

In the first-year calculus, we have discussed sequences and series with values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We have seen different criteria to describe their convergence. In the first term of this year, we have studied the notion of convergence of sequences in more general spaces, such as metric spaces. But if we want to discuss series, since an addition operation is needed, to make a more general theory, we will restrict ourselves to normed vector spaces. In this chapter, the sequences and series are considered to take their values in a normed vector space $(W, \|\cdot\|_W)$.

6.1 Basic notions

6.1.1 Reminders for real sequences

Definition 6.1.1 : Let $(a_n)_{n \geq 1}$ be a sequence of real numbers.

- We say that $(a_n)_{n \geq 1}$ converges to $\ell \in \mathbb{R}$, denoted $a_n \xrightarrow{n \rightarrow \infty} \ell$, if for every $\varepsilon > 0$, there exists $N \geq 1$ such that

$$\forall n \geq N, \quad |a_n - \ell| < \varepsilon.$$

- (Cauchy's condition) The above definition is equivalent to the following: for every $\varepsilon > 0$, there exists $N \geq 1$ such that

$$\forall m, n \geq N, \quad |a_m - a_n| < \varepsilon.$$

Proposition 6.1.2 : Let $(a_n)_{n \geq 1}$ be a sequence of real numbers.

- (1) If $(a_n)_{n \geq 1}$ is non-decreasing and is bounded from above by some $M < \infty$, then $(a_n)_{n \geq 1}$ converges to a limit $\ell \leq M$.

在大一微積分中，我們討論了取值在 $\mathbb{K} = \mathbb{R}$ 或 $\mathbb{C}$ 當中的序列與級數，我們也看到不同的方式，來描述他們的收斂。在這學年的第一學期，我們討論了在更一般空間中，像是賦距空間中，序列的收斂。但如果我們想要討論級數，由於我們需要使用的加法運算，如果要對更一般的情況做討論，我們需要限制在賦範向量空間中。在這一章中，序列和級數都會被假設取值在賦範向量空間 $(W, \|\cdot\|_W)$ 當中。

第一節 基本概念

第一小節 實數序列的回顧

定義 6.1.1 : 令 $(a_n)_{n \geq 1}$ 為實數序列。

- 如果對於每個 $\varepsilon > 0$ ，存在 $N \geq 1$ 使得

$$\forall n \geq N, \quad |a_n - \ell| < \varepsilon,$$

則我們說序列 $(a_n)_{n \geq 1}$ 收斂至 $\ell \in \mathbb{R}$ ，記作 $a_n \xrightarrow{n \rightarrow \infty} \ell$ 。

- 【Cauchy 條件】上面的定義也與下面等價：對於每個 $\varepsilon > 0$ ，存在 $N \geq 1$ 使得

$$\forall m, n \geq N, \quad |a_m - a_n| < \varepsilon.$$

命題 6.1.2 : 令 $(a_n)_{n \geq 1}$ 為實數序列。

- (1) 如果 $(a_n)_{n \geq 1}$ 非遞減且有上界 $M < \infty$ ，那麼 $(a_n)_{n \geq 1}$ 收斂至極限 $\ell \leq M$ 。
- (2) 如果 $(a_n)_{n \geq 1}$ 非遞增且有下界 $M > -\infty$ ，那麼 $(a_n)_{n \geq 1}$ 收斂至極限 $\ell \geq M$ 。

(2) If $(a_n)_{n \geq 1}$ is non-increasing and is bounded from below by some $M > -\infty$, then $(a_n)_{n \geq 1}$ converges to a limit $\ell \geq M$.

Definition 6.1.3 : Given two sequences $a = (a_n)_{n \geq 1}$ and $b = (b_n)_{n \geq 1}$ of real numbers. We say that they are *adjacent* (相伴序列) if one is increasing, the other one is decreasing, with $a_n - b_n \xrightarrow{n \rightarrow \infty} 0$.

Proposition 6.1.4 : If the sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ are adjacent, then they converge to the same limit.

Definition 6.1.5 : Given two sequences $a = (a_n)_{n \geq 1}$ and $b = (b_n)_{n \geq 1}$ of real numbers. We define the following asymptotic relations.

- (1) We say that a is *dominated* by b , denoted $a_n = \mathcal{O}(b_n)$, if there exists a bounded sequence $c = (c_n)_{n \geq 1}$ and $N \in \mathbb{N}$ such that $a_n = c_n b_n$ for all $n \geq N$.
- (2) We say that a is *negligible* compared to b , denoted $a_n = o(b_n)$, if there exists a sequence $\varepsilon = (\varepsilon_n)_{n \geq 1}$ that converges to 0 and $N \in \mathbb{N}$ such that $a_n = \varepsilon_n b_n$ for all $n \geq N$.
- (3) We say that a is *equivalent* to b , denoted $a_n \sim b_n$, if there exists a sequence $c = (c_n)_{n \geq 1}$ that converges to 1 and $N \in \mathbb{N}$ such that $a_n = c_n b_n$ for all $n \geq N$.

Remark 6.1.6 :

- (1) When we write these relations between a and b , we may add the condition $n \rightarrow \infty$ to emphasize that in the asymptotic relation, we are taking n to infinity (not some other value), or if there are other variables that might bring confusion.
- (2) It can be checked that the binary relation $\sim$ is an equivalence relation on the space of real-valued sequences $\mathbb{R}^{\mathbb{N}}$. However, the asymptotic notations $\mathcal{O}$ and o do not satisfy symmetry.

定義 6.1.3 : 給定兩個實數序列 $a = (a_n)_{n \geq 1}$ 和 $b = (b_n)_{n \geq 1}$ 。如果他們其中之一是遞增的，另一個是遞減的，而且 $a_n - b_n \xrightarrow{n \rightarrow \infty} 0$ ，則我們說他們是相伴序列 (adjacent sequences)。

命題 6.1.4 : 如果兩個序列 $(a_n)_{n \geq 1}$ 和 $(b_n)_{n \geq 1}$ 是相伴的，則他們會收斂到相同的極限。

定義 6.1.5 : 給定兩個實數序列 $a = (a_n)_{n \geq 1}$ 和 $b = (b_n)_{n \geq 1}$ ，我們定義下面幾個漸進關係。

- (1) 如果存在有界序列 $c = (c_n)_{n \geq 1}$ 以及 $N \in \mathbb{N}$ 使得 $a_n = c_n b_n$ 對於所有 $n \geq N$ ，則我們說 a 會被 b 控制，記作 $a_n = \mathcal{O}(b_n)$ 。
- (2) 如果存在序列 $\varepsilon = (\varepsilon_n)_{n \geq 1}$ 會收斂到 0 以及 $N \in \mathbb{N}$ 滿足 $a_n = \varepsilon_n b_n$ 對於所有 $n \geq N$ ，則我們說 a 相對於 b 是可忽略的，記作 $a_n = o(b_n)$ 。
- (3) 如果存在序列 $c = (c_n)_{n \geq 1}$ 收斂到 1 以及 $N \in \mathbb{N}$ 滿足 $a_n = c_n b_n$ 對於所有 $n \geq N$ ，則我們說 a 與 b 是等價的，記作 $a_n \sim b_n$ 。

註解 6.1.6 :

- (1) 當我們寫下上面 a 與 b 之間的關係時，我們可以增加條件 $n \rightarrow \infty$ 來強調我們的漸進式是在 n 趨近於無窮時的（而不是趨近於其他數），或是當我們有多個變數，可能會導致混淆，也可以這樣強調。
- (2) 我們可以檢查二元關係 $\sim$ 的確是個在實數序列空間 $\mathbb{R}^{\mathbb{N}}$ 上的等價關係。然而，漸進記號 $\mathcal{O}$ 和 o 並不滿足對稱性。

Example 6.1.7 :

- (1) Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be defined by

$$\forall n \geq 1, \quad a_n = \frac{1}{n} \quad \text{and} \quad b_n = \frac{1}{n} + \frac{1}{n^2}.$$

Then, $a_n = \mathcal{O}(b_n)$ and $a_n \sim b_n$.

- (2) Let $(a_n)_{n \geq 1} = (0, 1, 1, \dots)$ and $(b_n)_{n \geq 1} = (1, 1, 1, \dots)$. Then, $a_n = \mathcal{O}(b_n)$ and $a_n \sim b_n$.

- (3) Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be defined by

$$\forall n \geq 1, \quad a_n = n^2 \quad \text{and} \quad b_n = 2^n.$$

Then, $a_n = o(b_n)$ and $a_n = \mathcal{O}(b_n)$.

範例 6.1.7 :

- (1) 令 $(a_n)_{n \geq 1}$ 與 $(b_n)_{n \geq 1}$ 定義如下

$$\forall n \geq 1, \quad a_n = \frac{1}{n} \quad \text{以及} \quad b_n = \frac{1}{n} + \frac{1}{n^2}.$$

那麼 $a_n = \mathcal{O}(b_n)$ 且 $a_n \sim b_n$ 。

- (2) 令 $(a_n)_{n \geq 1} = (0, 1, 1, \dots)$ 與 $(b_n)_{n \geq 1} = (1, 1, 1, \dots)$ 。那麼 $a_n = \mathcal{O}(b_n)$ 且 $a_n \sim b_n$ 。

- (3) 令 $(a_n)_{n \geq 1}$ 與 $(b_n)_{n \geq 1}$ 定義如下

$$\forall n \geq 1, \quad a_n = n^2 \quad \text{以及} \quad b_n = 2^n.$$

那麼 $a_n = o(b_n)$ 且 $a_n = \mathcal{O}(b_n)$ 。

6.1.2 Definitions

Let $(u_n)_{n \geq 1}$ be a sequence with values in a normed vector space $(W, \|\cdot\|)$.

Definition 6.1.8 :

- The *series* (級數) with general term u_n is given by the sequence $(S_n)_{n \geq 0}$, defined by

$$S_0 = 0, \quad S_n = u_1 + \dots + u_n = \sum_{k=1}^n u_k, \quad \forall n \geq 1.$$

We may also denote this series by $\sum_{n \geq 1} u_n$ or $\sum u_n$.

- For each $n \geq 1$, u_n is called *the n -th term* of the series $\sum u_n$, S_n is called *the n -th partial sum* of the series $\sum u_n$.
- If the sequence $(S_n)_{n \geq 0}$ converges in $(W, \|\cdot\|)$, then we say that the series $\sum u_n$ converges. In this case, its limit is called the *sum of the series*, and is denoted by $\sum_{n=1}^{\infty} u_n$; for each $n \geq 1$, we

第二小節 定義

令 $(u_n)_{n \geq 1}$ 為取值在賦範向量空間 $(W, \|\cdot\|)$ 中的序列。

定義 6.1.8 :

- 一般項為 u_n 的級數 (series)，是由下列序列 $(S_n)_{n \geq 0}$ 所給定的：

$$S_0 = 0, \quad S_n = u_1 + \dots + u_n = \sum_{k=1}^n u_k, \quad \forall n \geq 1.$$

我們也可以把這個級數記作 $\sum_{n \geq 1} u_n$ 或 $\sum u_n$ 。

- 對於每個 $n \geq 1$ ，我們把 u_n 稱作級數 $\sum u_n$ 的第 n 項，把 S_n 稱作級數 $\sum u_n$ 的第 n 個部份和。
- 如果序列 $(S_n)_{n \geq 0}$ 在 $(W, \|\cdot\|)$ 中收斂，那麼我們說級數 $\sum u_n$ 收斂。在這個情況下，他的

denote by R_n the n -th remainder, defined by

$$R_n = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^n u_k = \sum_{k=n+1}^{\infty} u_k.$$

Remark 6.1.9 : We note that by definition, the convergence of a sequence $(S_n)_{n \geq 0}$ is equivalent to the convergence of the series $\sum (S_{n+1} - S_n)$, which are related by the relation

$$\sum_{n=0}^{N-1} (S_{n+1} - S_n) = S_N - S_0 = S_N.$$

Such a summation is called a *telescoping* summation.

Proposition 6.1.10 : We have the following two properties.

- (1) If the series $\sum u_n$ converges, then $(S_n)_{n \geq 0}$ is a Cauchy sequence.
- (2) Additionally, if $(W, \|\cdot\|)$ is a Banach space, then the series $\sum u_n$ converges if and only if $(S_n)_{n \geq 0}$ is a Cauchy sequence.

Proof : This proposition follows directly from the definition.

- (1) The series $\sum u_n$ converges means that the sequence $(S_n)_{n \geq 1}$ converges. And it follows from Proposition 2.4.6 that a convergent sequence is a Cauchy sequence.
- (2) It remains to show the converse. Suppose that $(S_n)_{n \geq 0}$ is a Cauchy sequence, since it takes value in a Banach space, it converges, so the series $\sum u_n$ is convergent. $\square$

Corollary 6.1.11 (Cauchy's condition) : Suppose that $(W, \|\cdot\|)$ is a Banach space. The series $\sum u_n$ converges if and only if for every $\varepsilon > 0$, there exists $N \geq 1$ such that

$$\forall n \geq N, \forall k \geq 1, \quad \|u_{n+1} + \cdots + u_{n+k}\| < \varepsilon. \quad (6.1)$$

This condition is called Cauchy's condition (柯西條件).

極限稱作級數的和，記作 $\sum_{n=1}^{\infty} u_n$ ；對於每個 $n \geq 1$ ，我們把他第 n 項餘項 R_n 定義為

$$R_n = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^n u_k = \sum_{k=n+1}^{\infty} u_k.$$

註解 6.1.9 : 我們注意到，根據定義，序列 $(S_n)_{n \geq 0}$ 的收斂與級數 $\sum (S_{n+1} - S_n)$ 的收斂等價，因為他們可以用下列關係式把他們連結在一起：

$$\sum_{n=0}^{N-1} (S_{n+1} - S_n) = S_N - S_0 = S_N.$$

這樣的和稱作裂項和。

命題 6.1.10 : 我們有下列兩個性質。

- (1) 如果級數 $\sum u_n$ 收斂，那麼 $(S_n)_{n \geq 0}$ 是個柯西序列。
- (2) 此外，如果 $(W, \|\cdot\|)$ 是個 Banach 空間，那麼級數 $\sum u_n$ 收斂若且唯若 $(S_n)_{n \geq 0}$ 是個柯西序列。

證明 : 這個命題可以直接藉由定義所得到。

- (1) 級數 $\sum u_n$ 收斂代表序列 $(S_n)_{n \geq 1}$ 收斂。從命題 2.4.6 我們知道收斂序列是個柯西序列。
- (2) 我們只需要證明逆命題即可。假設 $(S_n)_{n \geq 0}$ 是個柯西序列，由於他取值在 Banach 空間中，他會收斂，所以級數 $\sum u_n$ 收斂。 $\square$

系理 6.1.11 【柯西條件】：假設 $(W, \|\cdot\|)$ 是個 Banach 空間。級數 $\sum u_n$ 收斂若且唯若對於每個 $\varepsilon > 0$ ，存在 $N \geq 1$ 使得

$$\forall n \geq N, \forall k \geq 1, \quad \|u_{n+1} + \cdots + u_{n+k}\| < \varepsilon. \quad (6.1)$$

Proof : It is a direct consequence of Proposition 6.1.10 (2). $\square$

Corollary 6.1.12 : If $\sum u_n$ is a convergent series, then $\lim_{n \rightarrow \infty} u_n = 0$.

Proof : It is a direct consequence of Cauchy's condition by taking the special case $k = 1$ in Eq. (6.1). $\square$

Remark 6.1.13 : We note that the convergence to zero of the general term is *necessary but not sufficient* for a series to converge. For example, the harmonic series $\sum \frac{1}{n}$ diverges, but its general term tends to 0.

Definition 6.1.14 : Suppose that $(W, \|\cdot\|)$ is a Banach space, and let $\sum u_n$ be a series with general terms in W .

- If the series $\sum \|u_n\|$ converges, we say that the series $\sum u_n$ *converges absolutely* (絕對收斂).
- If the series $\sum u_n$ converges but does not converge absolutely, then we say that $\sum u_n$ *converges conditionally* (條件收斂).

Example 6.1.15 : The series $\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} = \ln 2$ is convergent but not absolutely convergent. We will have a more thorough study of such series, called *alternating series*, in Section 6.4.1.

Theorem 6.1.16 : Suppose that $(W, \|\cdot\|)$ is a Banach space. A series $\sum u_n$ that converges absolutely in W also converges.

Proof : For every $n, k \geq 1$, we have

$$\|u_{n+1} + \cdots + u_{n+k}\| \leq \|u_{n+1}\| + \cdots + \|u_{n+k}\|.$$

這條件稱作柯西條件 (Cauchy's condition) 。

證明 : 這是命題 6.1.10 (2) 的直接結果。 $\square$

系理 6.1.12 : 如果 $\sum u_n$ 是個收斂序列，那麼 $\lim_{n \rightarrow \infty} u_n = 0$ 。

證明 : 這可以看作是式 (6.1) 中柯西條件取 $k = 1$ 的特例。 $\square$

註解 6.1.13 : 我們注意到，一般項收斂到零是級數收斂的必要但不充分條件。例如，調和級數 $\sum \frac{1}{n}$ 發散，但他的一般項趨近於 0 。

定義 6.1.14 : 假設 $(W, \|\cdot\|)$ 是個 Banach 空間，並令 $\sum u_n$ 為一般項在 W 中的級數。

- 如果級數 $\sum \|u_n\|$ 收斂，那我們說級數 $\sum u_n$ 會絕對收斂 (converges absolutely) 。
- 如果級數 $\sum u_n$ 收斂但不絕對收斂，那我們說 $\sum u_n$ 條件收斂 (converges conditionally) 。

範例 6.1.15 : 級數 $\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} = \ln 2$ 收斂但不會絕對收斂。在 6.4.1 小節中，我們會比較完整地討論這種類型的級數，稱作交錯級數。

定理 6.1.16 : 假設 $(W, \|\cdot\|)$ 是個 Banach 空間。在 W 中會絕對收斂的級數 $\sum u_n$ 也會收斂。

證明 : 對於每個 $n, k \geq 1$ ，我們有

$$\|u_{n+1} + \cdots + u_{n+k}\| \leq \|u_{n+1}\| + \cdots + \|u_{n+k}\|.$$

Thus, Cauchy's condition for $\sum \|u_n\|$ implies Cauchy's condition for $\sum u_n$. $\square$

Remark 6.1.17 : From the above theorem, to show that a series converges in a Banach space, we may show that it converges absolutely, which reduces to the convergence of a series with non-negative terms. This is the reason why understanding the behavior of a series with non-negative terms is important. In the next subsection, we are going to study sufficient conditions for such series to converge.

6.2 Series with non-negative terms

6.2.1 Comparison between series

Proposition 6.2.1 : Let $\sum u_n$ be a series with non-negative terms. It converges if and only if the sequence $(S_n)_{n \geq 0}$ of partial sums is bounded from above.

Proof : It is a direct consequence of Proposition 6.1.2. $\square$

Proposition 6.2.2 (Comparison test) : We consider two non-negative series $\sum u_n$ and $\sum v_n$ satisfying

$$\forall n \geq 1, \quad 0 \leq u_n \leq v_n.$$

- (1) If $\sum v_n$ converges, then $\sum u_n$ converges.
- (2) If $\sum u_n$ diverges, then $\sum v_n$ diverges.

