^* := \mathcal{L}(V, \mathbb{K})$ are called linear forms (線性泛函), and the elements in the set $\mathcal{L}_c(V, \mathbb{K})$ are called continuous linear forms (連續線性泛函).

Example 3.2.11 :

(1) For any normed vector space V, we have $|||\mathrm{Id}_V||| = 1$.

(2) We may equip $\mathcal{M}_n(\mathbb{R})$ with the norm $\|\cdot\|_\infty$, given by

$$\forall A = (a_{i,j})_{1 \leqslant i,j \leqslant n}, \quad \|A\|_\infty = \sup_{1 \leqslant i,j \leqslant n} |a_{i,j}|.$$

Let us consider the linear form $\mathrm{tr} : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$.

- For $A \in \mathcal{M}_n(\mathbb{R})$, we have $|\mathrm{tr}(A)| \leqslant \sum_{k=1}^n |a_{k,k}| \leqslant n \|A\|_\infty$. This shows the continuity of tr and $|||\mathrm{tr}||| \leqslant n$.
- We have $|\mathrm{tr}(I_n)| = n$ and $\|I_n\|_\infty = 1$, this shows that $|||\mathrm{tr}||| = n$.

Theorem 3.2.12 : *Let $f \in \mathcal{L}(V, W)$ be a linear map from V to W. The following properties are equivalent.*

(a) f is continuous on V.

(b) f is continuous at 0.

(c) f is bounded on the closed unit ball $\overline{B}(0, 1)$ of V.

(d) f is bounded on the unit sphere $S(0, 1)$ of V.

(e) There exists $M > 0$ such that $\|f(x)\|_W \leqslant M \|x\|_V$ for all $x \in V$.

(f) f *is Lipschitz continuous on* V.

(g) f *is uniformly continuous on* V.

Remark 3.2.13 : In practice, to show that a linear map $f \in \mathcal{L}(V, W)$ is continuous, we prove (e), that is we look for a constant $M > 0$ such that

$$\forall x \in V, \quad \|f(x)\|_W \leqslant M \|x\|_V .$$

If we look at the definition of the operator norm in Eq. (3.7), we see that the best (smallest) constant M we can take is $M = |||f|||$. To show that a linear map $f \in \mathcal{L}(V, W)$ is not continuous, we may show that f is not continuous at 0. We achieve this by establishing a sequence $x_n \xrightarrow[n\to\infty]{} 0$ such that $f(x_n)$ does not converge to 0. See Exercise 3.29 for an example.

Proof :

- (a) $\Leftrightarrow$ (b). For $x \in V$ and $y \in B(0, \varepsilon)$, we have

$$f(x + y) - f(x) = f(y) - f(0).$$

Therefore, the continuity at x is equivalent to the continuity at 0.

- (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (c). There is nothing to show for (c) $\Rightarrow$ (d) and (e) $\Rightarrow$ (c). Let us show (d) $\Rightarrow$ (e). Let $M = \sup_{x \in S(0,1)} \|f(x)\|_V < \infty$. Fix $x \in V$, by linearity and the property of the norm, we have

$$\|f(x)\|_W = \|x\|_V \left\| f\Big(\frac{x}{\|x\|_V}\Big) \right\|_W \leqslant M \|x\|_W$$

because $\frac{x}{\|x\|_V} \in S(0, 1)$.

- (f) $\Leftrightarrow$ (g). We have already seen (f) $\Rightarrow$ (g) in Corollary 2.5.31. Suppose that (g) holds. Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\forall x, y \in V, \quad \|x - y\|_V \leqslant \delta \quad \Rightarrow \quad \|f(x - y)\|_W = \|f(x) - f(y)\|_W \leqslant \varepsilon.$$

Then, for any $x, y \in V$ with $x \neq y$, we have

$$\|f(x) - f(y)\|_W = \frac{\|x - y\|_V}{\delta} \cdot \left\| f\Big(\frac{\delta(x - y)}{\|x - y\|_V}\Big) \right\|_W \leqslant \frac{\varepsilon}{\delta} \cdot \|x - y\|_V .$$