Proof : Let $(S_n)_{n \geq 0}$ be the partial sums of $\sum u_n$ and $(T_n)_{n \geq 0}$ be the partial sums of $\sum v_n$. Then, for every $n \geq 0$, we have $S_n \leq T_n$. We conclude by Proposition 6.1.2. $\square$

因此，級數 $\sum \|u_n\|$ 的柯西條件會蘊含級數 $\sum u_n$ 的柯西條件。 $\square$

註解 6.1.17 : 根據上面定理，要證明在 Banach 空間中的級數會收斂，我們可以證明他會絕對收斂，也就是需要去討論一般項非負級數的收斂。這也就是為什麼，我們需要更有系統性地去了解這種一般項非負級數的收斂。在下個小節，我們會去討論讓這種級數收斂的一些充分條件。

第二節 一般項非負項的級數

第一小節 級數之間的比較

命題 6.2.1 : 令 $\sum u_n$ 為一般項非負項的級數。他會收斂若且唯若部份和序列 $(S_n)_{n \geq 0}$ 有上界。

證明 : 這是命題 6.1.2 的直接結果。 $\square$

命題 6.2.2 【比較法則】 : 我們考慮兩個非負序列 $\sum u_n$ 和 $\sum v_n$ 滿足

$$\forall n \geq 1, \quad 0 \leq u_n \leq v_n.$$

- (1) 如果 $\sum v_n$ 收斂，則 $\sum u_n$ 收斂。
- (2) 如果 $\sum u_n$ 發散，則 $\sum v_n$ 發散。

證明 : 令 $(S_n)_{n \geq 0}$ 為 $\sum u_n$ 的部份和，且 $(T_n)_{n \geq 0}$ 為 $\sum v_n$ 的部份和。那麼對於每個 $n \geq 0$ ，我們有 $S_n \leq T_n$ 。我們最後透過命題 6.1.2 來總結。 $\square$

Theorem 6.2.3 : Let $\sum u_n$ and $\sum v_n$ be two series with non-negative terms.

- (1) If $v_n = \mathcal{O}(u_n)$ and $\sum u_n$ converges, then $\sum v_n$ converges.
- (2) If $u_n \sim v_n$, then the series $\sum u_n$ and $\sum v_n$ are of the same behavior (i.e. both divergent or convergent).

Proof :

- (1) Suppose that $v_n = \mathcal{O}(u_n)$. Let $M > 0$ and $N \geq 1$ such that $v_n \leq Mu_n$ for all $n \geq N$. This means that for $n \geq N$, we have

$$\sum_{k=1}^n v_k = \sum_{k=1}^{N-1} v_k + \sum_{k=N}^n v_k \leq \sum_{k=1}^{N-1} v_k + M \sum_{k=N}^n u_k.$$

Since $\sum u_n$ converges, the sequence $(\sum_{k=N}^n u_k)_{n \geq N}$ is bounded from above. Therefore, the series $\sum v_n$ converges.

- (2) If $u_n \sim v_n$, it means that $u_n = \mathcal{O}(v_n)$ and $v_n = \mathcal{O}(u_n)$. So (1) implies that $\sum u_n$ converges if and only if $\sum v_n$ converges.

□

Remark 6.2.4 : We note that to apply Theorem 6.2.3, the assumption on non-negative terms is essential.

- (1) For $n \geq 1$, let $u_n = \frac{(-1)^n}{n}$ and $v_n = \frac{1}{n}$. It is clear that $v_n = \mathcal{O}(u_n)$. However, $\sum u_n$ converges by Theorem 6.4.2, but $\sum v_n$ diverges from Proposition 6.2.6.
- (2) For $n \geq 1$, let $u_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}$ and $v_n = \frac{(-1)^n}{\sqrt{n}}$. It is clear that $u_n \sim v_n$. However, $\sum v_n$ converges by Theorem 6.4.2, but $\sum u_n$ diverges because $\sum \frac{1}{n}$ diverges.

Example 6.2.5 : Let us study the behavior of the series $\sum \frac{1}{n^2}$. First, we note that

$$\forall k \geq 2, \quad \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \leq \frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

定理 6.2.3 : 令 $\sum u_n$ 與 $\sum v_n$ 為兩個一般項非負項的級數。

- (1) 如果 $v_n = \mathcal{O}(u_n)$ 而且 $\sum u_n$ 收斂，那麼 $\sum v_n$ 收斂。
- (2) 如果 $u_n \sim v_n$ ，那麼級數 $\sum u_n$ 和 $\sum v_n$ 會有相同行為（也就是兩者皆發散或兩者皆收斂）。

證明 :

- (1) 假設 $v_n = \mathcal{O}(u_n)$ 。令 $M > 0$ 以及 $N \geq 1$ 使得 $v_n \leq Mu_n$ 對於所有 $n \geq N$ 。這代表當 $n \geq N$ 時，我們有

$$\sum_{k=1}^n v_k = \sum_{k=1}^{N-1} v_k + \sum_{k=N}^n v_k \leq \sum_{k=1}^{N-1} v_k + M \sum_{k=N}^n u_k.$$

由於 $\sum u_n$ 收斂，序列 $(\sum_{k=N}^n u_k)_{n \geq N}$ 會有上界。因此，級數 $\sum v_n$ 收斂。

- (2) 如果 $u_n \sim v_n$ ，這代表 $u_n = \mathcal{O}(v_n)$ 而且 $v_n = \mathcal{O}(u_n)$ 。因此 (1) 蘊含 $\sum u_n$ 收斂若且唯若 $\sum v_n$ 收斂。

□

註解 6.2.4 : 我們注意到，如果想要使用定理 6.2.3，一般項非負的假設是很重要的。

- (1) 對於 $n \geq 1$ ，令 $u_n = \frac{(-1)^n}{n}$ 以及 $v_n = \frac{1}{n}$ 。顯然，我們有 $v_n = \mathcal{O}(u_n)$ 。然而，根據定理 6.4.2，級數 $\sum u_n$ 會收斂，但根據命題 6.2.6，級數 $\sum v_n$ 會發散。
- (2) 對於 $n \geq 1$ ，令 $u_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}$ 以及 $v_n = \frac{(-1)^n}{\sqrt{n}}$ 。顯然，我們有 $u_n \sim v_n$ 。然而，根據定理 6.4.2，級數 $\sum v_n$ 會收斂，但由於 $\sum \frac{1}{n}$ 會發散，級數 $\sum u_n$ 也會發散。

範例 6.2.5 : 我們來研究級數 $\sum \frac{1}{n^2}$ 的行為。首先，我們注意到

$$\forall k \geq 2, \quad \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \leq \frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

Since the telescoping series $\sum(\frac{1}{n-1} - \frac{1}{n})$ converges, it follows from Proposition 6.2.2 (1) that the series $\sum \frac{1}{n^2}$ also converges. Moreover, for $n \geq 2$, the following relation holds for the n -th remainder,

$$\frac{1}{n+1} \leq R_n := \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \frac{1}{n}.$$

By Cauchy's condition, the series $\sum \frac{1}{n^2}$ converges, and its n -th remainder is equivalent to $\frac{1}{n}$.

Another way to show the convergence is to note that $\frac{1}{n^2} \sim \frac{1}{n-1} - \frac{1}{n}$ and apply Theorem 6.2.3 (2). But to find an asymptotic formula for the n -th remainder, we need to apply Theorem 6.2.8 that we will see at a later stage.

Proposition 6.2.6 (Riemann series) : Let α be a real number. The Riemann series $\sum \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.

Proof : For real numbers $\alpha > \beta$, we have $\frac{1}{n^\alpha} < \frac{1}{n^\beta}$ for all $n \geq 1$. By Proposition 6.2.2, it is sufficient to show that $\sum \frac{1}{n}$ diverges, and for any $\alpha > 1$, the series $\sum \frac{1}{n^\alpha}$ converges.

- Let $\alpha = 1$. For every $k \geq 1$, we have

$$\frac{1}{k} = \int_k^{k+1} \frac{dt}{t} \geq \int_k^{k+1} \frac{dt}{t} = \ln(k+1) - \ln k.$$

This implies that

$$\forall n \geq 1, \quad \sum_{k=1}^n \frac{1}{k} \geq \sum_{k=1}^n (\ln(k+1) - \ln k) = \ln(n+1).$$

Since $\ln(n+1) \xrightarrow{n \rightarrow \infty} +\infty$, we deduce from Proposition 6.2.2 (2) that the series $\sum \frac{1}{n}$ diverges.

- Let $\alpha > 1$. Similarly, For every $k \geq 2$, we have

$$\frac{1}{k^\alpha} = \int_{k-1}^k \frac{dt}{t^\alpha} \leq \int_{k-1}^k \frac{dt}{t^\alpha} = \frac{1}{1-\alpha} \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}} \right]. \quad (6.2)$$

This gives that

$$\forall n \geq 2, \quad \sum_{k=2}^n \frac{1}{k^\alpha} \leq \frac{1}{1-\alpha} \sum_{k=2}^n \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}} \right] = \frac{1}{\alpha-1} \left[1 - \frac{1}{n^{\alpha-1}} \right] \leq \frac{1}{\alpha-1}.$$

由於裂項級數 $\sum(\frac{1}{n-1} - \frac{1}{n})$ 收斂，根據命題 6.2.2 (1)，級數 $\sum \frac{1}{n^2}$ 也會收斂。此外，對於 $n \geq 2$ ，我們可以控制第 n 項的餘項，也就是：

$$\frac{1}{n+1} \leq R_n := \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leq \frac{1}{n}.$$

根據柯西條件，級數 $\sum \frac{1}{n^2}$ 收斂，且他的第 n 項餘項與 $\frac{1}{n}$ 等價。

我們也可以使用定理 6.2.3 (2) 來討論收斂性，因為我們有 $\frac{1}{n^2} \sim \frac{1}{n-1} - \frac{1}{n}$ 。但如果想要找到第 n 項餘項的漸進式，我們需要使用稍後才會看到的定理 6.2.8。

命題 6.2.6 【黎曼級數】： 令 α 為實數。黎曼級數 $\sum \frac{1}{n^\alpha}$ 收斂若且唯若 $\alpha > 1$ 。

證明：對於實數 $\alpha > \beta$ ，我們有 $\frac{1}{n^\alpha} < \frac{1}{n^\beta}$ 對於所有 $n \geq 1$ 。根據命題 6.2.2，我們只需要證明 $\sum \frac{1}{n}$ 發散，且對於任意 $\alpha > 1$ ，級數 $\sum \frac{1}{n^\alpha}$ 會收斂即可。

- 令 $\alpha = 1$ 。對於每個 $k \geq 1$ ，我們有

$$\frac{1}{k} = \int_k^{k+1} \frac{dt}{t} \geq \int_k^{k+1} \frac{dt}{t} = \ln(k+1) - \ln k.$$

這蘊含

$$\forall n \geq 1, \quad \sum_{k=1}^n \frac{1}{k} \geq \sum_{k=1}^n (\ln(k+1) - \ln k) = \ln(n+1).$$

由於 $\ln(n+1) \xrightarrow{n \rightarrow \infty} +\infty$ ，我們從 命題 6.2.2 (2) 推得級數 $\sum \frac{1}{n}$ 會發散。

- 令 $\alpha > 1$ 。同理，對於每個 $k \geq 2$ ，我們有

$$\frac{1}{k^\alpha} = \int_{k-1}^k \frac{dt}{t^\alpha} \leq \int_{k-1}^k \frac{dt}{t^\alpha} = \frac{1}{1-\alpha} \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}} \right]. \quad (6.2)$$

這會給我們

$$\forall n \geq 2, \quad \sum_{k=2}^n \frac{1}{k^\alpha} \leq \frac{1}{1-\alpha} \sum_{k=2}^n \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}} \right] = \frac{1}{\alpha-1} \left[1 - \frac{1}{n^{\alpha-1}} \right] \leq \frac{1}{\alpha-1}.$$

So the series $\sum \frac{1}{n^\alpha}$ converges when $\alpha > 1$. $\square$

Remark 6.2.7 : For a fixed $\alpha > 1$, the Riemann series $\sum \frac{1}{n^\alpha}$ is convergent by Proposition 6.2.6. We may find an asymptotic expression for its remainder. For $k \geq 1$, we have

$$\frac{1}{k^\alpha} \geq \int_k^{k+1} \frac{dt}{t^\alpha} = \frac{1}{1-\alpha} \left[\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{k^{\alpha-1}} \right].$$

This gives that

$$\forall n \geq 1, \quad \sum_{k=n}^{\infty} \frac{1}{k^\alpha} \geq \frac{1}{1-\alpha} \sum_{k=n}^{\infty} \left[\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{k^{\alpha-1}} \right] = \frac{1}{\alpha-1} \frac{1}{n^{\alpha-1}}.$$

Similarly, from Eq. (6.2), we find,

$$\forall n \geq 1, \quad \sum_{k=n}^{\infty} \frac{1}{k^\alpha} \leq \frac{1}{n^\alpha} + \frac{1}{1-\alpha} \sum_{k=n+1}^{\infty} \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}} \right] = \frac{1}{\alpha-1} \frac{1}{n^{\alpha-1}} + \frac{1}{n^\alpha}.$$

The above two relations imply that

$$\sum_{k=n}^{\infty} \frac{1}{k^\alpha} = \frac{1}{\alpha-1} \frac{1}{n^{\alpha-1}} + \mathcal{O}\left(\frac{1}{n^\alpha}\right), \quad \text{when } n \rightarrow \infty. \quad (6.3)$$

6.2.2 Partial sums and remainders

Theorem 6.2.8 : Let $\sum u_n$ and $\sum v_n$ are two series with non-negative terms such that $u_n \sim v_n$. Then, the following properties hold.

(1) If $\sum u_n$ converges, then $\sum v_n$ converges and their remainders satisfy

$$\sum_{k=n}^{\infty} u_k \sim \sum_{k=n}^{\infty} v_k, \quad n \rightarrow \infty. \quad (6.4)$$

(2) If $\sum u_n$ diverges, $\sum v_n$ diverges and their partial sums satisfy

$$\sum_{k=1}^n u_k \sim \sum_{k=1}^n v_k, \quad n \rightarrow \infty. \quad (6.5)$$

因此當 $\alpha > 1$ 時，級數 $\sum \frac{1}{n^\alpha}$ 會收斂。 $\square$

註解 6.2.7 : 對一個固定的 $\alpha > 1$ 來說，根據命題 6.2.6，黎曼級數 $\sum \frac{1}{n^\alpha}$ 會收斂。我們可以對他的餘項找到漸進式。對於 $k \geq 1$ ，我們有

$$\frac{1}{k^\alpha} \geq \int_k^{k+1} \frac{dt}{t^\alpha} = \frac{1}{1-\alpha} \left[\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{k^{\alpha-1}} \right].$$

這給我們

$$\forall n \geq 1, \quad \sum_{k=n}^{\infty} \frac{1}{k^\alpha} \geq \frac{1}{1-\alpha} \sum_{k=n}^{\infty} \left[\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{k^{\alpha-1}} \right] = \frac{1}{\alpha-1} \frac{1}{n^{\alpha-1}}.$$

同理，從式 (6.2)，我們得到

$$\forall n \geq 1, \quad \sum_{k=n}^{\infty} \frac{1}{k^\alpha} \leq \frac{1}{n^\alpha} + \frac{1}{1-\alpha} \sum_{k=n+1}^{\infty} \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}} \right] = \frac{1}{\alpha-1} \frac{1}{n^{\alpha-1}} + \frac{1}{n^\alpha}.$$

上面兩個關係式讓我們得到

$$\sum_{k=n}^{\infty} \frac{1}{k^\alpha} = \frac{1}{\alpha-1} \frac{1}{n^{\alpha-1}} + \mathcal{O}\left(\frac{1}{n^\alpha}\right), \quad \text{當 } n \rightarrow \infty. \quad (6.3)$$

第二小節 部份和與餘項

定理 6.2.8 : 令 $\sum u_n$ 與 $\sum v_n$ 為兩個一般項非負的級數，且滿足 $u_n \sim v_n$ 。那麼下列性質成立。

(1) 如果 $\sum u_n$ 收斂，那麼 $\sum v_n$ 收斂，且他們的餘項滿足

$$\sum_{k=n}^{\infty} u_k \sim \sum_{k=n}^{\infty} v_k, \quad n \rightarrow \infty. \quad (6.4)$$

(2) 如果 $\sum u_n$ 發散，那麼 $\sum v_n$ 發散，且他們的部份和滿足

$$\sum_{k=1}^n u_k \sim \sum_{k=1}^n v_k, \quad n \rightarrow \infty. \quad (6.5)$$

Proof : By Theorem 6.2.3, we already know that $\sum u_n$ and $\sum v_n$ have the same behavior.

(1) Let $\varepsilon > 0$. It follows from the equivalence $u_n \sim v_n$ that we may find $N \geq 1$ such that

$$\forall k \geq N, \quad (1 - \varepsilon)u_k \leq v_k \leq (1 + \varepsilon)u_k. \quad (6.6)$$

By taking a summation over $k \geq n$ with $n \geq N$, we find

$$\forall n \geq N, \quad (1 - \varepsilon) \sum_{k=n}^{\infty} u_k \leq \sum_{k=n}^{\infty} v_k \leq (1 + \varepsilon) \sum_{k=n}^{\infty} u_k.$$

This is exactly what Eq. (6.4) means.

(2) Let $\varepsilon > 0$. As above, we may find $N \geq 1$ such that Eq. (6.6) holds. Then, for any $n \geq N$, we have

$$\sum_{k=1}^{N-1} v_k + (1 - \varepsilon) \sum_{k=N}^n u_k \leq \sum_{k=1}^n v_k \leq \sum_{k=1}^{N-1} v_k + (1 + \varepsilon) \sum_{k=N}^n u_k. \quad (6.7)$$

We use the fact that the general terms v_n are non-negative and the series $\sum u_k$ diverges, we may find $N' \geq N$ such that the two following inequalities hold,

$$\sum_{k=1}^{N-1} v_k \leq \varepsilon \sum_{k=N}^n u_k, \quad \forall n \geq N', \quad (6.8)$$

$$(1 - 2\varepsilon) \sum_{k=1}^{N-1} u_k - \sum_{k=1}^{N-1} v_k \leq \varepsilon \sum_{k=N}^n u_k, \quad \forall n \geq N'. \quad (6.9)$$

If we use Eq. (6.8) to the right side of Eq. (6.7), we find

$$\sum_{k=1}^n v_k \leq (1 + 2\varepsilon) \sum_{k=N}^n u_k \leq (1 + 2\varepsilon) \sum_{k=1}^n u_k,$$

where the last inequality follows because the general terms u_n are non-negative. Similarly, if we use Eq. (6.9) to the left side of Eq. (6.7), we find

$$\sum_{k=1}^n v_k \geq (1 - 2\varepsilon) \sum_{k=1}^{N-1} u_k - \varepsilon \sum_{k=N}^n u_k + (1 - \varepsilon) \sum_{k=N}^n u_k = (1 - 2\varepsilon) \sum_{k=1}^n u_k.$$

This is exactly what we need to show for Eq. (6.5).