- (g) $\Rightarrow$ (a) is clear.
- (b) $\Rightarrow$ (e). Given $\varepsilon > 0$, by the continuity of f at 0, we may find $\delta > 0$ such that

$$\forall y \in V, \quad \|y\|_V \leqslant \delta \quad \Rightarrow \quad \|f(y)\|_W \leqslant \varepsilon.$$

  Therefore, for any given $x \in V$, we have

$$\|f(x)\|_W = \frac{\|x\|_V}{\delta} \left\| f\Big(\frac{\delta x}{\|x\|_V}\Big) \right\|_W \leqslant \frac{\varepsilon}{\delta} \cdot \|x\|_V \,.$$

- (e) $\Rightarrow$ (f). For any $x, y \in V$, we have

$$\|f(x) - f(y)\|_W = \|f(x-y)\|_W \leqslant M \, \|x-y\|_V \,.$$

□

Proposition 3.2.14 : *Let U, V, W be three normed vector spaces. For $f \in \mathcal{L}_c(U,V)$ and $g \in \mathcal{L}_c(V,W)$, we have $g \circ f \in \mathcal{L}_c(U,W)$ and $|||g \circ f||| \leqslant |||g||| \cdot |||f|||$.*

Proof : For $x \in U$, we have

$$\|g \circ f(x)\|_W \leqslant |||g||| \, \|f(x)\|_V \leqslant |||g||| \cdot |||f||| \, \|x\|_U \,.$$

In other words,

$$|||g \circ f||| = \sup_{x \neq 0} \frac{\|g \circ f(x)\|_W}{\|x\|_U} \leqslant |||g||| \cdot |||f|||.$$

□

Remark 3.2.15 : In particular, for $U = V = W$, the proposition reduces to a submultiplicative inequality on the space of continuous endomorphisms (自同態) on U, that is

$$|||gf||| \leqslant |||g||| \cdot |||f|||, \quad \forall f, g \in \mathcal{L}_c(U) := \mathcal{L}_c(U,U),$$

where the composition law $\circ$ is the multiplication on the algebra $\mathcal{L}_c(U)$. This is an example of a normed algebra (賦範代數).

Remark 3.2.16 : Let $m, n \geqslant 1$ be integers. From a matrix $A \in \mathcal{M}_{m,n}(\mathbb{R})$, one can define a linear continuous operator $\mathbb{R}^n \to \mathbb{R}^m$, $X \mapsto AX$, where we identify the vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ as column vectors. This defines a norm on the space of matrices, that is

$$\forall A \in \mathcal{M}_{m,n}(\mathbb{R}), \quad |||A||| = \sup_{X \neq 0} \frac{\|AX\|}{\|X\|} = \sup_{\|X\|=1} \|AX\| = \sup_{\|X\| \leqslant 1} \|AX\| \in [0, \infty].$$

This norm on matrices is an operator norm. Additionally, when $m = n$, this operator norm satisfies

$$|||AB||| \leqslant |||A||| |||B|||, \quad \forall A, B \in \mathcal{M}_n(\mathbb{R}).$$

Definition 3.2.17 : A complete normed vector space is called a Banach space (Banach 空間).

Theorem 3.2.18 : *If W is a Banach space, then the normed vector space $(\mathcal{L}_c(V, W), |||\cdot|||)$ is also a Banach space.*

Proof : Let $(f_n)_{n \geqslant 1}$ be a Cauchy sequence in $\mathcal{L}_c(V, W)$. We are going to construct a potential limit f and check that it is indeed the limit in three steps.

- Fix $x \in V$. By observing that

$$\|f_p(x) - f_q(x)\| \leqslant |||f_p - f_q||| \cdot \|x\|_V, \quad \forall p, q \geqslant 1,$$

  we know $(f_n(x))_{n \geqslant 1}$ is a Cauchy sequence in W. W being a Banach space, the sequence $(f_n(x))_{n \geqslant 1}$ converges, and we denote its limit by $f(x)$. This defines a map $f : V \to W$.

- Let us check that the map f defined above is linear and continuous. For $x, y \in V$, we have

$$f(x + y) = \lim_{n \to \infty} f_n(x + y) = \lim_{n \to \infty} [f_n(x) + f_n(y)] = f(x) + f(y).$$

  For $x \in V$ and $\lambda \in \mathbb{K}$, we have

$$f(\lambda x) = \lim_{n \to \infty} f_n(\lambda x) = \lim_{n \to \infty} \lambda f_n(x) = \lambda f(x).$$

  For the continuity, since $(f_n)_{n \geqslant 1}$ is Cauchy, it is also bounded (Proposition 2.4.8), say $|||f_n||| \leqslant M$

for all $n \geqslant 1$ for some $M > 0$. For any $x \in V$, we have

$$\|f(x)\|_W = \left\|\lim_{n\to\infty} f_n(x)\right\|_W = \lim_{n\to\infty} \|f_n(x)\|_W \leqslant M \|x\|_V .$$

- To finish the proof, we need to show that $(f_n)_{n\geqslant 1}$ converges to f with respect to the norm $|||\cdot|||$. Let $\varepsilon > 0$. There exists $N \geqslant 1$ such that $|||f_p - f_q||| \leqslant \varepsilon$ for all $p, q \geqslant N$. Fix $p \geqslant N$, we have

$$\forall q \geqslant N, \quad \|f_p(x) - f_q(x)\|_W \leqslant |||f_p - f_q||| \cdot \|x\|_V \leqslant \varepsilon \|x\|_V .$$

  By taking the limit $q \to \infty$, due the the convergence of $f_q(x)$, we find

$$\|f_p(x) - f(x)\|_W \leqslant \varepsilon \|x\|_V .$$

  Since this inequality holds for all $x \in V$, we deduce that $|||f_p - f||| \leqslant \varepsilon$ for all $p \geqslant N$. □

Remark 3.2.19 : The proof we just see above is a standard procedure to show that a normed space is Banach. More precisely, the three steps are as follow.

(1) We construct a potential candidate for the limit (a function in this example).

(2) We check the properties of this candidate, in order to show that it belongs to the correct space.

(3) Check the potential candidate is indeed the limit.

In Exercise 3.31, you may also follow the same steps to complete the proof.

Proposition 3.2.20 : *Let U be a complete normed vector space (Banach space) and $u \in \mathcal{L}_c(U)$ satisfying $|||u||| < 1$. Then,* $\mathrm{Id} - u$ *is invertible, and its inverse writes*

$$\sum_{k=0}^{\infty} u^k := \lim_{n\to\infty} \sum_{k=0}^{n} u^k \in \mathcal{L}_c(U). \tag{3.8}$$

Proof : We are going to check that the limit in Eq. (3.8) is well defined, then check that it is the inverse of $\mathrm{Id} - u$.

- For every $n \geqslant 0$, let $S_n = \sum_{k=0}^{n} u^k$. For any $m \geqslant n \geqslant 0$, we have

$$S_m - S_n = \sum_{k=n+1}^{m} u^k,$$
$$|||S_m - S_n||| \leqslant \sum_{k=n+1}^{m} \left|\left|\left| u^k \right|\right|\right| \leqslant \sum_{k=n+1}^{m} |||u|||^k \leqslant \frac{|||u|||^{n+1}}{1 - |||u|||}.$$

  Thus, the sequence $(S_n)_{n \geqslant 0}$ is Cauchy in $\mathcal{L}_c(U)$, which is a complete space by Theorem 3.2.18, implying that $S := \lim_{n\to\infty} S_n$ exists and belongs to $\mathcal{L}_c(U)$.

- It is a simple computation to check that the limit S is the inverse of $\mathrm{Id} - u$. We write

$$(\mathrm{Id} - u)S = \lim_{n\to\infty} (\mathrm{Id} - u)S_n = \lim_{n\to\infty} (\mathrm{Id} - u^{n+1}) = \mathrm{Id}.$$

  And similarly, we also have $S(\mathrm{Id} - u) = \mathrm{Id}$. □

Remark 3.2.21 : Following a similar approach, we may also define the exponential of a continuous linear endomorphism $u \in \mathcal{L}_c(U)$ as below,

$$\exp(u) = \sum_{n \geqslant 0} \frac{u^n}{n!} \in \mathcal{L}_c(U).$$

We may also apply the same idea to construct other convergent series.