□

證明：根據定理 6.2.3，我們已經知道 $\sum u_n$ 和 $\sum v_n$ 有相同的行為。

(1) 令 $\varepsilon > 0$ 。從等價關係 $u_n \sim v_n$ 我們能找到 $N \geq 1$ 使得

$$\forall k \geq N, \quad (1 - \varepsilon)u_k \leq v_k \leq (1 + \varepsilon)u_k. \quad (6.6)$$

對任意 $n \geq N$ ，我們對 $k \geq n$ 取和，進而得到

$$\forall n \geq N, \quad (1 - \varepsilon) \sum_{k=n}^{\infty} u_k \leq \sum_{k=n}^{\infty} v_k \leq (1 + \varepsilon) \sum_{k=n}^{\infty} u_k.$$

這剛好就是式 (6.4) 所代表的意思。

(2) 令 $\varepsilon > 0$ 。同上，我們可以找到 $N \geq 1$ 使得式 (6.6) 成立。接著，對於任意 $n \geq N$ ，我們有

$$\sum_{k=1}^{N-1} v_k + (1 - \varepsilon) \sum_{k=N}^n u_k \leq \sum_{k=1}^n v_k \leq \sum_{k=1}^{N-1} v_k + (1 + \varepsilon) \sum_{k=N}^n u_k. \quad (6.7)$$

由於一般項 v_n 是非負的，而且級數 $\sum u_k$ 發散，我們可以找到 $N' \geq N$ 使得下面兩個不等式成立：

$$\sum_{k=1}^{N-1} v_k \leq \varepsilon \sum_{k=N}^n u_k, \quad \forall n \geq N', \quad (6.8)$$

$$(1 - 2\varepsilon) \sum_{k=1}^{N-1} u_k - \sum_{k=1}^{N-1} v_k \leq \varepsilon \sum_{k=N}^n u_k, \quad \forall n \geq N'. \quad (6.9)$$

在式 (6.7) 的右方，我們使用式 (6.8)，進而得到

$$\sum_{k=1}^n v_k \leq (1 + 2\varepsilon) \sum_{k=N}^n u_k \leq (1 + 2\varepsilon) \sum_{k=1}^n u_k,$$

其中最後一個不等式會成立，是因為一般項 u_n 是非負的。同理，在式 (6.7) 的左方我們使用式 (6.9)，會得到

$$\sum_{k=1}^n v_k \geq (1 - 2\varepsilon) \sum_{k=1}^{N-1} u_k - \varepsilon \sum_{k=N}^n u_k + (1 - \varepsilon) \sum_{k=N}^n u_k = (1 - 2\varepsilon) \sum_{k=1}^n u_k.$$

這會是檢查式 (6.5) 所需要證明的。

□

The above result has useful applications when it comes to asymptotic expansion of sequences or series. We are going to look at an important example below.

Example 6.2.9 : The sequence $(H_n)_{n \geq 1}$ of harmonic numbers is defined by

$$\forall n \geq 1, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(1) We first note that when $n \rightarrow \infty$, we have the following equivalence,

$$\frac{1}{n} \sim \ln\left(1 + \frac{1}{n}\right).$$

Since both series $\sum \frac{1}{n}$ and $\sum \ln(1 + \frac{1}{n})$ are with non-negative terms, it follows from Theorem 6.2.3 (2) that they are of the same behavior. It is not hard to see that

$$\sum_{k=1}^n \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^n [\ln(k+1) - \ln(k)] = \ln(n+1)$$

diverges when $n \rightarrow \infty$, so we deduce that $\sum \frac{1}{n}$ also diverges. Moreover, it follows from Theorem 6.2.8 (2) that their partial sums are equivalent. In other words, for $n \rightarrow \infty$, we have

$$H_n \sim \sum_{k=1}^n \ln\left(1 + \frac{1}{k}\right) = \ln(n+1) \sim \ln n.$$

This gives the first term in the asymptotic expansion of the harmonic numbers.

(2) To get the following terms in the asymptotic expansion of $(H_n)_{n \geq 1}$, let us consider the sequence $(A_n)_{n \geq 1}$ defined by $A_n = H_n - \ln n$ for $n \geq 1$. Then, for $n \geq 2$, we may write

$$A_n - A_{n-1} = \frac{1}{n} - \ln n + \ln(n-1) = \frac{1}{n} + \ln\left(1 - \frac{1}{n}\right) \sim -\frac{1}{2n^2}. \quad (6.10)$$

By the result on the Riemann series in Proposition 6.2.6, we know that the series $\sum \frac{1}{n^2}$ converges, and again by Theorem 6.2.3 (2), we know that the series $\sum (A_n - A_{n-1})$ converges. Additionally, since

$$\forall n \geq 2, \quad \sum_{k=2}^n (A_k - A_{k-1}) = A_n - A_1,$$

we deduce that the sequence $(A_n)_{n \geq 1}$ converges. Let us define $\gamma := \lim_{n \rightarrow \infty} A_n$, called *Euler's*

當我們要找出序列或級數的漸進表示式時，我們可以應用上面的結果。下面我們給一個重要的例子。

範例 6.2.9 : 由調和數所構成的序列 $(H_n)_{n \geq 1}$ 定義如下：

$$\forall n \geq 1, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(1) 我們首先注意到，當 $n \rightarrow \infty$ 時，我們有下列等價關係：

$$\frac{1}{n} \sim \ln\left(1 + \frac{1}{n}\right).$$

由於級數 $\sum \frac{1}{n}$ 和 $\sum \ln(1 + \frac{1}{n})$ 當中的項都是非負的，從定理 6.2.3 (2) 我們知道他們有相同的行為。接著，我們不難看出

$$\sum_{k=1}^n \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^n [\ln(k+1) - \ln(k)] = \ln(n+1)$$

會在 $n \rightarrow \infty$ 時發散，因此我們得知 $\sum \frac{1}{n}$ 也會發散。此外，從定理 6.2.8 (2) 我們知道，他們的部份和會是等價的。換句話說，當 $n \rightarrow \infty$ 時，我們有

$$H_n \sim \sum_{k=1}^n \ln\left(1 + \frac{1}{k}\right) = \ln(n+1) \sim \ln n.$$

這給我們調和數漸進展開中的第一項。

(2) 要找到 $(H_n)_{n \geq 1}$ 漸進展開後面的項，我們考慮序列 $(A_n)_{n \geq 1}$ 定義做 $A_n = H_n - \ln n$ 對於 $n \geq 1$ 。那麼對於 $n \geq 2$ ，我們可以寫

$$A_n - A_{n-1} = \frac{1}{n} - \ln n + \ln(n-1) = \frac{1}{n} + \ln\left(1 - \frac{1}{n}\right) \sim -\frac{1}{2n^2}. \quad (6.10)$$

根據命題 6.2.6 中黎曼級數的結果，我們知道級數 $\sum \frac{1}{n^2}$ 收斂，接著使用定理 6.2.3 (2)，我們知道級數 $\sum (A_n - A_{n-1})$ 收斂。此外，由於

$$\forall n \geq 2, \quad \sum_{k=2}^n (A_k - A_{k-1}) = A_n - A_1,$$

constant, and we have

$$H_n = \ln n + A_n = \ln n + \gamma + o(1), \quad \text{when } n \rightarrow \infty.$$

- (3) Following the computations in (2), due to the equivalence in Eq. (6.10) and Theorem 6.2.8 (1), we know that the following equivalence holds,

$$\gamma - A_n = \sum_{k=n+1}^{\infty} (A_k - A_{k-1}) \sim -\frac{1}{2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \sim -\frac{1}{2n}.$$

where the last equivalence comes from Example 6.2.5. This gives the asymptotic expansion of H_n below

$$H_n = \ln n + \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right), \quad \text{when } n \rightarrow \infty.$$

- (4) We may go further in the above asymptotic expansion. Let us consider the sequence $(D_n)_{n \geq 1}$ defined by

$$D_n = H_n - \ln n - \gamma - \frac{1}{2n} = A_n - \gamma - \frac{1}{2n}, \quad \forall n \geq 1.$$

Then, when $n \rightarrow \infty$, we have

$$\begin{aligned} D_n - D_{n-1} &= \frac{1}{n} + \ln\left(1 - \frac{1}{n}\right) + \frac{1}{2(n-1)} - \frac{1}{2n} \\ &= \frac{1}{n} - \left(\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + o\left(\frac{1}{n^3}\right)\right) + \frac{1}{2n}\left(1 + \frac{1}{n} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right) - \frac{1}{2n} \\ &= \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right). \end{aligned}$$

Since the Riemann series $\sum \frac{1}{n^3}$ converges, we know that the series $\sum (D_n - D_{n-1})$ also converges, and when $n \rightarrow \infty$, we have

$$\sum_{k=n+1}^{\infty} (D_k - D_{k-1}) \sim \frac{1}{6} \sum_{k=n+1}^{\infty} \frac{1}{k^3}.$$

By applying Eq. (6.3), we deduce that $\sum_{k=n+1}^{\infty} \frac{1}{k^3} \sim \frac{1}{2n^2}$, and the left-hand side is equal to $\lim_{k \rightarrow \infty} D_k - D_n = -D_n$. This allows us to obtain that $D_n \sim -\frac{1}{12} \frac{1}{n^2}$ when $n \rightarrow \infty$. The asymptotic expansion of the harmonic numbers writes

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12} \frac{1}{n^2} + o\left(\frac{1}{n^2}\right), \quad \text{when } n \rightarrow \infty.$$

我們推得序列 $(A_n)_{n \geq 1}$ 收斂。我們定義 $\gamma := \lim_{n \rightarrow \infty} A_n$ ，稱作歐拉常數，這讓我們可以寫

$$H_n = \ln n + A_n = \ln n + \gamma + o(1), \quad \text{當 } n \rightarrow \infty.$$

- (3) 接續 (2) 中的計算，由於我們有式 (6.10) 中的等價關係，透過定理 6.2.8 (1) 我們知道下面等價關係成立：

$$\gamma - A_n = \sum_{k=n+1}^{\infty} (A_k - A_{k-1}) \sim -\frac{1}{2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \sim -\frac{1}{2n}.$$

其中最後一個等價關係式來自範例 6.2.5。這給我們下面關於 H_n 的漸進展開：

$$H_n = \ln n + \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right), \quad \text{當 } n \rightarrow \infty.$$

- (4) 我們可以把上面的漸進展開式繼續寫下去。我們考慮序列 $(D_n)_{n \geq 1}$ 定義做

$$D_n = H_n - \ln n - \gamma - \frac{1}{2n} = A_n - \gamma - \frac{1}{2n}, \quad \forall n \geq 1.$$

那麼，當 $n \rightarrow \infty$ 時，我們有

$$\begin{aligned} D_n - D_{n-1} &= \frac{1}{n} + \ln\left(1 - \frac{1}{n}\right) + \frac{1}{2(n-1)} - \frac{1}{2n} \\ &= \frac{1}{n} - \left(\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + o\left(\frac{1}{n^3}\right)\right) + \frac{1}{2n}\left(1 + \frac{1}{n} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right) - \frac{1}{2n} \\ &= \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right). \end{aligned}$$

由於黎曼級數 $\sum \frac{1}{n^3}$ 收斂，我們知道級數 $\sum (D_n - D_{n-1})$ 也會收斂，而且當 $n \rightarrow \infty$ 時，我們有

$$\sum_{k=n+1}^{\infty} (D_k - D_{k-1}) \sim \frac{1}{6} \sum_{k=n+1}^{\infty} \frac{1}{k^3}.$$

如果我們使用式 (6.3)，我們推得 $\sum_{k=n+1}^{\infty} \frac{1}{k^3} \sim \frac{1}{2n^2}$ ，而左方會等於 $\lim_{k \rightarrow \infty} D_k - D_n = -D_n$ 。把他們放在一起，我們得到 $D_n \sim -\frac{1}{12} \frac{1}{n^2}$ 當 $n \rightarrow \infty$ 。調和數的漸進展開式可以寫

(5) You may repeat the above procedure to find the following asymptotic expansion

$$H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k \geq 1}^N \frac{B_{2k}}{2k n^{2k}} + o\left(\frac{1}{n^{2N}}\right), \quad \text{when } n \rightarrow \infty,$$

where $(B_{2k})_{k \geq 1}$ are Bernoulli numbers whose first terms are given by $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, etc.

Remark 6.2.10 : In Exercise 6.12, you may apply the same method as in Example 6.2.9 to deduce an asymptotic formula for $n!$, called *Stirling's formula*.

6.2.3 Comparison between series and integrals

Integrals and series are closely related: a series is a discrete summation, whereas an integral is the limit of such discrete summations. When a series converges, it is usually not trivial to get an exact formula or value for its limit; however, there are a lot of functions who have a nice primitive that we can compute (Appendix 1). In this subsection, we introduce a method allowing us to study the behavior of a series with the help of integrals. The idea behind is similar to what we have done in the proof of Proposition 6.2.6, but here we give a more general setting and a more precise result for such a method.

Proposition 6.2.11 : Let $f : [1, +\infty) \rightarrow \mathbb{R}_+$ be a non-increasing function with $\lim_{x \rightarrow \infty} f(x) = 0$. For every integer $n \geq 1$, let us define

$$S_n = \sum_{k=1}^n f(k), \quad I_n = \int_1^n f(t) dt, \quad D_n = S_n - I_n.$$

Then, the following properties hold.

- (1) For $n \geq 1$, we have $0 \leq f(n+1) \leq D_{n+1} \leq D_n \leq f(1)$.
- (2) The sequence $(D_n)_{n \geq 1}$ converges, and denote $D := \lim_{n \rightarrow \infty} D_n$.
- (3) The series $\sum f(n)$ and the integral $\int_1^\infty f(t) dt := \lim_{x \rightarrow \infty} \int_1^x f(t) dt$ have the same behavior, that

做

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + o\left(\frac{1}{n^2}\right), \quad \text{當 } n \rightarrow \infty.$$

(5) 你可以重複上面的步驟，進而推得下列漸進展開式：

$$H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k \geq 1}^N \frac{B_{2k}}{2k n^{2k}} + o\left(\frac{1}{n^{2N}}\right), \quad \text{當 } n \rightarrow \infty,$$

其中 $(B_{2k})_{k \geq 1}$ 是 Bernoulli 數，他的最前面幾項是 $B_2 = \frac{1}{6}$ 、 $B_4 = -\frac{1}{30}$ 、 $B_6 = \frac{1}{42}$ 等等。

註解 6.2.10 : 在習題 6.12 中，你可以使用與範例 6.2.9 中相同的方法，來推得 $n!$ 的漸進展開式，稱作 Stirling 公式。

第三小節 級數與積分的比較

積分和級數有密切的關係：級數是個離散和，積分則是離散和的極限。當級數收斂時，一般來講要得到他極限的值不是容易的事情；然而，很多函數有好的原函數是可以計算的（附錄一）。在這個小節中，我們會介紹一個方法，讓我們可以藉由積分來討論級數的行為。背後的想法與我們在命題 6.2.6 當中類似，但這裡我們會假設的是更一般的情況，並且會給出更精確的結果。

命題 6.2.11 : 令 $f : [1, +\infty) \rightarrow \mathbb{R}_+$ 為非遞增函數，滿足 $\lim_{x \rightarrow \infty} f(x) = 0$ 。對於每個整數 $n \geq 1$ ，我們定義

$$S_n = \sum_{k=1}^n f(k), \quad I_n = \int_1^n f(t) dt, \quad D_n = S_n - I_n.$$

那麼，下列性質成立。

- (1) 對於 $n \geq 1$ ，我們有 $0 \leq f(n+1) \leq D_{n+1} \leq D_n \leq f(1)$ 。
- (2) 序列 $(D_n)_{n \geq 1}$ 會收斂，我們記 $D := \lim_{n \rightarrow \infty} D_n$ 。

is both are either convergent or divergent.

(4) For $n \geq 1$, we have $0 \leq D_n - D \leq f(n)$.

Proof :

(1) Since f is non-increasing, we find

$$\forall k \geq 0, \quad f(k+1) \leq \int_k^{k+1} f(t) dt \leq f(k).$$

Let us fix $n \geq 1$. We have

$$I_{n+1} = \int_1^{n+1} f(t) dt = \sum_{k=1}^n \int_k^{k+1} f(t) dt \leq \sum_{k=1}^n f(k) = S_n$$

Therefore, $f(n+1) = S_{n+1} - S_n \leq S_{n+1} - I_{n+1} = D_{n+1}$, which shows the first part of the inequality.

Next, we write

$$D_n - D_{n+1} = \int_n^{n+1} f(t) dt - f(n+1) \geq 0,$$

This shows that $D_{n+1} \leq D_n$. To conclude, we note that $D_1 = f(1)$, and by induction, we find $D_n \leq D_1 = f(1)$ for all $n \geq 1$.

(2) From (1), we know that $(D_n)_{n \geq 1}$ is a non-negative sequence bounded from below by 0, so it converges.

(3) From (2), we know that $\lim_{n \rightarrow \infty} (S_n - I_n)$ exists. Therefore, both sequences $(S_n)_{n \geq 1}$ and $(I_n)_{n \geq 1}$ have the same behavior.

(4) For $n \geq 1$, we write the telescoping summation

$$D_n - D = \lim_{N \rightarrow \infty} (D_n - D_N) = \lim_{N \rightarrow \infty} \sum_{k=n}^{N-1} (D_k - D_{k+1}).$$

From (1), we know that $D_k - D_{k+1} \geq 0$ for all $k \geq 1$, so $D_n - D \geq 0$. For every $k \geq 1$, we also have

$$D_k - D_{k+1} = \int_k^{k+1} f(t) dt - f(k+1) \leq f(k) - f(k+1),$$

(3) 級數 $\sum f(n)$ 和積分 $\int_1^\infty f(t) dt := \lim_{x \rightarrow \infty} \int_1^x f(t) dt$ 有相同的行為，也就是兩者皆收斂或兩者皆發散。

(4) 對於 $n \geq 1$ ，我們有 $0 \leq D_n - D \leq f(n)$ 。

證明：

(1) 由於 f 是非遞增的，我們有

$$\forall k \geq 0, \quad f(k+1) \leq \int_k^{k+1} f(t) dt \leq f(k).$$

讓我們固定 $n \geq 1$ 。我們有

$$I_{n+1} = \int_1^{n+1} f(t) dt = \sum_{k=1}^n \int_k^{k+1} f(t) dt \leq \sum_{k=1}^n f(k) = S_n$$

因此， $f(n+1) = S_{n+1} - S_n \leq S_{n+1} - I_{n+1} = D_{n+1}$ ，這證明了不等式的第一部份。

接著，我們寫

$$D_n - D_{n+1} = \int_n^{n+1} f(t) dt - f(n+1) \geq 0,$$

這證明了 $D_{n+1} \leq D_n$ 。最後，我們注意到 $D_1 = f(1)$ ，再透過數學歸納法，對於所有 $n \geq 1$ ，我們得到 $D_n \leq D_1 = f(1)$ 。

(2) 從 (1)，我們知道 $(D_n)_{n \geq 1}$ 是個非負序列，且有 0 為下界，所以會收斂。

(3) 從 (2)，我們知道 $\lim_{n \rightarrow \infty} (S_n - I_n)$ 存在。因此，序列 $(S_n)_{n \geq 1}$ 和 $(I_n)_{n \geq 1}$ 有相同的行為。

(4) 對於 $n \geq 1$ ，我們寫下裂項和：

$$D_n - D = \lim_{N \rightarrow \infty} (D_n - D_N) = \lim_{N \rightarrow \infty} \sum_{k=n}^{N-1} (D_k - D_{k+1}).$$

從 (1)，我們知道對於所有 $k \geq 1$ ，我們有 $D_k - D_{k+1} \geq 0$ ，因此 $D_n - D \geq 0$ 。對於每個

so

$$D_n - D \leq \lim_{N \rightarrow \infty} \sum_{k=n}^{N-1} (f(k) - f(k+1)) = \lim_{N \rightarrow \infty} (f(n) - f(N)) = f(n).$$

□

Remark 6.2.12 :

- (1) In Proposition 6.2.11, if f is non-increasing on $[M, +\infty)$ for some $M > 0$, then the qualitative statements such as (2) and (3) still hold, whereas the bounds in (1) and (4) need to be adjusted.
- (2) From Proposition 6.2.11 (4), we also know that

$$0 \leq \sum_{k=1}^n f(k) - \int_1^n f(t) dt - D \leq f(n).$$

In other words, we have an asymptotic expansion

$$\sum_{k=1}^n f(k) = \int_1^n f(t) dt + D + \mathcal{O}(f(n)), \quad \text{when } n \rightarrow \infty. \quad (6.11)$$

If we apply this to the function $f(x) = \frac{1}{x}$, then we find

$$H_n = \sum_{k=1}^n \frac{1}{k} = \ln n + D + \mathcal{O}\left(\frac{1}{n}\right), \quad \text{when } n \rightarrow \infty,$$

which is similar to the result obtained in Example 6.2.9, stronger than (2), but weaker than (3).