Theorem 3.2.22 : *In a finite dimensional normed vector space, all the norms are equivalent.*

Proof : Let V be a finite dimensional vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We write n for its dimension, and consider a basis of V, denoted by $(e_1, \ldots, e_n)$. Let N_∞ be a norm on V defined as follows,

$$N_\infty(x) = \sup_{1 \leqslant i \leqslant n} |x_i|, \quad x = \sum_{i=1}^{n} x_i e_i \in V.$$

Let N be a norm on V. We want to show that N_∞ and N are equivalent.

For $x \in V$, we may write it as $x = \sum_{i=1}^{n} x_i e_i$, and its norm satisfies

$$N(x) \leqslant \sum_{i=1}^{n} N(x_i e_i) = \sum_{i=1}^{n} |x_i| N(e_i) \leqslant a N_\infty(x), \quad a = \sum_{i=1}^{n} N(e_i).$$

Let $S = \{x \in V : N_\infty(x) = 1\}$. In the normed space (V, N_∞), S is clearly bounded, it is also closed, being the preimage of the closed set $\{1\}$ under the continuous map $x \mapsto N_\infty(x)$. It follows from Remark 3.1.32 that S is a compact subset of (V, N_∞). Additionally, the map $N : (V, N_\infty) \to \mathbb{R}$ is continuous because it is Lipschitz continuous (Corollary 2.5.31),

$$|N(x) - N(y)| \leqslant N(x - y) \leqslant aN_\infty(x - y).$$

Therefore, the infimum of N on S is attained (Proposition 3.1.12), so needs to be strictly larger than 0. We write $b = \inf_{x \in S} N(x) > 0$, and we have

$$\forall x \in V \backslash \{0\}, \quad N(x) = N_\infty(x) \cdot N\Big(\frac{x}{N_\infty(x)}\Big) \geqslant bN_\infty(x).$$ □

Remark 3.2.23 : Theorem 3.2.22 basically tells us that on a finite dimension vector space, we may choose whichever norm we want, since many topological notions do not depend on the chosen norm anymore.

- Different norms give rise to normed spaces which are topologically equivalent (Remark 2.5.33). This means that the notions, such as open sets, closed sets, interior, closure, boundary, compact sets, connected sets, limit, and continuity of a function, etc. are the same for all the norms.
- Some stronger notions which depend on the distance (norm), and not only on the topology, are also the same, such as the boundedness of a set, and the uniform continuity of a function.

Corollary 3.2.24 : *Following are consequences of Theorem 3.2.22.*

(1) *Any linear map from a finite dimensional normed vector space to any normed vector space is continuous. In other words, $\mathcal{L}(V, W) = \mathcal{L}_c(V, W)$ if $\dim V < \infty$.*

(2) *Every finite dimensional normed vector space is complete.*

(3) *Every finite dimensional subvector space of a normed vector space is closed.*

(4) *In a finite dimensional normed vector space, compact subsets are exactly closed and bounded subsets.*

Remark 3.2.25 : In an infinite dimensional normed space, these properties do not hold anymore.

(1) We may equip the vector space $\mathbb{R}[X]$ of polynomials with the norm

$$\forall P = \sum_{n \geqslant 0} a_n X^n, \qquad \|P\| = \sup_{n \geqslant 0} |a_n|.$$

Then, the following map

$$\begin{array}{rccc} f: & \mathbb{R}[X] & \to & \mathbb{R}[X] \\ & P & \mapsto & P' \end{array}$$

is linear but not continuous. Actually, for every $n \geqslant 1$, we have $\|f(X^n)\| = n$ and $\|X^n\| = 1$.

(2) In the space of continuous functions $\mathcal{C}([0,1],\mathbb{R})$, we may consider the following sequence of affine functions,

$$\forall n \geqslant 1, \forall x \in [0,1], \quad f(x) = \begin{cases} \sqrt{n} - n^{3/2}x, & \text{if } 0 \leqslant x \leqslant \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \leqslant x \leqslant 1. \end{cases}$$

Then, we see that

- $\|f_n\|_\infty = \sqrt{n}$ for all $n \geqslant 1$, so f_n does not converge under the norm $\|\cdot\|_\infty$;
- $\|f_n\|_1 \xrightarrow[n\to\infty]{} 0$, so f_n converges to the constant function 0 under the norm $\|\cdot\|_1$;
- $\|f_n\|_2 = \frac{1}{\sqrt{3}}$ for all $n \geqslant 1$, so the sequence $(f_n)_{n\geqslant 1}$ is bounded under the norm $\|\cdot\|_2$. However, it does not converge to the constant function 0, and cannot converge to any other function either by uniqueness of limit (and Cauchy-Schwarz inequality).

In the second semester, we will discuss more about different notions of convergence of sequences of functions.

(3) In Remark 3.1.34, we saw that in $\ell^\infty(\mathbb{R})$, the unit closed ball $\overline{B}(0,1)$ is bounded and closed, but not compact.

3.3 Completion of a metric space

3.3.1 Using Cauchy sequences

A metric space (M,d) is not necessarily complete, it is possible to make it complete by adding those missing limiting points (of Cauchy sequences). To this end, we are going to consider the space consisting of *all* the Cauchy sequences in M. This space is much larger (up to an isometry) than the original space M itself, and some of its elements are actually represented by multiple Cauchy sequences. Therefore, we need to identify some of its elements, and show that the resulting space is complete. Such as a space is unique, and is the smallest (up to an isometry) complete space containing the original space M, and we call it the *completion* of (M,d).

Theorem 3.3.1 (Completion of a metric space) **:** *Let (M, d) be a metric space. There exists a metric space $(\hat{M}, \hat{\delta})$ and an isometric injection $i : M \to \hat{M}$ such that the following properties are satisfied.*

(1) $i(M)$ is dense in $\hat{M}$.

(2) The metric space $(\hat{M}, \hat{\delta})$ is complete.

Moreover, such a metric space $(\hat{M}, \hat{\delta})$ is unique in the following sense. Let (M_1, δ_1) and (M_2, δ_2) be two complete metric spaces, and $i_1 : M \to M_1$ and $i_2 : M \to M_2$ be two isometric injections such that $i_1(M)$ is dense in M_1 and $i_2(M)$ is dense in M_2. Then, there exists a bijective isometry $\varphi : M_1 \to M_2$ such that $\varphi(i_1(x)) = i_2(x)$ for all $x \in M$.

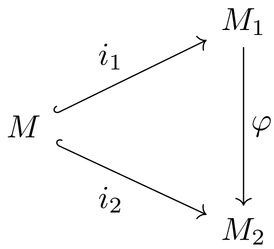

Figure 3.1: This diagram illustrates the uniqueness of the completion of (M, d) up to a bijective isometry φ.

Example 3.3.2 : Below are a few examples of completion of metric spaces.

(1) If we equip $M = (0, 1]$ with the usual distance $|\cdot|$, then its completion writes $\hat{M} = [0, 1]$.

(2) If we equip $M = (0, 1]$ with the distance d as in Exercise 2.23, then $\hat{M} = M$ because (M, d) is complete.

(3) If we equip $\mathbb{Q}$ with the same distance $|\cdot|$, then $\hat{\mathbb{Q}} = \mathbb{R}$, corresponding to Cantor's construction of the real numbers. In this construction, each real number can be identified to a Cauchy sequence.

If we equip $\mathbb{Q}$ with the discrete distance d_{discrete}, then $\hat{\mathbb{Q}} = \mathbb{Q}$.

We decompose the proof of Theorem 3.3.1 into several lemmas as below. To start with, we fix a metric space (M, d). Let us denote by $\mathcal{C}$ the set of all the Cauchy sequences $U = (u_n)_{n \geqslant 1}$ with values in M.

Lemma 3.3.3 : *Let us define the function $\delta : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ as follows. For $U = (u_n)_{n\geqslant 1}, V = (v_n)_{n\geqslant 1} \in \mathcal{C}$, let*

$$\delta(U, V) = \lim_{n\to\infty} d(u_n, v_n).$$

Then, δ is well defined, symmetric, and satisfies the triangle inequality.