Example 6.2.13 : Take $s \in \mathbb{R}$ and $f(x) = x^{-s}$ in Proposition 6.2.11. Along with Proposition 6.2.6, we know that $\sum n^{-s}$ converges if $s > 1$ and diverges if $s \leq 1$. For $s > 1$, this series is called the *Riemann zeta function*,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

$k \geq 1$, 我們也會有

$$D_k - D_{k+1} = \int_k^{k+1} f(t) dt - f(k+1) \leq f(k) - f(k+1),$$

所以

$$D_n - D \leq \lim_{N \rightarrow \infty} \sum_{k=n}^{N-1} (f(k) - f(k+1)) = \lim_{N \rightarrow \infty} (f(n) - f(N)) = f(n).$$

□

註解 6.2.12 :

- (1) 在命題 6.2.11 中，如果對於某個 $M > 0$ ， f 在 $[M, +\infty)$ 上是非遞增的，那麼定性的敘述像是 (2) 和 (3) 仍然成立，但 (1) 和 (4) 當中的界需要調整。
- (2) 從命題 6.2.11 (4)，我們也可以得知

$$0 \leq \sum_{k=1}^n f(k) - \int_1^n f(t) dt - D \leq f(n).$$

換句話說，我們會有漸進展開

$$\sum_{k=1}^n f(k) = \int_1^n f(t) dt + D + \mathcal{O}(f(n)), \quad \text{當 } n \rightarrow \infty. \quad (6.11)$$

如果我們把這用在函數 $f(x) = \frac{1}{x}$ ，那麼我們得到

$$H_n = \sum_{k=1}^n \frac{1}{k} = \ln n + D + \mathcal{O}\left(\frac{1}{n}\right), \quad \text{當 } n \rightarrow \infty,$$

這會與範例 6.2.9 中的結果相似，比 (2) 強，但比 (3) 弱。

範例 6.2.13 : 在命題 6.2.11 中，我們取 $s \in \mathbb{R}$ 以及 $f(x) = x^{-s}$ 。使用命題 6.2.6，我們知道如果 $s > 1$ ，那麼 $\sum n^{-s}$ 會收斂；如果 $s \leq 1$ ，那麼他則會發散。對於 $s > 1$ ，這樣的級數稱作

For $s \in (0, 1)$, Eq. (6.11) gives us

$$\sum_{k=1}^n \frac{1}{k^s} = \frac{n^{1-s} - 1}{1-s} + C(s) + \mathcal{O}\left(\frac{1}{n^s}\right), \quad \text{when } n \rightarrow \infty,$$

where $C(s)$ is a constant depending on s . The constant term in the above summation is equal to $\zeta(s)$, so the remainder of the series writes,

$$\sum_{k=n+1}^{\infty} \frac{1}{k^s} = \frac{n^{1-s}}{s-1} + \mathcal{O}\left(\frac{1}{n^s}\right) \quad \text{when } n \rightarrow \infty,$$

which is the result we found in Eq. (6.2).

Proposition 6.2.14 (Bertrand's series) : For $\alpha, \beta \in \mathbb{R}$, the corresponding Bertrand series is given by

$$\sum_{n \geq 2} \frac{1}{n^\alpha (\ln n)^\beta}.$$

- (1) When $\alpha > 1$, the Bertrand series converges.
- (2) When $\alpha = 1$ and $\beta > 1$, the Bertrand series converges.
- (3) Otherwise, the Bertrand series diverges.

Proof : We are going to apply the comparison test (Proposition 6.2.2) and Proposition 6.2.11 to show these properties.

- (1) Let $\alpha > 1$. Note that we have the comparison

$$\frac{1}{n^\alpha (\ln n)^\beta} = o\left(\frac{1}{n^{(1+\alpha)/2}}\right) \quad \text{when } n \rightarrow \infty.$$

We know that the Riemann series $\sum \frac{1}{n^{(1+\alpha)/2}}$ converges, because $\frac{1+\alpha}{2} > 1$. By comparison (Proposition 6.2.2), we deduce that the series $\sum \frac{1}{n^\alpha (\ln n)^\beta}$ is convergent.

- (2) Let $\alpha = 1$ and $\beta > 1$, and let us apply Proposition 6.2.11 to the non-increasing function $f(x) =$

黎曼 ζ 函數 :

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

對於 $s \in (0, 1)$ ，式 (6.11) 給我們

$$\sum_{k=1}^n \frac{1}{k^s} = \frac{n^{1-s} - 1}{1-s} + C(s) + \mathcal{O}\left(\frac{1}{n^s}\right), \quad \text{當 } n \rightarrow \infty,$$

其中 $C(s)$ 是個取決於 s 的常數。上式中的常數項會等於 $\zeta(s)$ ，因此級數的餘項寫做

$$\sum_{k=n+1}^{\infty} \frac{1}{k^s} = \frac{n^{1-s}}{s-1} + \mathcal{O}\left(\frac{1}{n^s}\right) \quad \text{當 } n \rightarrow \infty,$$

這也是式 (6.2) 所得到的結果。

命題 6.2.14 【Bertrand 級數】 : 對於 $\alpha, \beta \in \mathbb{R}$ ，他所對應到的 Bertrand 級數寫做

$$\sum_{n \geq 2} \frac{1}{n^\alpha (\ln n)^\beta}.$$

- (1) 當 $\alpha > 1$ ，此 Bertrand 級數收斂。
- (2) 當 $\alpha = 1$ 且 $\beta > 1$ ，此 Bertrand 級數收斂。
- (3) 其他狀況下，此 Bertrand 級數發散。

證明 : 我們會使用比較法則 (命題 6.2.2) 以及命題 6.2.11 來證明這些性質。

- (1) 令 $\alpha > 1$ 。注意到我們有下面的比較 :

$$\frac{1}{n^\alpha (\ln n)^\beta} = o\left(\frac{1}{n^{(1+\alpha)/2}}\right) \quad \text{當 } n \rightarrow \infty.$$

因為 $\frac{1+\alpha}{2} > 1$ ，我們知道黎曼級數 $\sum \frac{1}{n^{(1+\alpha)/2}}$ 會收斂。藉由比較法則 (命題 6.2.2)，我們推得級數 $\sum \frac{1}{n^\alpha (\ln n)^\beta}$ 是會收斂的。

- (2) 令 $\alpha = 1$ 以及 $\beta > 1$ ，並把命題 6.2.11 用在非遞增函數 $f(x) = \frac{1}{x(\ln x)^\beta}$ 上。 f 的積分寫起來

$\frac{1}{x(\ln x)^\beta}$. The integral of f writes as below,

$$\int_2^n f(t) dt = \int_2^n \frac{1}{t(\ln t)^\beta} dt = \int_{\ln 2}^{\ln n} \frac{1}{s^\beta} ds = \frac{1}{1-\beta} \left[\frac{1}{(\ln n)^{\beta-1}} - \frac{1}{(\ln 2)^{\beta-1}} \right].$$

The right hand side of the above integral converges to some finite limit, so the series $\sum \frac{1}{n(\ln n)^\beta}$ is convergent.

- (3) Let us first deal with the case $\alpha = \beta = 1$. We apply Proposition 6.2.11 to the non-increasing function $f(x) = \frac{1}{x \ln x}$. The integral of f writes as below,

$$\int_2^n f(t) dt = \int_2^n \frac{1}{t \ln t} dt = \int_{\ln 2}^{\ln n} \frac{1}{s} ds = \ln \ln n - \ln \ln 2.$$

The right hand side of the above integral diverges, so the series $\sum \frac{1}{n \ln n}$ is divergent.

When $\alpha = 1$ and $\beta < 1$, we note that we have

$$\frac{1}{n \ln n} \leq \frac{1}{n(\ln n)^\beta}.$$

We conclude by comparison (Proposition 6.2.2) that the series $\sum \frac{1}{n(\ln n)^\beta}$ diverges.

When $\alpha < 1$, we have the relation

$$\frac{1}{n^{(1+\alpha)/2}} = o\left(\frac{1}{n^\alpha (\ln n)^\beta}\right) \quad \text{when } n \rightarrow \infty,$$

and the result follows from the divergence of the Riemann series $\sum \frac{1}{n^{(1+\alpha)/2}}$ with $\frac{1+\alpha}{2} < 1$. $\square$

6.3 Tests of convergence

Theorem 6.3.1 (D'Alembert's criterion, ratio test): Let $(u_n)_{n \geq 1}$ be a sequence of real numbers. Suppose that it is strictly positive from a certain index. Additionally, assume that the following limit exists

$$\ell := \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} \in [0, +\infty].$$

Then, the following statements hold.

- (1) If $\ell < 1$, then the series $\sum u_n$ is convergent.

如下：

$$\int_2^n f(t) dt = \int_2^n \frac{1}{t(\ln t)^\beta} dt = \int_{\ln 2}^{\ln n} \frac{1}{s^\beta} ds = \frac{1}{1-\beta} \left[\frac{1}{(\ln n)^{\beta-1}} - \frac{1}{(\ln 2)^{\beta-1}} \right].$$

上面積分的右方收斂到某個有限極限，因此級數 $\sum \frac{1}{n(\ln n)^\beta}$ 會收斂。

- (3) 讓我們先處理 $\alpha = \beta = 1$ 的情況。我們把命題 6.2.11 用在非遞增函數 $f(x) = \frac{1}{x \ln x}$ 上。 f 的積分寫起來如下：

$$\int_2^n f(t) dt = \int_2^n \frac{1}{t \ln t} dt = \int_{\ln 2}^{\ln n} \frac{1}{s} ds = \ln \ln n - \ln \ln 2.$$

上面積分的右方發散，因此級數 $\sum \frac{1}{n \ln n}$ 會發散。

當 $\alpha = 1$ 而且 $\beta < 1$ 時，我們會有

$$\frac{1}{n \ln n} \leq \frac{1}{n(\ln n)^\beta}.$$

藉由比較法則（命題 6.2.2），我們知道級數 $\sum \frac{1}{n(\ln n)^\beta}$ 發散。

當 $\alpha < 1$ ，我們有下列關係式：

$$\frac{1}{n^{(1+\alpha)/2}} = o\left(\frac{1}{n^\alpha (\ln n)^\beta}\right) \quad \text{當 } n \rightarrow \infty,$$

我們就可以藉由黎曼級數 $\sum \frac{1}{n^{(1+\alpha)/2}}$ 由於 $\frac{1+\alpha}{2} < 1$ 而發散的結果，推得我們所要的。 $\square$

第三節 收斂檢測法

定理 6.3.1 【D'Alembert 法則，商檢測法】：令 $(u_n)_{n \geq 1}$ 為實數序列。假設從某項開始，他會是嚴格為正的。此外，假設下列極限存在：

$$\ell := \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} \in [0, +\infty].$$

那麼，下面敘述成立。

(2) If $\ell > 1$, then $u_n \xrightarrow{n \rightarrow \infty} +\infty$ and the series $\sum u_n$ is divergent.

(3) If $\ell = 1$ and the ratio $\frac{u_{n+1}}{u_n}$ stays above 1 for all large enough n , then the series $\sum u_n$ is divergent.

Remark 6.3.2 : When $\ell = 1$, even if the ratio $\frac{u_{n+1}}{u_n}$ always stays below 1 for all n , d'Alembert's criterion does not allow us to conclude. For example, we may consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. In both cases, the limit is $\ell = 1$, and the ratio $\frac{u_{n+1}}{u_n}$ is smaller than 1 for all $n \geq 1$. However, the former series is divergent, and the latter series is convergent.

Proof :

(1) Suppose that $\ell < 1$. Let $N \geq 1$ such that for $n \geq N$, we have $u_n > 0$ and $\frac{u_{n+1}}{u_n} \leq \frac{1+\ell}{2} =: r < 1$. Then, for $n \geq N$, we find $u_n \leq r^{n-N} u_N$, and

$$\sum_{k=1}^n u_k = \sum_{k=1}^{N-1} u_k + \sum_{k=N}^n u_k \leq \sum_{k=1}^{N-1} u_k + \sum_{k=N}^n r^{k-N} u_N.$$

In the above summation, the first term is a constant, and the second term is a geometric series with ratio $r < 1$, so converges.

(2) Suppose that $\ell > 1$. We proceed in a similar way as above, with inequalities reversed. Let $N \geq 1$ such that for $n \geq N$, we have $u_n > 0$ and $\frac{u_{n+1}}{u_n} \geq \frac{1+\ell}{2} =: r > 1$. Then, for $n \geq N$, we find $u_n \geq r^{n-N} u_N$, and

$$\sum_{k=1}^n u_k = \sum_{k=1}^{N-1} u_k + \sum_{k=N}^n u_k \geq \sum_{k=1}^{N-1} u_k + \sum_{k=N}^n r^{k-N} u_N.$$

In the above summation, the first term is a constant, and the second term is a geometric series with ratio $r > 1$, so diverges.

(3) Suppose that $\ell = 1$ and let $N \geq 1$ such that $u_n > 0$ and the ratio $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq N$. Then, for all $n \geq N$, we have $u_n \geq u_N > 0$. Clearly, this implies that the series $\sum u_n$ is divergent. $\square$

(1) 如果 $\ell < 1$ ，那麼級數 $\sum u_n$ 會收斂。

(2) 如果 $\ell > 1$ ，那麼 $u_n \xrightarrow{n \rightarrow \infty} +\infty$ 而且級數 $\sum u_n$ 會發散。

(3) 如果 $\ell = 1$ 而且對於所有夠大的 n 來說，商 $\frac{u_{n+1}}{u_n}$ 會保持在 1 之上，那麼級數 $\sum u_n$ 會發散。

註解 6.3.2 : 當 $\ell = 1$ ，即使對於所有 n ，商 $\frac{u_{n+1}}{u_n}$ 永遠維持在 1 以下，D'Alembert 法則無法讓我們得到結論。例如，我們可以考慮級數 $\sum \frac{1}{n}$ 和 $\sum \frac{1}{n^2}$ 。在兩種情況下，極限都是 $\ell = 1$ ，而且對於所有 $n \geq 1$ ，商 $\frac{u_{n+1}}{u_n}$ 會比 1 來得小。然而，前者級數發散，而後者級數收斂。

證明 :

(1) 假設 $\ell < 1$ 。令 $N \geq 1$ 使得對於 $n \geq N$ ，我們會有 $u_n > 0$ 以及 $\frac{u_{n+1}}{u_n} \leq \frac{1+\ell}{2} =: r < 1$ 。那麼，對於 $n \geq N$ ，我們得到 $u_n \leq r^{n-N} u_N$ ，而且

$$\sum_{k=1}^n u_k = \sum_{k=1}^{N-1} u_k + \sum_{k=N}^n u_k \leq \sum_{k=1}^{N-1} u_k + \sum_{k=N}^n r^{k-N} u_N.$$

在上面的和裡面，第一項是個常數，第二項是公比為 $r < 1$ 的幾何級數，所以會收斂。

(2) 假設 $\ell > 1$ 。我們使用相同的方法，差別在於得到的不等式方向不同。令 $N \geq 1$ 使得對於 $n \geq N$ ，我們會有 $u_n > 0$ 以及 $\frac{u_{n+1}}{u_n} \geq \frac{1+\ell}{2} =: r > 1$ 。那麼，對於 $n \geq N$ ，我們得到 $u_n \geq r^{n-N} u_N$ ，而且

$$\sum_{k=1}^n u_k = \sum_{k=1}^{N-1} u_k + \sum_{k=N}^n u_k \geq \sum_{k=1}^{N-1} u_k + \sum_{k=N}^n r^{k-N} u_N.$$

在上面的和裡面，第一項是個常數，第二項是公比為 $r > 1$ 的幾何級數，所以會發散。

(3) 假設 $\ell = 1$ 並令 $N \geq 1$ 使得對於所有 $n \geq N$ ，我們有 $u_n > 0$ 而且商滿足 $\frac{u_{n+1}}{u_n} \geq 1$ 。那麼，對於所有 $n \geq N$ ，我們會有 $u_n \geq u_N > 0$ 。顯然，這蘊含級數 $\sum u_n$ 會發散。 $\square$

Example 6.3.3 : For a given $z \in \mathbb{C}^*$, let us look at the series $\sum \frac{z^n}{n!}$. The ratio of two consecutive terms writes

$$\frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|}{n+1} \xrightarrow{n \rightarrow \infty} 0 < 1.$$

By the d'Alembert's criterion, the series $\sum \frac{z^n}{n!}$ converges absolutely, so it also converges. Therefore, this series converges for all $z \in \mathbb{C}$. This is a direct consequence of Theorem 6.1.16 if we see $\mathbb{C}$ as a two-dimensional vector space over $\mathbb{R}$.

A simple generalization of Theorem 6.3.1 is stated in the following corollary. To get the absolute convergence of a complex-valued series, we may look at the $\liminf$ and $\limsup$ of the ratio.

Corollary 6.3.4 : Let $\sum u_n$ be a series with nonzero terms in a Banach space $(W, \|\cdot\|)$. Let

$$r = \liminf_{n \rightarrow \infty} \frac{\|u_{n+1}\|}{\|u_n\|} \quad \text{and} \quad R = \limsup_{n \rightarrow \infty} \frac{\|u_{n+1}\|}{\|u_n\|}.$$

- (1) If $R < 1$, then the series $\sum u_n$ converges absolutely.
- (2) If $r > 1$, then the series $\sum u_n$ diverges.
- (3) If $r \leq 1 \leq R$, then we cannot conclude.