Proof : See Exercise 2.26 (1). □

Lemma 3.3.4 : *On $\mathcal{C}$, we may define the following equivalence relation,*

$$U \sim V \quad \Leftrightarrow \quad \delta(U, V) = 0.$$

Then, we define the quotient space $\hat{M} := \mathcal{C}/\sim$, and for an element $U \in \mathcal{C}$, we write $\hat{U} \in \hat{M}$ for its equivalence class. On $\hat{M}$, we may define a distance $\hat{\delta}$ induced by δ, which does not depend on the representant chosen from the equivalence classes, in the sense that for $U \sim V$ and $S \sim W$, we have $\delta(U, S) = \delta(V, W)$, and thus we can set $\hat{\delta}(\hat{U}, \hat{S}) = \delta(U, S)$ for $U, S \in \mathcal{C}$. Then, $(\hat{M}, \hat{\delta})$ is a metric space.

Proof : To show that $\sim$ is an equivalence relation, we need to check that it is reflexive, symmetric, and transitive. The reflexivity is trivial, the symmetry can be obtained by its definition in Lemma 3.3.3, and the transitivity is checked in Exercise 2.26 (2).

To check that $(\hat{M}, \hat{\delta})$ is a metric space, we need to start by checking that the definition of $\hat{\delta}$ does not depend on the element chosen from the equivalence class. This is checked in Exercise 2.26 (3). Then, the distance $\hat{\delta}$ is positive definite due to the equivalence relation, it is symmetric and satisfies the triangle inequality due to Lemma 3.3.3. □

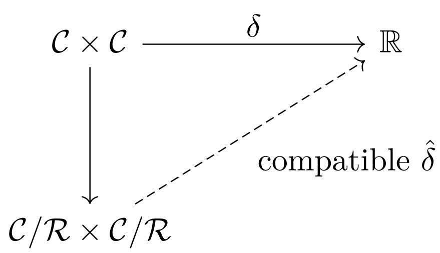

Figure 3.2: This diagram illustrates how to make δ defined on $\mathcal{C} \times \mathcal{C}$, which is not a distance, into a distance δ on $\mathcal{C}/\mathcal{R}$ so that it does not depend on the representant from each equivalence class.

Proposition 3.3.5 : *We define $i : M \to \hat{M}$ as below. For any $x \in M$, let $i(x) = \hat{X}$, where $X = (x)_{n \geqslant 1}$ be given by the constant sequence. Then, i is an isometric injection, and $i(M)$ is dense in $\hat{M}$.*

Proof : Given $u, v \in M$, we have

$$\hat{\delta}(i(u), i(v)) = \delta((u)_{n\geqslant 1}, (v)_{n\geqslant 1}) = d(u, v).$$

Therefore, i is an isometry, so it is also an injection.

Then, let us show that $i(M)$ is dense in $\hat{M}$. Let $\hat{U} \in \hat{M}$ with $\hat{U} = (u_n)_{n\geqslant 1}$. We want to conclude by showing that $\hat{U}$ is the limit of $(i(u_n))_{n\geqslant 1}$. Let $\varepsilon > 0$. Since $U = (u_n)_{n\geqslant 1}$ is a Cauchy sequence, we may find $N \geqslant 1$ such that

$$d(u_m, u_n) < \varepsilon, \quad \forall m, n \geqslant N.$$

For a fixed $m \geqslant N$, we have

$$\hat{\delta}(\hat{U}, i(u_m)) = \delta(U, (u_m)_{n\geqslant 1}) = \lim_{n\to\infty} d(u_n, u_m) \leqslant \varepsilon.$$

Therefore,

$$\lim_{m\to\infty} i(u_m) = \hat{U}.$$

This shows that any point in $\hat{M}$ can be obtained as the sequential limit of points in the image $i(M)$, that is $i(M)$ is dense in $\hat{M}$. □

Proposition 3.3.6 : *The metric space $(\hat{M}, \hat{\delta})$ is complete.*

Proof : Let $(\alpha_n)_{n\geqslant 1}$ be a Cauchy sequence in $\hat{M}$. Using the fact that $i(M)$ is dense in $\hat{M}$, for every $n \geqslant 1$, we may find $x_n \in M$ such that $\hat{\delta}(\alpha_n, i(x_n)) < \frac{1}{n}$. Then, for any $m, n \geqslant 1$, we have

$$d(x_m, x_n) = \hat{\delta}(i(x_m), i(x_n)) \leqslant \hat{\delta}(i(x_m), \alpha_m) + \hat{\delta}(\alpha_m, \alpha_n) + \hat{\delta}(\alpha_n, i(x_n)) \leqslant \hat{\delta}(\alpha_m, \alpha_n) + \frac{1}{m} + \frac{1}{n},$$

which means that $(x_n)_{n\geqslant 1}$ is also a Cauchy sequence (in M).