Remark 6.3.5 : Do not forget that this corollary is useful especially when $(W, \|\cdot\|) = (\mathbb{C}, |\cdot|)$.

Proof : The proof is very similar to that of Theorem 6.3.1. Let us prove it for (1) as an example.

Let $\sum u_n$ be a series with nonzero terms in $(W, \|\cdot\|)$ such that $R < 1$. By the definition of $\limsup$, there exists $N \geq 1$ such that

$$\forall n \geq N, \quad \frac{\|u_{n+1}\|}{\|u_n\|} \leq \frac{1+R}{2} =: x < 1.$$

Then, we may follow the same argument to conclude. $\square$

範例 6.3.3 : 對於給定的 $z \in \mathbb{C}^*$, 讓我們來看級數 $\sum \frac{z^n}{n!}$ 。相鄰兩項的商滿足

$$\frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|}{n+1} \xrightarrow{n \rightarrow \infty} 0 < 1.$$

D'Alembert's 法則告訴我們，級數 $\sum \frac{z^n}{n!}$ 會絕對收斂，所以會收斂。因此，這個級數會對於所有 $z \in \mathbb{C}$ 收斂。當我們把 $\mathbb{C}$ 看作是 $\mathbb{R}$ 上面的二維空間時，這個是定理 6.1.16 的直接結果。

下面的系理是定理 6.3.1 的一個簡單推廣。如果要得到複數級數的絕對收斂，我們只需要去看商的 $\liminf$ 和 $\limsup$ 即可。

系理 6.3.4 : 令 $\sum u_n$ 為一般項非零，取值在 Banach 空間 $(W, \|\cdot\|)$ 中的級數。令

$$r = \liminf_{n \rightarrow \infty} \frac{\|u_{n+1}\|}{\|u_n\|} \quad \text{以及} \quad R = \limsup_{n \rightarrow \infty} \frac{\|u_{n+1}\|}{\|u_n\|}.$$

- (1) 如果 $R < 1$ ，那麼級數 $\sum u_n$ 會絕對收斂。
- (2) 如果 $r > 1$ ，那麼級數 $\sum u_n$ 會發散。
- (3) 如果 $r \leq 1 \leq R$ ，那麼我們無法總結。

註解 6.3.5 : 不要忘記，在特殊情況 $(W, \|\cdot\|) = (\mathbb{C}, |\cdot|)$ 之下，這個系理非常好用。

證明 : 這個證明與定理 6.3.1 的證明非常相似。我們這裡拿 (1) 的證明為例。

令 $\sum u_n$ 為一般項非零，取值在 $(W, \|\cdot\|)$ 中的級數，並滿足 $R < 1$ 。根據 $\limsup$ 的定義，會存在 $N \geq 1$ 使得

$$\forall n \geq N, \quad \frac{\|u_{n+1}\|}{\|u_n\|} \leq \frac{1+R}{2} =: x < 1.$$

接著，我們可以使用相同方法總結。 $\square$

Theorem 6.3.6 (Cauchy's criterion, root test) : Let $(u_n)_{n \geq 1}$ be a sequence of real numbers. Suppose that it is non-negative from a certain index. Additionally, assume that the following limit exists,

$$\lambda := \lim_{n \rightarrow \infty} (u_n)^{1/n} \in [0, +\infty].$$

Then, the following properties hold.

- (1) If $\lambda < 1$, then the series $\sum u_n$ is convergent.
- (2) If $\lambda > 1$, then the series $\sum u_n$ is divergent.
- (3) If $\lambda = 1$, and $(u_n)^{1/n}$ stays above 1 for all large enough n , then the series $\sum u_n$ is divergent.

Proof :

- (1) Suppose that $\lambda < 1$. Let $\mu = \frac{1+\lambda}{2} \in (\lambda, 1)$. Take $N \geq 1$ such that $u_n > 0$ and $(u_n)^{1/n} \leq \mu$ for all $n \geq N$. This shows that for any $n \geq N$, we have

$$\sum_{k=N}^n u_k \leq \sum_{k=N}^n \mu^k \leq \frac{\mu^N}{1-\mu} < \infty.$$

Therefore, the series $\sum_{k \geq N} u_k$ converges, and so does the series $\sum_{n \geq 1} u_n$.

- (2) Suppose that $\lambda > 1$. Let $\mu = \frac{1+\lambda}{2} \in (1, \lambda)$. Take $N \geq 1$ such that $(u_n)^{1/n} \geq \mu$ for all $n \geq N$. This shows that for any $n \geq N$, we have

$$\sum_{k=N}^n u_k \geq \sum_{k=N}^n \mu^k \geq \mu^n \xrightarrow{n \rightarrow \infty} +\infty.$$

Clearly, the series $\sum u_n$ is divergent.

- (3) Suppose that $\lambda = 1$ and there exists $N \geq 1$ such that $(u_n)^{1/n} \geq 1$ for all $n \geq N$. Let us fix such an $N \geq 1$. Then, for all $n \geq N$, we also have $u_n \geq 1$, so the series $\sum u_n$ is divergent. $\square$

定理 6.3.6 【柯西法則，根檢測法】：令 $(u_n)_{n \geq 1}$ 為實數序列。假設從某項開始，他會是非負的。此外，假設下列極限存在：

$$\lambda := \lim_{n \rightarrow \infty} (u_n)^{1/n} \in [0, +\infty].$$

那麼，下面敘述成立。

- (1) 如果 $\lambda < 1$ ，那麼級數 $\sum u_n$ 會收斂。
- (2) 如果 $\lambda > 1$ ，那麼級數 $\sum u_n$ 會發散。
- (3) 如果 $\lambda = 1$ 而且對於所有夠大的 n 來說， $(u_n)^{1/n}$ 會保持在 1 之上，那麼級數 $\sum u_n$ 會發散。

證明：

- (1) 假設 $\lambda < 1$ 。令 $\mu = \frac{1+\lambda}{2} \in (\lambda, 1)$ 。取 $N \geq 1$ 使得對於所有 $n \geq N$ ，我們有 $u_n > 0$ 以及 $(u_n)^{1/n} \leq \mu$ 。這證明了對於任意 $n \geq N$ ，我們有

$$\sum_{k=N}^n u_k \leq \sum_{k=N}^n \mu^k \leq \frac{\mu^N}{1-\mu} < \infty.$$

因此，級數 $\sum_{k \geq N} u_k$ 收斂，所以級數 $\sum_{n \geq 1} u_n$ 也會收斂。

- (2) 假設 $\lambda > 1$ 。令 $\mu = \frac{1+\lambda}{2} \in (1, \lambda)$ 。取 $N \geq 1$ 使得對於所有 $n \geq N$ ，我們有 $(u_n)^{1/n} \geq \mu$ 。這證明了對於任意 $n \geq N$ ，我們有

$$\sum_{k=N}^n u_k \geq \sum_{k=N}^n \mu^k \geq \mu^n \xrightarrow{n \rightarrow \infty} +\infty.$$

顯然，級數 $\sum u_n$ 是發散的。

- (3) 假設 $\lambda = 1$ 而且存在 $N \geq 1$ 使得 $(u_n)^{1/n} \geq 1$ 對於所有 $n \geq N$ 。讓我們固定一個這樣的 $N \geq 1$ 。那麼，對於所有 $n \geq N$ ，我們也會有 $u_n \geq 1$ ，所以級數 $\sum u_n$ 會發散。 $\square$

Remark 6.3.7 : Similar to Remark 6.3.2, when $\lambda = 1$ and $(u_n)^{1/n}$ always stays below 1 for all n , Cauchy's criterion does not allow us to conclude. We may again take the same series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ as examples.

Corollary 6.3.8 : Let $(u_n)_{n \geq 1}$ be a sequence in a Banach space $(W, \|\cdot\|)$ and

$$\lambda := \limsup_{n \rightarrow \infty} \|u_n\|^{1/n} \in [0, +\infty].$$

Then, the following properties hold.

- (1) If $\lambda < 1$, then the series $\sum u_n$ is absolutely convergent.
- (2) If $\lambda > 1$, then the series $\sum u_n$ is divergent.
- (3) If $\lambda = 1$, then we cannot conclude.

Proof : The proof is similar to that of Theorem 6.3.6, which is left as an exercise, see Exercise 6.16. $\square$

Remark 6.3.9 : In Exercise 2.31, we saw the following inequality for a sequence $(a_n)_{n \geq 1}$ with general strictly positive terms,

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} (a_n)^{1/n} \leq \limsup_{n \rightarrow \infty} (a_n)^{1/n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

This means that the root test is stronger than the ratio test. For example, we may look at the sequence $(a_n)_{n \geq 1}$ defined by

$$\forall n \geq 1, \quad a_n = (1 + (-1)^n)2^n + 1 = \begin{cases} 2^{n+1} + 1 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Then, we have

$$0 = \liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < \liminf_{n \rightarrow \infty} (a_n)^{1/n} = 1 < \limsup_{n \rightarrow \infty} (a_n)^{1/n} = 2 < \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = +\infty.$$

If we want to apply the ratio test (Corollary 6.3.4), we see that we are in the third scenario; whereas if we apply the root test (Corollary 6.3.8), we are in the second scenario, that is the series $\sum a_n$ is divergent. However, we may note that by applying the ratio test to $\sum a_{2n}$, we find the divergence of the series $\sum a_{2n}$, leading to the divergence of $\sum a_n$.

註解 6.3.7 : 與註解 6.3.2 相似，當 $\lambda = 1$ 而且對於所有 n ，我們有 $(u_n)^{1/n}$ 一直處在 1 之下，柯西法則仍然無法讓我們得到任何結論。我們可以取相同的級數 $\sum \frac{1}{n}$ 和 $\sum \frac{1}{n^2}$ 當作例子。

系理 6.3.8 : 令 $(u_n)_{n \geq 1}$ 為在 Banach 空間 $(W, \|\cdot\|)$ 中的數列，以及

$$\lambda := \limsup_{n \rightarrow \infty} \|u_n\|^{1/n} \in [0, +\infty].$$

那麼，下列性質成立。

- (1) 如果 $\lambda < 1$ ，那麼級數 $\sum u_n$ 會絕對收斂。
- (2) 如果 $\lambda > 1$ ，那麼級數 $\sum u_n$ 會發散。
- (3) 如果 $\lambda = 1$ ，那麼我們無法總結。

證明 : 證明與定理 6.3.6 的證明相似，我們留在習題，見習題 6.16。 $\square$

註解 6.3.9 : 在習題 2.31 中，我們看到如果數列 $(a_n)_{n \geq 1}$ 的一般項嚴格為正，我們會有下列不等式：

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} (a_n)^{1/n} \leq \limsup_{n \rightarrow \infty} (a_n)^{1/n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

這代表著，根檢測法比商檢測法來得強。例如，我們可以考慮定義如下的序列 $(a_n)_{n \geq 1}$ ：

$$\forall n \geq 1, \quad a_n = (1 + (-1)^n)2^n + 1 = \begin{cases} 2^{n+1} + 1 & \text{若 } n \text{ 為偶數,} \\ 1 & \text{若 } n \text{ 為奇數.} \end{cases}$$

那麼，我們有

$$0 = \liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < \liminf_{n \rightarrow \infty} (a_n)^{1/n} = 1 < \limsup_{n \rightarrow \infty} (a_n)^{1/n} = 2 < \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = +\infty.$$

如果我們想要使用商檢測法（系理 6.3.4），我們會在第三個情境中；但如果我們使用根檢測法（系理 6.3.8），我們會在第二個情境中，也就是說級數 $\sum a_n$ 會發散。但可以注意到，我們能夠對 $\sum a_{2n}$

6.4 Conditionally convergent series

In this section, we still consider series with terms in a Banach space $(W, \|\cdot\|)$, which converge conditionally. We recall that if a series converges but does not converge absolutely, then we say that it converges conditionally, see Definition 6.1.14. We note that the special cases $W = \mathbb{R}$ or $\mathbb{C}$ are useful in practice.

6.4.1 Alternating series

Definition 6.4.1 : Let $\sum u_n$ be a series with terms in $\mathbb{R}$. We say that it is an *alternating series* (交錯級數) if $(-1)^n u_n$ has the same sign for all $n \geq 1$. Up to a global sign change, we may rewrite the series $\sum u_n$ as $\sum (-1)^n a_n$, where $a_n \geq 0$ for all $n \geq 1$.

Theorem 6.4.2 : Let $(a_n)_{n \geq 1}$ be a non-negative sequence. Suppose that it is non-increasing and tends to 0. Then, the alternating series $\sum (-1)^n a_n$ converges, and its remainder satisfies

$$\forall n \geq 1, \quad |R_n| \leq a_{n+1}, \quad \text{where } R_n = \sum_{k=n+1}^{\infty} (-1)^k a_k.$$

Remark 6.4.3 : In Exercise 6.22, you can see that under some additional mild assumptions, we get a finer estimation on the remainder of an alternating series. In particular, you may apply this result to the alternating series $\sum \frac{(-1)^{n+1}}{n}$.

Proof : Since (a_n) is non-increasing, we find, for all $n \geq 1$,

$$S_{2n+2} - S_{2n} = a_{2n+2} - a_{2n+1} \leq 0 \quad \text{and} \quad S_{2n+1} - S_{2n-1} = a_{2n} - a_{2n+1} \geq 0.$$

In other words, the sequence $(S_{2n})_{n \geq 1}$ is non-increasing, and the sequence $(S_{2n-1})_{n \geq 1}$ is non-

使用商檢測法，來推得級數 $\sum a_{2n}$ 的發散性，進而推得 $\sum a_n$ 的發散性。

第四節 條件收斂級數

在這個小節，我們要考慮的級數一般項仍然會是在 Banach 空間 $(W, \|\cdot\|)$ 中，而且會條件收斂。我們回顧在定義 6.1.14 中的定義：如果級數收斂但不會絕對收斂，那麼我們說他絕對收斂。我們可以注意到的是，在實際應用中，我們常常會取 $W = \mathbb{R}$ 或 $\mathbb{C}$ 這樣的例子。

第一小節 交錯級數

定義 6.4.1 : 令 $\sum u_n$ 為一般項在 $\mathbb{R}$ 中的級數。如果對於所有 $n \geq 1$ ， $(-1)^n u_n$ 有相同的正負號，則我們說他是個交錯級數 (alternating series)。有需要的話，我們可以同時對所有的項改變正負號，我們可以把級數 $\sum u_n$ 寫成 $\sum (-1)^n a_n$ ，其中 $a_n \geq 0$ 對於所有 $n \geq 1$ 。

定理 6.4.2 : 令 $(a_n)_{n \geq 1}$ 為非負序列。假設他是非遞增，且趨近於 0。那麼，交錯級數 $\sum (-1)^n a_n$ 會收斂，而且他的餘項滿足

$$\forall n \geq 1, \quad |R_n| \leq a_{n+1}, \quad \text{其中 } R_n = \sum_{k=n+1}^{\infty} (-1)^k a_k.$$

註解 6.4.3 : 在習題 6.22 中，你可以看到，在一些微弱的額外假設之下，我們可以對交錯級數的餘項得到更好的估計。這個結果可以用在交錯級數 $\sum \frac{(-1)^{n+1}}{n}$ 這個範例上。

證明 : 由於 (a_n) 是非遞增的，對於所有 $n \geq 1$ ，我們有

$$S_{2n+2} - S_{2n} = a_{2n+2} - a_{2n+1} \leq 0 \quad \text{以及} \quad S_{2n+1} - S_{2n-1} = a_{2n} - a_{2n+1} \geq 0.$$

換句話說，序列 $(S_{2n})_{n \geq 1}$ 是非遞增的，而且序列 $(S_{2n-1})_{n \geq 1}$ 是非遞減的。由於 $S_{2n} - S_{2n-1} =$

decreasing. Since $S_{2n} - S_{2n-1} = a_{2n} \xrightarrow{n \rightarrow \infty} 0$, the sequences $(S_{2n})_{n \geq 1}$ and $(S_{2n-1})_{n \geq 1}$ are adjacent. Then, it follows from Proposition 6.1.4 that they converge to the same limit, denoted S . Therefore, $S_n \xrightarrow{n \rightarrow \infty} S$ and

$$\forall n \geq 1, \quad S_{2n-1} \leq S_{2n+1} \leq S \leq S_{2n}$$

This implies that

$$\forall n \geq 1, \quad |R_{2n}| = |S - S_{2n}| \leq S_{2n} - S_{2n+1} = a_{2n+1}.$$

Similarly,

$$\forall n \geq 1, \quad |R_{2n-1}| = |S - S_{2n-1}| \leq S_{2n} - S_{2n-1} = a_{2n}. \quad \square$$

Example 6.4.4 : By Theorem 6.4.2, the series $\sum \frac{(-1)^{n+1}}{n}$ is convergent. Let us compute its sum. By Example 6.2.9, we know that the harmonic numbers have the following asymptotic behavior,

$$H_n = \sum_{k=1}^n \frac{1}{k} = \ln n + \gamma + o(1), \quad \text{when } n \rightarrow \infty.$$

Let us denote the partial sums of the alternating series as below,

$$\forall n \geq 1, \quad S_n = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}.$$

Then, for every $n \geq 1$, we find

$$H_{2n} - S_{2n} = \sum_{k=1}^n \frac{2}{2k} = H_n.$$

In other words, we have the following asymptotic behavior for S_{2n} ,

$$S_{2n} = H_{2n} - H_n = (\ln(2n) + \gamma + o(1)) - (\ln n + \gamma + o(1)) = \ln 2 + o(1), \quad \text{when } n \rightarrow \infty.$$

This means that the series $\sum \frac{(-1)^{n+1}}{n}$ converges to $\ln 2$.

$a_{2n} \xrightarrow{n \rightarrow \infty} 0$ ，數列 $(S_{2n})_{n \geq 1}$ 和 $(S_{2n-1})_{n \geq 1}$ 是相伴序列，根據命題 6.1.4，他們會收斂到相同極限，記作 S 。因此， $S_n \xrightarrow{n \rightarrow \infty} S$ 而且

$$\forall n \geq 1, \quad S_{2n-1} \leq S_{2n+1} \leq S \leq S_{2n}$$

這蘊含

$$\forall n \geq 1, \quad |R_{2n}| = |S - S_{2n}| \leq S_{2n} - S_{2n+1} = a_{2n+1}.$$

同理，我們也有

$$\forall n \geq 1, \quad |R_{2n-1}| = |S - S_{2n-1}| \leq S_{2n} - S_{2n-1} = a_{2n}. \quad \square$$

範例 6.4.4 : 根據定理 6.4.2，級數 $\sum \frac{(-1)^{n+1}}{n}$ 會收斂，接著讓我們來計算他的和。根據範例 6.2.9，我們知道調和數會有下面這個漸進行為：

$$H_n = \sum_{k=1}^n \frac{1}{k} = \ln n + \gamma + o(1), \quad \text{當 } n \rightarrow \infty.$$

我們把交錯級數的部份和記作

$$\forall n \geq 1, \quad S_n = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}.$$

那麼，對於每個 $n \geq 1$ ，我們有

$$H_{2n} - S_{2n} = \sum_{k=1}^n \frac{2}{2k} = H_n.$$

換句話說，對於 S_{2n} 我們會有下面這個漸進行為：

$$S_{2n} = H_{2n} - H_n = (\ln(2n) + \gamma + o(1)) - (\ln n + \gamma + o(1)) = \ln 2 + o(1), \quad \text{當 } n \rightarrow \infty.$$

這代表著級數 $\sum \frac{(-1)^{n+1}}{n}$ 會收斂到 $\ln 2$ 。

6.4.2 Dirichlet's test

Let us consider a series $\sum u_n$ whose general term can be rewritten as $u_n = a_n b_n$ for $n \geq 1$. We write $S_n = \sum_{k=1}^n b_k$ for $n \geq 1$ and $S_0 = 0$.

Proposition 6.4.5 (Abel's transform) : For every $n \geq 0$, we have

$$\sum_{k=1}^n u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n. \quad (6.12)$$

Proof : For every $n \geq 0$, we have

$$\begin{aligned} \sum_{k=1}^n u_k &= \sum_{k=1}^n a_k b_k = \sum_{k=1}^n a_k (S_k - S_{k-1}) \\ &= \sum_{k=1}^n a_k S_k - \sum_{k=0}^{n-1} a_{k+1} S_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n. \end{aligned} \quad \square$$

Remark 6.4.6 : This is exactly the integration by parts for the Riemann–Stieltjes integral when the integrator is given by the Gauss function $\lfloor \cdot \rfloor$, see Corollary 5.2.24, and try to compare the two formulas.

Theorem 6.4.7 (Dirichlet's test) : Let $\sum u_n$ be a series with general terms in a Banach space $(W, \|\cdot\|)$. Suppose that its general term u_n writes $u_n = a_n b_n$ with $a_n \in \mathbb{R}$ and $b_n \in W$ for all $n \geq 1$, and satisfies

- (i) the sequence $(a_n)_{n \geq 1}$ is non-negative, non-increasing, and tends to 0;
- (ii) the series $\sum b_n$ is bounded.

Then, the series $\sum u_n$ is convergent.

Remark 6.4.8 : We can make the same observation as in Remark 6.4.6. We have already seen an application of Corollary 5.2.24 to show the convergence of series in Exercise 5.21. This theorem is based on the Abel's transform (Proposition 6.4.5), and the result is exactly the same as in Exercise 5.21.

第二小節 Dirichlet 檢測法

我們考慮級數 $\sum u_n$ 其中一般項可以寫做 $u_n = a_n b_n$ 對於所有 $n \geq 1$ 。我們記 $S_0 = 0$ 以及對於 $n \geq 1$ ，記 $S_n = \sum_{k=1}^n b_k$ 。

命題 6.4.5 【Abel 變換】：對於每個 $n \geq 0$ ，我們有

$$\sum_{k=1}^n u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n. \quad (6.12)$$

證明：對於每個 $n \geq 0$ ，我們有

$$\begin{aligned} \sum_{k=1}^n u_k &= \sum_{k=1}^n a_k b_k = \sum_{k=1}^n a_k (S_k - S_{k-1}) \\ &= \sum_{k=1}^n a_k S_k - \sum_{k=0}^{n-1} a_{k+1} S_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n. \end{aligned} \quad \square$$

註解 6.4.6：在系理 5.2.24 裡面的 Riemann–Stieltjes 積分理論中，當我們把積分函數取做高斯函數 $\lfloor \cdot \rfloor$ ，所得到的分部積分就是這個結果。你可以嘗試比較兩邊的式子。

定理 6.4.7 【Dirichlet 檢測法】：令 $\sum u_n$ 為一般項取值在 Banach 空間 $(W, \|\cdot\|)$ 中的級數。假設他的一般項 u_n 可以寫做 $u_n = a_n b_n$ ，其中 $a_n \in \mathbb{R}$ 且 $b_n \in W$ 對於所有 $n \geq 1$ ，且滿足：

- (i) 序列 $(a_n)_{n \geq 1}$ 是非負、非遞增，且趨近於 0；
- (ii) 級數 $\sum b_n$ 有界。

那麼級數 $\sum u_n$ 會收斂。

註解 6.4.8：我們可以做出與註解 6.4.6 相同的觀察。在習題 5.21 中，我們有看過系理 5.2.24 的應用，可以來證明級數的收斂。這個定理是建立在 Abel 變換（命題 6.4.5）的基礎上，而且他的結果與習題 5.21 當中的結果完全一樣。

Proof : Let us apply Abel transform Eq. (6.12) to the series $\sum u_n$. For every $n \geq 0$, we have

$$\sum_{k=1}^n u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n,$$

where S_n is the n -th partial sum of the series $\sum b_n$. Let $M > 0$ such that $|S_n| = |\sum_{k=1}^n b_k| \leq M$ for all $n \geq 1$. Then, we have $|a_n S_n| \leq |a_n| M \xrightarrow{n \rightarrow \infty} 0$, so the series $\sum u_n$ and $\sum (a_n - a_{n+1}) S_n$ have the same behavior. Moreover, for every $k \geq 0$, we have

$$|(a_k - a_{k+1}) S_k| \leq (a_k - a_{k+1}) M,$$

since $(a_k)_{k \geq 1}$ is non-increasing. Thus, for every $n \geq 0$, we have

$$\sum_{k=1}^n |(a_k - a_{k+1}) S_k| \leq \sum_{k=1}^n (a_k - a_{k+1}) M = (a_1 - a_{n+1}) M \leq a_1 M.$$

This shows that the series $\sum (a_n - a_{n+1}) S_n$ is absolutely convergent, so convergent. $\square$

Example 6.4.9 : Applying Theorem 6.4.7, we obtain the convergence of the following series.

- (1) Let $(a_n)_{n \geq 0}$ be a non-increasing sequence that tends to 0. The alternating series $\sum (-1)^n a_n$ is convergent because the following partial sum is bounded,

$$\forall n \geq 1, \quad |(-1)^1 + (-1)^2 + \cdots + (-1)^n| \leq 1.$$

This is exactly the result in the first part of Theorem 6.4.2 for alternating series. However, the Dirichlet's test does not give us any estimate on the remainders of the series.

- (2) Let $(a_n)_{n \geq 0}$ be a non-increasing sequence that tends to 0. Let $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Consider the series $\sum a_n e^{in\theta}$. For all $n \geq 0$, we have

$$\forall n \geq 0, \quad |1 + e^{i\theta} + \cdots + e^{in\theta}| = \left| \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \right| = \left| \frac{\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \right| \leq \frac{1}{\left| \sin\left(\frac{\theta}{2}\right) \right|}.$$

Therefore, the series $\sum a_n e^{in\theta}$ converges if $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.

證明：我們對級數 $\sum u_n$ 使用 Abel 變換式 (6.12)。對於每個 $n \geq 0$ ，我們有

$$\sum_{k=1}^n u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n,$$

其中 S_n 是級數 $\sum b_n$ 的第 n 個部份和。令 $M > 0$ 使得 $|S_n| = |\sum_{k=1}^n b_k| \leq M$ 對於所有 $n \geq 1$ 。那麼，我們有 $|a_n S_n| \leq |a_n| M \xrightarrow{n \rightarrow \infty} 0$ ，所以級數 $\sum u_n$ 和 $\sum (a_n - a_{n+1}) S_n$ 有相同的行為。此外，對於每個 $k \geq 0$ ，我們有

$$|(a_k - a_{k+1}) S_k| \leq (a_k - a_{k+1}) M,$$

因為 $(a_k)_{k \geq 1}$ 是非遞增的。因此，對於每個 $n \geq 0$ ，我們有

$$\sum_{k=1}^n |(a_k - a_{k+1}) S_k| \leq \sum_{k=1}^n (a_k - a_{k+1}) M = (a_1 - a_{n+1}) M \leq a_1 M.$$

這證明了級數 $\sum (a_n - a_{n+1}) S_n$ 會絕對收斂，所以會收斂。 $\square$

範例 6.4.9 : 使用定理 6.4.7，我們可以得到下列級數的收斂。

- (1) 令 $(a_n)_{n \geq 0}$ 為非遞增且趨近於 0 的序列。交錯級數 $\sum (-1)^n a_n$ 會收斂，因為下面這個部份和是有界的：

$$\forall n \geq 1, \quad |(-1)^1 + (-1)^2 + \cdots + (-1)^n| \leq 1.$$

這是定理 6.4.2 中第一部份關於交錯級數的結果。然而，Dirichlet 檢測法無法給我們此級數餘項的估計。

- (2) 令 $(a_n)_{n \geq 0}$ 為非遞增且趨近於 0 的序列。令 $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ 。考慮級數 $\sum a_n e^{in\theta}$ 。對於所有 $n \geq 0$ ，我們有

$$\forall n \geq 0, \quad |1 + e^{i\theta} + \cdots + e^{in\theta}| = \left| \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \right| = \left| \frac{\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \right| \leq \frac{1}{\left| \sin\left(\frac{\theta}{2}\right) \right|}.$$

因此，如果 $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ ，則級數 $\sum a_n e^{in\theta}$ 收斂。

6.5 Rearrangement of series

Let $(W, \|\cdot\|)$ be a Banach space and $\sum u_n$ be a series with general terms in W .

Definition 6.5.1 : We say that the series $\sum v_n$ is a *rearrangement* (重新排列) of $\sum u_n$ if there exists a bijection $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$v_n = u_{\varphi(n)}, \quad \forall n \in \mathbb{N}.$$

Theorem 6.5.2 : Suppose that the series $\sum u_n$ is absolutely convergent with sum s . Then, any rearrangement of $\sum u_n$ is also absolutely convergent with sum s .

Proof : Let $\sum v_n$ be a rearrangement of $\sum u_n$ defined by $v_n = u_{\varphi(n)}$ for all $n \in \mathbb{N}$, where $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ is a bijection. We note that for every $n \geq 1$, we have

$$\sum_{k=1}^n \|v_k\| = \sum_{k=1}^n \|u_{\varphi(k)}\| \leq \sum_{k=1}^{\infty} \|u_k\|.$$

The series $\sum \|v_k\|$ has non-negative terms and bounded from above, so it converges, that is $\sum v_k$ converges absolutely.

Let $\varepsilon > 0$. Since $\sum u_n$ converges absolutely, we may find $N \geq 1$ such that

$$\sum_{k=N+1}^{\infty} \|u_k\| \leq \varepsilon.$$

We write $(S_n)_{n \geq 0}$ for the partial sums of $\sum u_n$ and $(T_n)_{n \geq 0}$ for the partial sums of $\sum v_n$. Let $M \geq 1$ be such that

$$\{1, \dots, N\} \subseteq \{\varphi(1), \dots, \varphi(M)\}, \quad (6.13)$$

then for any $n \geq M + 1$, we have $\varphi(n) \geq N + 1$. Take $n \geq M + 1$, we have

$$\|T_n - S_N\| = \left\| \sum_{k=1}^n v_k - \sum_{k=1}^N u_k \right\| = \left\| \sum_{k=1}^n u_{\varphi(k)} - \sum_{k=1}^N u_k \right\| \leq \sum_{k=N+1}^{\infty} \|u_k\| \leq \varepsilon,$$

第五節 級數的重排

令 $(W, \|\cdot\|)$ 為 Banach 空間且 $\sum u_n$ 為一般項在 W 中的級數。

定義 6.5.1 : 給定級數 $\sum v_n$ ，如果存在雙射函數 $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ 使得

$$v_n = u_{\varphi(n)}, \quad \forall n \in \mathbb{N},$$

則我們說他是 $\sum u_n$ 的重新排列 (rearrangement)。

定理 6.5.2 : 假設級數 $\sum u_n$ 會絕對收斂，且和為 s 。那麼，任何 $\sum u_n$ 的重新排列也會絕對收斂，且和為 s 。

證明 : 令 $\sum v_n$ 為 $\sum u_n$ 的重新排列，定義做 $v_n = u_{\varphi(n)}$ 對於所有 $n \in \mathbb{N}$ ，其中 $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ 是個雙射函數。我們注意到，對於每個 $n \geq 1$ ，我們有

$$\sum_{k=1}^n \|v_k\| = \sum_{k=1}^n \|u_{\varphi(k)}\| \leq \sum_{k=1}^{\infty} \|u_k\|.$$

級數 $\sum \|v_k\|$ 當中的項非負，且有上界，所以會收斂，也就是說 $\sum v_k$ 會絕對收斂。

令 $\varepsilon > 0$ 。由於 $\sum u_n$ 會絕對收斂，我們能找到 $N \geq 1$ 使得

$$\sum_{k=N+1}^{\infty} \|u_k\| \leq \varepsilon.$$

我們把 $\sum u_n$ 的部份和記作 $(S_n)_{n \geq 0}$ ，還有 $\sum v_n$ 的部份和記作 $(T_n)_{n \geq 0}$ 。令 $M \geq 1$ 滿足

$$\{1, \dots, N\} \subseteq \{\varphi(1), \dots, \varphi(M)\}, \quad (6.13)$$

那麼對於任意 $n \geq M + 1$ ，我們有 $\varphi(n) \geq N + 1$ 。取 $n \geq M + 1$ ，我們有

$$\|T_n - S_N\| = \left\| \sum_{k=1}^n v_k - \sum_{k=1}^N u_k \right\| = \left\| \sum_{k=1}^n u_{\varphi(k)} - \sum_{k=1}^N u_k \right\| \leq \sum_{k=N+1}^{\infty} \|u_k\| \leq \varepsilon,$$

where in the last equality, we use the inclusion from Eq. (6.13). To conclude, we write

$$\|T_n - s\| \leq \|T_n - S_N\| + \|S_N - s\| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Thus, the series $\sum v_n$ also converges to s . $\square$

Remark 6.5.3 : In Theorem 6.5.2, it is important to assume that the series converges absolutely. Below, we provide a counterexample in Example 6.5.4 and give a general result in Theorem 6.5.5.

Example 6.5.4 : We already know that the following series converges (Example 6.4.4)

$$\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

If we rearrange the terms in the following way, we get a different sum,

$$\begin{aligned} & \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2} \ln 2. \end{aligned}$$

Theorem 6.5.5 (Riemann series theorem) : Let $\sum u_n$ be a real-valued series. Suppose that it converges conditionally. Let $-\infty \leq x \leq y \leq +\infty$. Then, there exists a rearrangement $\sum v_n$ of $\sum u_n$ such that

$$\liminf_{n \rightarrow \infty} T_n = x \quad \text{and} \quad \limsup_{n \rightarrow \infty} T_n = y,$$

where for every $n \geq 1$, $T_n = v_1 + \cdots + v_n$ is the n -th partial sum of $\sum v_n$.

Remark 6.5.6 : In particular, if we take $x = y$ in Theorem 6.5.5, then the theorem says that we can find a rearrangement whose sum is equal to $x = y$.

當中我們在最後一個等式中，使用了式 (6.13) 中的包含關係。最後，我們可以得到

$$\|T_n - s\| \leq \|T_n - S_N\| + \|S_N - s\| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

因此，級數 $\sum v_n$ 也會收斂到 s 。 $\square$

註解 6.5.3 : 在定理 6.5.2 中，假設級數會絕對收斂是很重要的。接下來，我們會在範例 6.5.4 中給一個反例，並且在定理 6.5.5 中討論一般性的結果。

範例 6.5.4 : 我們已經知道下面的級數會收斂（範例 6.4.4）：

$$\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

如果我們把當中的項用下列方式重新排列，我們會得到不同的和：

$$\begin{aligned} & \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2} \ln 2. \end{aligned}$$

定理 6.5.5 【黎曼級數定理】 : 令 $\sum u_n$ 為實數序列，並假設他會條件收斂。令 $-\infty \leq x \leq y \leq +\infty$ 。那麼，存在 $\sum u_n$ 的重新排列 $\sum v_n$ 使得

$$\liminf_{n \rightarrow \infty} T_n = x \quad \text{以及} \quad \limsup_{n \rightarrow \infty} T_n = y,$$

其中對於每個 $n \geq 1$ ，我們把 $T_n = v_1 + \cdots + v_n$ 記作是 $\sum v_n$ 的第 n 個部份和。

註解 6.5.6 : 如果在定理 6.5.5 中，我們取特例 $x = y$ ，那麼定理告訴我們可以找到一個重新排列，使得他的和會是 $x = y$ 。

Proof : This statement can be shown by construction. We do not give the details here. $\square$

6.6 Cauchy series

Definition 6.6.1 : Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $(\mathcal{A}, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. Consider a binary operator $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$.

- (1) We say that $(\mathcal{A}, \cdot)$ is an *algebra* if $\cdot$ is bilinear, that is, the following are satisfied.
 - (a) (Right distributivity) For $x, y, z \in \mathcal{A}$, we have $(x + y) \cdot z = x \cdot z + y \cdot z$.
 - (b) (Left distributivity) For $x, y, z \in \mathcal{A}$, we have $z \cdot (x + y) = z \cdot x + z \cdot y$.
 - (c) (Scalar multiplication) For $x, y \in \mathcal{A}$ and $a, b \in \mathbb{K}$, we have $(ax) \cdot (by) = (ab)(x \cdot y)$.
- (2) We say that $(\mathcal{A}, \cdot, \|\cdot\|)$ is a *normed algebra* (賦範代數) if $(\mathcal{A}, \cdot)$ is an algebra, and the norm $\|\cdot\|$ is submultiplicative, i.e.,

$$\forall x, y \in \mathcal{A}, \quad \|xy\| \leq \|x\| \|y\|.$$

Example 6.6.2 :

- (1) The simplest examples of normed algebras are $(\mathbb{R}, \times, |\cdot|)$ and $(\mathbb{C}, \times, |\cdot|)$.
- (2) We have seen in Remark 3.2.15 that $\mathcal{L}_c(U)$ equipped with the operator norm $\|\cdot\|$ is a normed algebra for any normed vector space U . In particular, if U is a finite-dimensional normed vector space, then $\mathcal{L}(U)$ equipped with the operator norm $\|\cdot\|$ is a normed algebra.
- (3) Equivalently, for an integer $n \geq 1$, the space of $n \times n$ matrices $\mathcal{M}_{n \times n}(\mathbb{K})$ equipped with the matrix norm $\|\cdot\|$ is also a normed algebra.