Let $(x_n)_{n\geqslant 1} \in \mathcal{C}$, and α be its equivalence class with respect to the relation $\sim$. We want to prove that $(\alpha_n)_{n\geqslant 1}$ converges to α. Let $\varepsilon > 0$. We may find $N \geqslant 1$ such that

$$d(x_n, x_m) \leqslant \varepsilon, \quad \forall n, m \geqslant N.$$

Then, for $n \geqslant N$, we find

$$\hat{\delta}(\alpha_n, \alpha) \leqslant \hat{\delta}(\alpha_n, i(x_n)) + \hat{\delta}(i(x_n), \alpha) \leqslant \frac{1}{n} + \varepsilon,$$

because by definition, we have

$$\hat{\delta}(i(x_n), \alpha) = \lim_{m\to\infty} d(x_n, x_m) \leqslant \varepsilon.$$

Therefore, by taking $\limsup$, we find

$$\limsup_{n\to\infty} \hat{\delta}(\alpha_n, \alpha) \leqslant \varepsilon.$$

Since $\varepsilon > 0$ can be taken to be arbitrarily small, we find $\limsup_{n\to\infty} \hat{\delta}(\alpha_n, \alpha) = 0$, in other words, $\lim_{n\to\infty} \hat{\delta}(\alpha_n, \alpha) = 0$. □

Proposition 3.3.7 : *The completion $(\hat{M}, \hat{\delta})$ is unique in the sense of Theorem 3.3.1.*

Proof : Let $(\hat{M}_1, \hat{\delta}_1)$ and $(\hat{M}_2, \hat{\delta}_2)$ be two completions of (M, d), and i_1 and i_2 be the corresponding isometric injections as in Theorem 3.3.1.

Let $\varphi(i_1(x)) = i_2(x)$ for all $x \in M$, which defines φ on the image $i_1(M)$. It is easy to check that $\varphi_{|i_1(M)}$ is an isometry,

$$\forall x, y \in M, \quad \hat{\delta}_2(\varphi(i_1(x)), \varphi(i_2(y))) = \hat{\delta}_2(i_2(x), i_2(y)) = d(x, y) = \hat{\delta}_1(i_1(x), i_1(y)).$$

Therefore, φ is uniformly continuous on $i_1(M)$. Since $i_1(M) \subseteq \hat{M}_1$ is a dense subset and $\hat{M}_2$ is copmlete, it follows from Exercise 3.23 that there exists a *unique* uniform continuation of φ on $\hat{M}_1$, that we still call φ by abuse of notation. Moreover, due to the fact that φ is isometric on $i_1(M)$ and $i_1(M)$ is dense in $\hat{M}_1$, the continuity of φ shows that φ is isometric on $\hat{M}_1$. In particular, this also shows that φ is injective.

To show that φ is surjective, we are given $y \in \hat{M}_2$ and we need to construct its preimage under φ. We use the fact that $i_2(M)$ is dense in $\hat{M}_2$ to find a sequence $(y_n = i_2(x_n))_{n\geqslant 1}$, where $x_n \in M$ for all

$n \geqslant 1$, and such that $y_n \xrightarrow[n\to\infty]{} y$. For any $p, q \geqslant 1$, we have

$$\hat{\delta}_1(i_1(x_p), i_1(x_q)) = d(x_p, x_q) = \hat{\delta}_2(i_2(x_p), i_2(x_q)) = \hat{\delta}_2(y_p, y_q),$$

we see that the sequence $(i_1(x_n))_{n\geqslant 1}$ is Cauchy in $\hat{M}_1$. Since $\hat{M}_1$ is complete, it converges to a limit $\alpha := \lim_{n\to\infty} i_1(x_n)$. Then, by the continuity of φ, we find

$$\varphi(\alpha) = \lim_{n\to\infty} \varphi(i_1(x_n)) = \lim_{n\to\infty} i_2(x_n) = \lim_{n\to\infty} y_n = y.$$

This concludes that φ is surjective. □

3.3.2 Completion of a normed space

Now, we discuss another construction in the special case of a normed space. We note that a finite dimensional normed vector space is always complete, so the interesting cases concern infinite dimensional normed vector spaces, such as spaces of sequences $\ell^p(\mathbb{K})$ for $p = 1, 2, \infty$, or functional spaces such as $\mathcal{C}([0,1], \mathbb{K})$ or $\mathcal{B}([0,1], \mathbb{K})$. The construction is quite simple, but involves a theorem from functional analysis, whose proof will be omitted here.

Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Write $V^* = \mathcal{L}_c(V, \mathbb{K})$ for the dual space of V, which is the space of the continuous linear forms on V, that we equip with the operator norm

$$\forall f \in V^*, \quad |||f|||_{V^*} := \sup\{|f(x)| : x \in V, \|x\| \leqslant 1\}.$$

We also consider the double dual space $V^{**} := (V^*)^* = \mathcal{L}_c(V^*, \mathbb{K})$, that we equip with the operator norm

$$\forall \Phi \in V^{**}, \quad |||\Phi|||_{V^{**}} := \sup\{|\Phi(f)| : f \in V^*, |||f|||_{V^*} \leqslant 1\}.$$

It follows from Theorem 3.2.18 that both $(V^*, |||\cdot|||_{V^*})$ and $(V^{**}, |||\cdot|||_{V^{**}})$ are Banach spaces.

Then, we consider the map $J : V \to V^{**}$, defined by

$$J(x)(f) := f(x), \quad \forall x \in V, \forall f \in V^*.$$

We first check that J is well defined, that is, for every $x \in V$, we need to verify that $J(x) \in V^{**}$. Let

$f, g \in V^*$ and $\lambda \in \mathbb{K}$. Indeed, we have

$$J(x)(f + \lambda g) = (f + \lambda g)(x) = f(x) + \lambda g(x) = J(x)(f) + \lambda J(x)(g).$$

Then, we check that J is a linear map. Let $\lambda \in \mathbb{K}$ and $x, y \in V$. For any $f \in V^*$, we have

$$J(x + \lambda y)(f) = f(x + \lambda y) = f(x) + \lambda f(y) = J(x)(f) + \lambda J(y)(f),$$

that is,

$$J(x + \lambda y) = J(x) + \lambda J(y).$$

For every $x \in V$, we have

$$\begin{aligned} \lVert\!| J(x) |\!\rVert_{V^{**}} &= \sup\{|J(x)(f)| : f \in V^*, \lVert\!| f |\!\rVert_{V^*} \leqslant 1\} \\ &= \sup\{|f(x)| : f \in V^*, \lVert\!| f |\!\rVert_{V^*} \leqslant 1\} \leqslant \|x\| . \end{aligned} \tag{3.9}$$

By a theorem from functional analysis, called Hahn-Banach theorem, for any given $x \in V$, we can actually find a linear form $f_x \in V^*$ such that $f_x(x) = \|x\|$ and $\lVert\!| f_x |\!\rVert_{V^*} \leqslant 1$. This gives us

$$\begin{aligned} \lVert\!| J(x) |\!\rVert_{V^{**}} &= \sup\{|J(x)(f)| : f \in V^*, \lVert\!| f |\!\rVert_{V^*} \leqslant 1\} \\ &\geqslant J(x)(f_x) = f_x(x) = \|x\| . \end{aligned} \tag{3.10}$$

The above Eq. (3.9) and Eq. (3.10) give us $\lVert\!| J(x) |\!\rVert_{V^{**}} = \|x\|$, that is J is an isometry.

To conclude, we may take $\hat{V} := \overline{J(V)} \subseteq V^{**}$, which is a linear subspace of V^{**}. It is obvious that $J(V)$ is dense in $\overline{J(V)}$. Moreover, since V^{**} is a complete normed vector space, the closed subset $\overline{J(V)}$ is also complete by Proposition 3.2.1. This allows us to conclude that $(\hat{V}, \lVert\!| \cdot |\!\rVert_{V^{**}})$ is a completion of the normed vector space $(V, \|\cdot\|)$.