證明 : 這個敘述可以藉由構造法來證明，我們這裡不提供細節。 $\square$

第六節 柯西級數

定義 6.6.1 : 令 $\mathbb{K} = \mathbb{R}$ 或 $\mathbb{C}$ 且 $(\mathcal{A}, \|\cdot\|)$ 為 $\mathbb{K}$ 上的賦範向量空間。考慮二元算子 $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ 。

- (1) 如果 $\cdot$ 是雙線性的，也就是滿足下列條件，那我們說 $(\mathcal{A}, \cdot)$ 是個代數：
 - (a) 【右分配律】對於 $x, y, z \in \mathcal{A}$ ，我們有 $(x + y) \cdot z = x \cdot z + y \cdot z$ 。
 - (b) 【左分配律】對於 $x, y, z \in \mathcal{A}$ ，我們有 $z \cdot (x + y) = z \cdot x + z \cdot y$ 。
 - (c) 【純量乘法】對於 $x, y \in \mathcal{A}$ 以及 $a, b \in \mathbb{K}$ ，我們有 $(ax) \cdot (by) = (ab)(x \cdot y)$ 。
- (2) 如果 $(\mathcal{A}, \cdot)$ 是個代數，且範數 $\|\cdot\|$ 具有次可乘性，也就是說

$$\forall x, y \in \mathcal{A}, \quad \|xy\| \leq \|x\| \|y\|,$$

那我們說 $(\mathcal{A}, \cdot, \|\cdot\|)$ 是個賦範代數 (normed algebra)。

範例 6.6.2 :

- (1) 賦範代數最簡單的例子是 $(\mathbb{R}, \times, |\cdot|)$ 和 $(\mathbb{C}, \times, |\cdot|)$ 。
- (2) 在註解 3.2.15 中，我們有看過對於任意賦範向量空間 U ，在 $\mathcal{L}_c(U)$ 上賦予算子範數 $\|\cdot\|$ 時，會是個賦範代數。特別來說，如果 U 是個有限維度的賦範向量空間，那麼在 $\mathcal{L}(U)$ 上賦予算子範數 $\|\cdot\|$ 時，他會是個賦範代數。
- (3) 等價來說，對於任意整數 $n \geq 1$ ，由 $n \times n$ 矩陣所構成的空間 $\mathcal{M}_{n \times n}(\mathbb{K})$ 在賦予矩陣範數 $\|\cdot\|$ 時，也會是個賦範代數。

Theorem 6.6.3 (Cauchy product) : Let $\sum_{n \geq 0} a_n$ and $\sum_{n \geq 0} b_n$ be two absolutely convergent series with terms in a complete normed algebra $(\mathcal{A}, \|\cdot\|)$. We define their Cauchy product to be the series $\sum_{n \geq 0} c_n$ given by

$$\forall n \in \mathbb{N}_0, \quad c_n = \sum_{k=0}^n a_k b_{n-k}.$$

The series $\sum c_n$ is absolutely convergent, and its sum equals

$$\sum_{n \geq 0} c_n = \left(\sum_{p \geq 0} a_p \right) \left(\sum_{q \geq 0} b_q \right). \quad (6.14)$$

Proof : Let us denote the following sums,

$$A := \sum_{p=0}^{\infty} \|a_p\| \quad \text{and} \quad B := \sum_{q=0}^{\infty} \|b_q\|.$$

Let $n \geq 0$. The n -th partial sum of $\sum c_n$ writes

$$\begin{aligned} \sum_{k=0}^n \|c_k\| &\leq \sum_{k=0}^n \left(\sum_{\substack{p+q=k \\ p,q \geq 0}} \|a_p\| \cdot \|b_q\| \right) \leq \sum_{0 \leq p,q \leq n} \|a_p\| \cdot \|b_q\| \\ &= \left(\sum_{p=0}^n \|a_p\| \right) \left(\sum_{q=0}^n \|b_q\| \right) \leq AB. \end{aligned}$$

Thus, the series $\sum c_n$ is absolutely convergent.

To compute the sum of the series $\sum c_n$, let us define the following quantities,

$$\forall n \geq 0, \quad \Delta_n = \sum_{k=0}^{2n} c_k - \left(\sum_{p=0}^n a_p \right) \left(\sum_{q=0}^n b_q \right).$$

Then, for any $n \geq 0$, we have

$$\Delta_n = \sum_{\substack{p+q \leq 2n \\ p,q \geq 0}} a_p b_q - \sum_{0 \leq p,q \leq n} a_p b_q = \sum_{\substack{p \geq n+1, q \geq 0 \\ p+q \leq 2n}} a_p b_q + \sum_{\substack{q \geq n+1, p \geq 0 \\ p+q \leq 2n}} a_p b_q.$$

定理 6.6.3 【柯西積】 : 令 $\sum_{n \geq 0} a_n$ 及 $\sum_{n \geq 0} b_n$ 為兩個會絕對收斂的級數，且他們的一般項在完備賦範代數 $(\mathcal{A}, \|\cdot\|)$ 中。我們定義他們的柯西積為定義如下的級數 $\sum_{n \geq 0} c_n$:

$$\forall n \in \mathbb{N}_0, \quad c_n = \sum_{k=0}^n a_k b_{n-k}.$$

級數 $\sum c_n$ 會絕對收斂，而且他的和等於

$$\sum_{n \geq 0} c_n = \left(\sum_{p \geq 0} a_p \right) \left(\sum_{q \geq 0} b_q \right). \quad (6.14)$$

證明 : 我們定義下面的和：

$$A := \sum_{p=0}^{\infty} \|a_p\| \quad \text{以及} \quad B := \sum_{q=0}^{\infty} \|b_q\|.$$

令 $n \geq 0$ 。 $\sum c_n$ 的第 n 個部份和寫做

$$\begin{aligned} \sum_{k=0}^n \|c_k\| &\leq \sum_{k=0}^n \left(\sum_{\substack{p+q=k \\ p,q \geq 0}} \|a_p\| \cdot \|b_q\| \right) \leq \sum_{0 \leq p,q \leq n} \|a_p\| \cdot \|b_q\| \\ &= \left(\sum_{p=0}^n \|a_p\| \right) \left(\sum_{q=0}^n \|b_q\| \right) \leq AB. \end{aligned}$$

所以，級數 $\sum c_n$ 會絕對收斂。

要計算級數 $\sum c_n$ 的和，我們定義下面的量：

$$\forall n \geq 0, \quad \Delta_n = \sum_{k=0}^{2n} c_k - \left(\sum_{p=0}^n a_p \right) \left(\sum_{q=0}^n b_q \right).$$

那麼，對於任意 $n \geq 0$ ，我們有

$$\Delta_n = \sum_{\substack{p+q \leq 2n \\ p,q \geq 0}} a_p b_q - \sum_{0 \leq p,q \leq n} a_p b_q = \sum_{\substack{p \geq n+1, q \geq 0 \\ p+q \leq 2n}} a_p b_q + \sum_{\substack{q \geq n+1, p \geq 0 \\ p+q \leq 2n}} a_p b_q.$$

Therefore, for any $n \geq 0$, the triangle inequality gives,

$$\begin{aligned} \|\Delta_n\| &\leq \sum_{\substack{p \geq n+1, q \geq 0 \\ p+q \leq 2n}} \|a_p\| \|b_q\| + \sum_{\substack{q \geq n+1, p \geq 0 \\ p+q \leq 2n}} \|a_p\| \|b_q\| \\ &\leq \sum_{p \geq n+1, q \geq 0} \|a_p\| \|b_q\| + \sum_{q \geq n+1, p \geq 0} \|a_p\| \|b_q\| \\ &= B \cdot \sum_{p=n+1}^{\infty} \|a_p\| + A \cdot \sum_{q=n+1}^{\infty} \|b_q\| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

This proves Eq. (6.14). $\square$

6.7 Double sequences, double series

6.7.1 Double sequences and double limits

Let $(W, \|\cdot\|)$ be a Banach space. A sequence $(u_{m,n})_{m,n \geq 1}$, taking values in W with two indices, is called a *double sequence* (雙下標序列).

Definition 6.7.1: Let $(u_{m,n})_{m,n \geq 1}$ be a double sequence. Let $\ell \in W$. We say that the double sequence $(u_{m,n})_{m,n \geq 1}$ converges to ℓ , denoted

$$\lim_{m,n \rightarrow \infty} u_{m,n} = \ell,$$

if for every $\varepsilon > 0$, there exists $N \geq 1$ such that

$$\forall m, n \geq N, \quad \|u_{m,n} - \ell\| < \varepsilon. \quad (6.15)$$

We call ℓ the *limit* or the *double limit* of the double sequence $(u_{m,n})_{m,n \geq 1}$.

Example 6.7.2: Let $(u_{m,n})_{m,n \geq 1}$ be a real-valued double sequence defined by

$$\forall m, n \geq 1, \quad u_{m,n} = \mathbb{1}_{m \geq n}.$$

因此，對於任意 $n \geq 0$ ，三角不等式會給我們：

$$\begin{aligned} \|\Delta_n\| &\leq \sum_{\substack{p \geq n+1, q \geq 0 \\ p+q \leq 2n}} \|a_p\| \|b_q\| + \sum_{\substack{q \geq n+1, p \geq 0 \\ p+q \leq 2n}} \|a_p\| \|b_q\| \\ &\leq \sum_{p \geq n+1, q \geq 0} \|a_p\| \|b_q\| + \sum_{q \geq n+1, p \geq 0} \|a_p\| \|b_q\| \\ &= B \cdot \sum_{p=n+1}^{\infty} \|a_p\| + A \cdot \sum_{q=n+1}^{\infty} \|b_q\| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

這證明了式 (6.14)。 $\square$

第七節 雙下標序列、雙下標級數

第一小節 雙下標序列及雙下標極限

令 $(W, \|\cdot\|)$ 為 Banach 空間。有兩個下標，且取值在 W 中的序列 $(u_{m,n})_{m,n \geq 1}$ ，我們把他稱作 雙下標序列 (double sequence)。

定義 6.7.1 : 令 $(u_{m,n})_{m,n \geq 1}$ 為雙下標序列。令 $\ell \in W$ 。如果對於每個 $\varepsilon > 0$ ，都會存在 $N \geq 1$ 使得

$$\forall m, n \geq N, \quad \|u_{m,n} - \ell\| < \varepsilon, \quad (6.15)$$

則我們說雙下標序列 $(u_{m,n})_{m,n \geq 1}$ 會收斂到 ℓ ，記作

$$\lim_{m,n \rightarrow \infty} u_{m,n} = \ell.$$

我們說 ℓ 是雙下標序列 $(u_{m,n})_{m,n \geq 1}$ 的極限或是雙下標極限。

範例 6.7.2 : 令 $(u_{m,n})_{m,n \geq 1}$ 為取值為實數的雙下標序列，定義做

$$\forall m, n \geq 1, \quad u_{m,n} = \mathbb{1}_{m \geq n}.$$

Then, we have,

$$\forall n \geq 1, \quad \lim_{m \rightarrow \infty} u_{m,n} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} u_{m,n} = 1, \quad (6.16)$$

and

$$\forall m \geq 1, \quad \lim_{n \rightarrow \infty} u_{m,n} = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} u_{m,n} = 0. \quad (6.17)$$

However, this double sequence does not converge in the sense of Definition 6.7.1, where the uniformity in both indices m and n is required. We call the limits in Eq. (6.16) and Eq. (6.17) *iterated limits* of the double sequence $(u_{m,n})_{m,n \geq 1}$. This shows that when taking an *iterated limit* in a double sequence, the order in which the limits are taken is important.

Theorem 6.7.3 : Let $(u_{m,n})_{m,n \geq 1}$ be a double sequence. Suppose that

- (i) the limit $\lim_{m,n \rightarrow \infty} u_{m,n}$ exists and equals $\ell \in W$;
- (ii) for every $m \geq 1$, the limit $\lim_{n \rightarrow \infty} u_{m,n}$ exists.

Then, the following iterated limit exists and satisfies

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} u_{m,n} = \ell.$$

Proof : From the assumption (ii), we may define $\ell_m := \lim_{n \rightarrow \infty} u_{m,n}$ for every $m \geq 1$. Let $\varepsilon > 0$. From the assumption (i), we may find $N \geq 0$ such that

$$\|u_{m,n} - \ell\| \leq \varepsilon, \quad \forall m, n \geq N.$$

Fix $m \geq N$, by the definition of ℓ_m just above, we may find $N' = N'(m) \geq 1$ such that

$$\|\ell_m - u_{m,n}\| \leq \varepsilon, \quad \forall n \geq N'.$$

Then, for any $n \geq \max(N, N')$, we have

$$\|\ell - \ell_m\| \leq \|\ell - u_{m,n}\| + \|u_{m,n} - \ell_m\| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

那麼，我們有

$$\forall n \geq 1, \quad \lim_{m \rightarrow \infty} u_{m,n} = 1 \quad \text{以及} \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} u_{m,n} = 1, \quad (6.16)$$

以及

$$\forall m \geq 1, \quad \lim_{n \rightarrow \infty} u_{m,n} = 0 \quad \text{以及} \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} u_{m,n} = 0. \quad (6.17)$$

然而，這個雙下標序列並不會在定義 6.7.1 的意義下收斂，因為我們需要對於兩個下標 m 和 n 的均勻性。我們稱式 (6.16) 和式 (6.17) 當中的極限為雙下標序列 $(u_{m,n})_{m,n \geq 1}$ 的迭代極限。這表示，在對雙下標序列取迭代極限時，先後順序是很重要的。

定理 6.7.3 : 令 $(u_{m,n})_{m,n \geq 1}$ 為雙下標序列。假設

- (i) 極限 $\lim_{m,n \rightarrow \infty} u_{m,n}$ 存在且等於 $\ell \in W$;
- (ii) 對於每個 $m \geq 1$ ，極限 $\lim_{n \rightarrow \infty} u_{m,n}$ 存在。

那麼下面的迭代極限存在，且滿足

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} u_{m,n} = \ell.$$

證明：藉由假設 (ii)，我們可以對所有 $m \geq 1$ 定義 $\ell_m := \lim_{n \rightarrow \infty} u_{m,n}$ 。令 $\varepsilon > 0$ 。藉由假設

(i)，我們可以找到 $N \geq 0$ 使得

$$\|u_{m,n} - \ell\| \leq \varepsilon, \quad \forall m, n \geq N.$$

固定 $m \geq N$ ，根據上面 ℓ_m 的定義，我們能找到 $N' = N'(m) \geq 1$ 使得

$$\|\ell_m - u_{m,n}\| \leq \varepsilon, \quad \forall n \geq N'.$$

This shows that $\lim_{m \rightarrow \infty} \ell_m = \ell$. $\square$

6.7.2 Double series

Due to the Riemann series theorem (Theorem 6.5.5), when we want to discuss the order of summations in double series, we are only interested in absolutely convergent ones.

Theorem 6.7.4 : Let $(u_{m,n})_{m,n \geq 1}$ be a double sequence with general terms in a Banach space. Then, the two following properties are equivalent.

- (1) For every $n \geq 1$, the series $\sum_m u_{m,n}$ is absolutely convergent, and the series $\sum_n (\sum_m \|u_{m,n}\|)$ converges.
- (2) For every $m \geq 1$, the series $\sum_n u_{m,n}$ is absolutely convergent, and the series $\sum_m (\sum_n \|u_{m,n}\|)$ converges.

Moreover, when one of the above properties holds, we have

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} u_{m,n} \right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} u_{m,n} \right). \quad (6.18)$$

Proof : By symmetry, it is enough to show that (1) $\Rightarrow$ (2). Assume that (1) holds. For every $n \geq 1$, let $A_n = \sum_{m \geq 1} \|u_{m,n}\|$. Then, the property (1) states that $\sum A_n$ converges. Let us fix $m \geq 1$, then $\|u_{m,n}\| \leq A_n$ for every $n \geq 1$, so $\sum_{n \geq 1} u_{m,n}$ converges absolutely. Let $B_m := \sum_{n \geq 1} \|u_{m,n}\|$ for every $m \geq 1$. Then, for any $M \geq 1$, we have

$$\sum_{m=1}^M B_m = \sum_{m=1}^M \sum_{n \geq 1} \|u_{m,n}\| = \sum_{n \geq 1} \left(\sum_{m=1}^M \|u_{m,n}\| \right) \leq \sum_{n \geq 1} A_n,$$

where in the second equality, we use the linearity on convergent series. On the left-hand side, we have a series with non-negative terms, and on the right-hand side, we have an upper bound that is

那麼，對於任意 $n \geq \max(N, N')$ ，我們有

$$\|\ell - \ell_m\| \leq \|\ell - u_{m,n}\| + \|u_{m,n} - \ell_m\| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

這證明了 $\lim_{m \rightarrow \infty} \ell_m = \ell$ 。 $\square$

第二小節 雙下標級數

根據黎曼級數定理（定理 6.5.5），當我們想要討論雙下標級數取和的順序時，我們只對會絕對收斂的有興趣。

定理 6.7.4 : 令 $(u_{m,n})_{m,n \geq 1}$ 為一般項在 Banach 空間中的雙下標序列。那麼，下面兩個性質是等價的。

- (1) 對於每個 $n \geq 1$ ，級數 $\sum_m u_{m,n}$ 會絕對收斂，且級數 $\sum_n (\sum_m \|u_{m,n}\|)$ 收斂。
- (2) 對於每個 $m \geq 1$ ，級數 $\sum_n u_{m,n}$ 會絕對收斂，且級數 $\sum_m (\sum_n \|u_{m,n}\|)$ 收斂。

此外，當上面其中一個性質成立時，我們會有

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} u_{m,n} \right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} u_{m,n} \right). \quad (6.18)$$

證明：根據對稱性，只需要證明 (1) $\Rightarrow$ (2)。假設 (1) 成立。對於每個 $n \geq 1$ ，令 $A_n = \sum_{m \geq 1} \|u_{m,n}\|$ 。那麼，性質 (1) 說的是 $\sum A_n$ 會收斂。讓我們固定 $m \geq 1$ ，那麼 $\|u_{m,n}\| \leq A_n$ 對於每個 $n \geq 1$ ，所以 $\sum_{n \geq 1} u_{m,n}$ 會絕對收斂。對於每個 $m \geq 1$ ，令 $B_m := \sum_{n \geq 1} \|u_{m,n}\|$ 。那麼，對於任意 $M \geq 1$ ，我們有

$$\sum_{m=1}^M B_m = \sum_{m=1}^M \sum_{n \geq 1} \|u_{m,n}\| = \sum_{n \geq 1} \left(\sum_{m=1}^M \|u_{m,n}\| \right) \leq \sum_{n \geq 1} A_n,$$

在上方的第二個等式中，我們使用收斂級數的線性性質。在左方，我們有個一般項非負的級

independent of M , so the series $\sum B_m$ converges. This allows us to conclude that (2) holds.

We note that when (1) is satisfied, the right-hand side of Eq. (6.18) is well defined, because for every $n \geq 1$, the absolute convergence of $\sum_m u_{m,n}$ implies that $\sum_m u_{m,n}$ converges and satisfies

$$\left\| \sum_m u_{m,n} \right\| \leq \sum_m \|u_{m,n}\|.$$

Then, the convergence of $\sum_n \sum_m \|u_{m,n}\|$ implies that of $\sum_n \|\sum_m u_{m,n}\|$, which implies the convergence of $\sum_n (\sum_m u_{m,n})$. Since we have shown that (1) and (2) are equivalent, the left-hand side of Eq. (6.18) is also well defined.

Now, we are going to show Eq. (6.18). Define

$$\forall n \geq 1, \quad S_n = \sum_{p=1}^n \sum_{q=1}^n u_{p,q},$$

and let us show that S_n converges to the right-hand side of Eq. (6.18), then we conclude by symmetry.

Let

$$\forall m, q \geq 1, \quad a_{m,q} = \sum_{p=1}^m u_{p,q} \quad \text{and} \quad \forall q \geq 1, \quad a_q = \sum_{p \geq 1} u_{p,q}.$$

Let $\varepsilon > 0$ and $Q \geq 1$ such that $\sum_{q \geq Q} A_q \leq \varepsilon$. Then, for $n \geq Q$, we have

$$\sum_{q=1}^{\infty} a_q - S_n = \sum_{q=1}^{\infty} a_q - \sum_{q=1}^n a_{n,q} = \sum_{q=1}^Q (a_q - a_{n,q}) + \sum_{q=Q+1}^n (a_q - a_{n,q}) + \sum_{q=n+1}^{\infty} a_q.$$

We note that for $q \geq 1$, we have $\|a_q\| \leq A_q$ and for $q \geq Q$, we have $\|a_q - a_{n,q}\| = \left\| \sum_{p \geq n+1} u_{p,q} \right\| \leq A_q$. Thus, the above inequality gives

$$\left\| \sum_{q=1}^{\infty} a_q - S_n \right\| \leq \left\| \sum_{q=1}^Q (a_q - a_{n,q}) \right\| + \sum_{q=Q+1}^{\infty} A_q \leq \left\| \sum_{q=1}^Q (a_q - a_{n,q}) \right\| + \varepsilon.$$

Since $a_{n,q} \xrightarrow{n \rightarrow \infty} a_q$ for every $q \geq 1$, by taking $\limsup$ when $n \rightarrow \infty$, we find

$$\limsup_{n \rightarrow \infty} \left\| \sum_{q=1}^{\infty} a_q - S_n \right\| \leq \varepsilon.$$

數，且在右方，我們有個不取決於 M 的上界，所以級數 $\sum B_m$ 收斂。這讓我們可以總結 (2) 會成立。

我們注意到，當 (1) 成立時，式 (6.18) 的右方是定義良好的，因為對於每個 $n \geq 1$ ，級數 $\sum_m u_{m,n}$ 的絕對收斂性蘊含 $\sum_m u_{m,n}$ 收斂，而且滿足

$$\left\| \sum_m u_{m,n} \right\| \leq \sum_m \|u_{m,n}\|.$$

那麼， $\sum_n \sum_m \|u_{m,n}\|$ 的收斂會蘊含 $\sum_n \|\sum_m u_{m,n}\|$ 的收斂，再蘊含 $\sum_n (\sum_m u_{m,n})$ 的收斂。由於我們證明了 (1) 和 (2) 是等價的，式 (6.18) 的左方也是定義良好的。

再來，我們要證明式 (6.18)。定義

$$\forall n \geq 1, \quad S_n = \sum_{p=1}^n \sum_{q=1}^n u_{p,q},$$

並證明 S_n 會收斂到式 (6.18) 的右方，我們再使用對稱性來總結。令

$$\forall m, q \geq 1, \quad a_{m,q} = \sum_{p=1}^m u_{p,q} \quad \text{以及} \quad \forall q \geq 1, \quad a_q = \sum_{p \geq 1} u_{p,q}.$$

令 $\varepsilon > 0$ 以及 $Q \geq 1$ 滿足 $\sum_{q \geq Q} A_q \leq \varepsilon$ 。那麼，對於 $n \geq Q$ ，我們有

$$\sum_{q=1}^{\infty} a_q - S_n = \sum_{q=1}^{\infty} a_q - \sum_{q=1}^n a_{n,q} = \sum_{q=1}^Q (a_q - a_{n,q}) + \sum_{q=Q+1}^n (a_q - a_{n,q}) + \sum_{q=n+1}^{\infty} a_q.$$

我們注意到，對於 $q \geq 1$ ，我們有 $\|a_q\| \leq A_q$ 而且對於 $q \geq Q$ ，我們有 $\|a_q - a_{n,q}\| = \left\| \sum_{p \geq n+1} u_{p,q} \right\| \leq A_q$ 。因此，上面的不等式給我們

$$\left\| \sum_{q=1}^{\infty} a_q - S_n \right\| \leq \left\| \sum_{q=1}^Q (a_q - a_{n,q}) \right\| + \sum_{q=Q+1}^{\infty} A_q \leq \left\| \sum_{q=1}^Q (a_q - a_{n,q}) \right\| + \varepsilon.$$

由於對每個 $q \geq 1$ ，我們有 $a_{n,q} \xrightarrow{n \rightarrow \infty} a_q$ ，藉由對 $n \rightarrow \infty$ 取 $\limsup$ ，我們得到

$$\limsup_{n \rightarrow \infty} \left\| \sum_{q=1}^{\infty} a_q - S_n \right\| \leq \varepsilon.$$

The choice of $\varepsilon > 0$ is arbitrary, so we find

$$\limsup_{n \rightarrow \infty} \left\| \sum_{q=1}^{\infty} a_q - S_n \right\| = 0 \quad \Rightarrow \quad \lim_{n \rightarrow \infty} \left\| \sum_{q=1}^{\infty} a_q - S_n \right\| = 0,$$

that is, $S_n \xrightarrow{n \rightarrow \infty} \sum_{q=1}^{\infty} a_q$. $\square$

由於 $\varepsilon > 0$ 的選擇可以任意小，我們得到

$$\limsup_{n \rightarrow \infty} \left\| \sum_{q=1}^{\infty} a_q - S_n \right\| = 0 \quad \Rightarrow \quad \lim_{n \rightarrow \infty} \left\| \sum_{q=1}^{\infty} a_q - S_n \right\| = 0,$$

也就是說 $S_n \xrightarrow{n \rightarrow \infty} \sum_{q=1}^{\infty} a_q$ 。 $\square$

6.8 Infinite products

6.8.1 Convergence and divergence

Let $(u_n)_{n \geq 1}$ be a sequence with values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We may define the sequence $(P_n)_{n \geq 0}$ as below

$$P_0 = 1, \quad P_n = \prod_{k=1}^n u_k, \quad \forall n \geq 1.$$

For each $n \geq 1$, we call u_n the n -th factor of the infinite product $\prod u_n$, and P_n the n -th partial product of the infinite product $\prod u_n$.

Definition 6.8.1 : The notion of convergence and divergence of an infinite product $\prod u_n$ is given below.

- (1) If there are infinitely many factors u_n that are zero, we say that the infinite product $\prod u_n$ diverges to zero.
- (2) If $u_n \neq 0$ for all $n \geq 1$, we say that
 - (a) the infinite product converges to $P \neq 0$ if $P_n \xrightarrow{n \rightarrow \infty} P$, and we write $P = \prod_{n=1}^{\infty} u_n$;
 - (b) the infinite product diverges to 0 if $P_n \xrightarrow{n \rightarrow \infty} 0$;
 - (c) the infinite series diverge otherwise.
- (3) If there exists $N \geq 1$ such that $u_n \neq 0$ for all $n \geq N$, let us define

$$\forall n \geq 1, \quad v_n = u_{n+N-1},$$

第八節 無窮積

第一小節 收斂性與發散性

令 $(u_n)_{n \geq 1}$ 為取值在 $\mathbb{K} = \mathbb{R}$ 或 $\mathbb{C}$ 中的序列。我們可以定義序列 $(P_n)_{n \geq 0}$ 如下：

$$P_0 = 1, \quad P_n = \prod_{k=1}^n u_k, \quad \forall n \geq 1.$$

對於每個 $n \geq 1$ ，我們稱 u_n 為無窮積 $\prod u_n$ 的第 n 個因子，以及 P_n 為無窮積 $\prod u_n$ 的第 n 個部份積。

定義 6.8.1 : 無窮積 $\prod u_n$ 收斂與發散的概念由下面敘述所給定。

- (1) 如果有無窮多個因子 u_n 為零，那麼我們說無窮積 $\prod u_n$ 發散至零。
- (2) 如果對於所有 $n \geq 1$ ，我們有 $u_n \neq 0$ ，我們有下面情況：
 - (a) 如果 $P_n \xrightarrow{n \rightarrow \infty} P \neq 0$ ，則我們說無窮積收斂至 P ，並記 $P = \prod_{n=1}^{\infty} u_n$ ；
 - (b) 如果 $P_n \xrightarrow{n \rightarrow \infty} 0$ ，則我們說無窮積發散至 0；
 - (c) 其他狀況下，我們說無窮積發散。
- (3) 如果存在 $N \geq 1$ 使得對於所有 $n \geq N$ ，我們有 $u_n \neq 0$ ，讓我們定義

$$\forall n \geq 1, \quad v_n = u_{n+N-1},$$

and its corresponding partial products $(P'_n)_{n \geq 0}$ given by

$$P'_0 = 1, \quad P'_n = \prod_{k=1}^n v_k = \prod_{k=N}^{N+n-1} u_k, \quad \forall n \geq 1.$$

(a) If the infinite product $\prod v_n$ converges to $P \neq 0$, then we say that the infinite product $\prod u_n$ converges to $u_1 \dots u_{N-1}P$ and write its limit as

$$\prod_{n \geq 1} u_n := u_1 \dots u_{N-1} \prod_{n \geq N} u_n = u_1 \dots u_{N-1} \prod_{n \geq 1} v_n;$$

(b) If the infinite product $\prod v_n$ diverges to 0, we say that the infinite product $\prod u_n$ diverges to 0;

(c) Otherwise, we say that the infinite product $\prod u_n$ diverges.

Remark 6.8.2 : From Definition 6.8.1, we know that by adding or removing finitely many zeros to an infinite product, we do not change its convergent or divergent behavior.

Proposition 6.8.3 (Cauchy's condition) : *The infinite product $\prod u_n$ converges if and only if for every $\varepsilon > 0$, there exists $N \geq 1$ such that*

$$\forall n \geq N, \forall k \geq 1, \quad |u_{n+1} \dots u_{n+k} - 1| < \varepsilon. \quad (6.19)$$

Proof : Since the notion of convergence of $\prod u_n$ and the condition Eq. (6.19) are not changed if we remove finitely zero terms from $(u_n)_{n \geq 1}$, we may assume that $u_n \neq 0$ for all $n \geq 1$.

• Suppose that $\prod u_n$ converges. Let

$$P = \lim_{n \rightarrow \infty} P_n = \lim_{n \rightarrow \infty} \prod_{k=1}^n u_k \neq 0.$$

This means that the partial products $(P_n)_{n \geq 0}$ of $\prod u_n$ are bounded from below by some constant $M > 0$. Let $\varepsilon > 0$, by the Cauchy's condition for sequences from Corollary 6.1.11, we may find $N \geq 1$ such that

$$\forall n \geq N, \forall k \geq 1, \quad |P_{n+k} - P_n| < \varepsilon M,$$

以及他所對應到的部份積 $(P'_n)_{n \geq 0}$ 如下：

$$P'_0 = 1, \quad P'_n = \prod_{k=1}^n v_k = \prod_{k=N}^{N+n-1} u_k, \quad \forall n \geq 1.$$

(a) 如果無窮積 $\prod v_n$ 收斂到 $P \neq 0$ ，則我們說無窮積 $\prod u_n$ 收斂到 $u_1 \dots u_{N-1}P$ 並把他的極限記作

$$\prod_{n \geq 1} u_n := u_1 \dots u_{N-1} \prod_{n \geq N} u_n = u_1 \dots u_{N-1} \prod_{n \geq 1} v_n;$$

(b) 如果無窮積 $\prod v_n$ 發散至 0，則我們說無窮積 $\prod u_n$ 發散至 0；

(c) 其他狀況下，我們說無窮積 $\prod u_n$ 發散。

註解 6.8.2 : 從定義 6.8.1，我們看到如果我們在無窮積中加入或移除有限多個零，我們不會改變他收斂或發散的行為。

命題 6.8.3 【柯西條件】： 無窮積 $\prod u_n$ 收斂若且唯若對於所有 $\varepsilon > 0$ ，存在 $N \geq 1$ 使得

$$\forall n \geq N, \forall k \geq 1, \quad |u_{n+1} \dots u_{n+k} - 1| < \varepsilon. \quad (6.19)$$

證明： 如果我們從 $(u_n)_{n \geq 1}$ 中移除有限多個零，我們並不會改變 $\prod u_n$ 收斂的概念以及條件式 (6.19)，因此我們可以假設對於所有 $n \geq 1$ ，我們有 $u_n \neq 0$ 。

• 假設 $\prod u_n$ 收斂。令

$$P = \lim_{n \rightarrow \infty} P_n = \lim_{n \rightarrow \infty} \prod_{k=1}^n u_k \neq 0.$$

這代表著 $\prod u_n$ 的部份積 $(P_n)_{n \geq 0}$ 有下界，我們記 $M > 0$ 為其中一個下界。令 $\varepsilon > 0$ ，根據系理 6.1.11 中，對於序列的柯西條件，我們能找到 $N \geq 1$ 使得

$$\forall n \geq N, \forall k \geq 1, \quad |P_{n+k} - P_n| < \varepsilon M,$$

which, by dividing by $|P_n|$, implies

$$\forall n \geq N, \forall k \geq 1, \left| \frac{P_{n+k}}{P_n} - 1 \right| < \varepsilon,$$

which is exactly Eq. (6.19).

- Suppose that for every $\varepsilon > 0$, there exists $N \geq 1$ such that the condition Eq. (6.19) holds. Let $\varepsilon = \frac{1}{2}$, and take $N \geq 1$ such that Eq. (6.19) holds. This shows that for every $n \geq N$, we have $u_n \neq 0$. For $n \geq N$, let $Q_n = \prod_{k=N+1}^n u_k$. From Eq. (6.19), we also deduce that $\frac{1}{2} < |Q_n| < \frac{3}{2}$ for all $n \geq N$. Additionally, for every $n \geq N$ and $k \geq 1$, we also have

$$\left| \frac{Q_{n+k}}{Q_n} - 1 \right| < \varepsilon \Rightarrow |Q_{n+k} - Q_n| < \varepsilon |Q_n| < \frac{3}{2} \varepsilon.$$

This means that the sequence $(Q_n)_{n \geq N}$ satisfies Cauchy's condition, so converges. This also means that the product $\prod u_n$ converges.

□

Theorem 6.8.4 : Let $(a_n)_{n \geq 1}$ be a sequence with strictly positive general terms. Then, the infinite product $\prod(1 + a_n)$ converges if and only if the series $\sum a_n$ converges.

Proof : The convergence of the infinite product $\prod(1 + a_n)$ is equivalent to the convergence of the series $\sum \ln(1 + a_n)$.

- If $\sum \ln(1 + a_n)$ converges, it means that $\ln(1 + a_n) \xrightarrow{n \rightarrow \infty} 0$, so $a_n \xrightarrow{n \rightarrow \infty} 0$. Thus, we have the equivalence relation $\ln(1 + a_n) \sim a_n$ when $n \rightarrow \infty$. We apply Theorem 6.2.8 and we find that the series $\sum a_n$ converges.
- If the series $\sum a_n$ converges, then $a_n \xrightarrow{n \rightarrow \infty} 0$. Then, we conclude in a similar way. □

Remark 6.8.5 :

- (1) If some of the terms in $(a_n)_{n \geq 1}$ are zero, both the values of $\sum a_n$ and $\prod(1 + a_n)$ are not changed. Therefore, it is reasonable to assume that the sequence $(a_n)_{n \geq 1}$ does not contain any zero.
- (2) In the case that $(a_n)_{n \geq 1}$ is a sequence with strictly negative general terms, the same statement also

如果把上式除掉 $|P_n|$ ，我們會得到

$$\forall n \geq N, \forall k \geq 1, \left| \frac{P_{n+k}}{P_n} - 1 \right| < \varepsilon,$$

這剛好就是式 (6.19)。

- 假設對於每個 $\varepsilon > 0$ ，存在 $N \geq 1$ 使得條件式 (6.19) 成立。令 $\varepsilon = \frac{1}{2}$ ，並取 $N \geq 1$ 使得式 (6.19) 會成立。這證明了對於每個 $n \geq N$ ，我們有 $u_n \neq 0$ 。對於 $n \geq N$ ，令 $Q_n = \prod_{k=N+1}^n u_k$ 。從式 (6.19)，我們能推得 $\frac{1}{2} < |Q_n| < \frac{3}{2}$ 對於所有 $n \geq N$ 。此外，對於每個 $n \geq N$ 和 $k \geq 1$ ，我們也有

$$\left| \frac{Q_{n+k}}{Q_n} - 1 \right| < \varepsilon \Rightarrow |Q_{n+k} - Q_n| < \varepsilon |Q_n| < \frac{3}{2} \varepsilon.$$

這代表著序列 $(Q_n)_{n \geq N}$ 滿足柯西條件，所以收斂。這也代表積 $\prod u_n$ 收斂。

□

定理 6.8.4 : 令 $(a_n)_{n \geq 1}$ 為一般項嚴格為正的序列。那麼，無窮積 $\prod(1 + a_n)$ 收斂若且唯若級數 $\sum a_n$ 收斂。

證明：無窮積 $\prod(1 + a_n)$ 的收斂與級數 $\sum \ln(1 + a_n)$ 的收斂等價。

- 如果 $\sum \ln(1 + a_n)$ 收斂，這代表著 $\ln(1 + a_n) \xrightarrow{n \rightarrow \infty} 0$ ，所以 $a_n \xrightarrow{n \rightarrow \infty} 0$ 。因此，當 $n \rightarrow \infty$ 時，我們有等價關係 $\ln(1 + a_n) \sim a_n$ 。我們利用定理 6.2.8 來得到級數 $\sum a_n$ 會收斂。
- 如果級數 $\sum a_n$ 收斂，那麼 $a_n \xrightarrow{n \rightarrow \infty} 0$ 。那麼，我們可以以相同方式總結。 □

註解 6.8.5 :

- (1) 如果 $(a_n)_{n \geq 1}$ 當中有些項為零，那麼 $\sum a_n$ 和 $\prod(1 + a_n)$ 的值都不會改變。因此，我們可以合理假設序列 $(a_n)_{n \geq 1}$ 不包含任何零項。

holds.

(3) It is important to assume that the sequence $(a_n)_{n \geq 1}$ has a constant sign. We may consider the example

$$a_n = \frac{(-1)^n}{\sqrt{n}} \text{ for } n \geq 1.$$

- The series $\sum a_n$ is an alternating series, and by Theorem 6.4.2, it converges.
- For every $n \geq 1$, we have

$$(1 + a_{2n})(1 + a_{2n+1}) = \left(1 + \frac{1}{\sqrt{2n}}\right)\left(1 - \frac{1}{\sqrt{2n+1}}\right) = 1 - \frac{1}{2n} + o\left(\frac{1}{n}\right), \text{ when } n \rightarrow \infty.$$

Since $\sum \frac{1}{n}$ diverges, the infinite product $\prod(1 + a_{2n})(1 + a_{2n+1})$ also diverges.

Definition 6.8.6 : Let $(a_n)_{n \geq 1}$ be a complex-valued nonzero sequence. We say that the infinite product $\prod(1 + a_n)$ converges absolutely if $\prod(1 + |a_n|)$ converges.

Theorem 6.8.7 : Let $(a_n)_{n \geq 1}$ be a complex-valued nonzero sequence. If the infinite product $\prod(1 + a_n)$ converges absolutely, then it converges.

Proof : Let us check Cauchy's condition provided in Proposition 6.8.3. For every $n, k \geq 1$, by the triangle inequality, we have

$$\left| \prod_{j=1}^k (1 + a_{n+j}) - 1 \right| \leq \prod_{j=1}^k (1 + |a_{n+j}|) - 1.$$

Therefore, if the infinite product $\prod(1 + |a_n|)$ satisfies the Cauchy's condition, so does $\prod(1 + a_n)$. $\square$

6.8.2 Application to the Riemann zeta function

Let the sequence $(p_k)_{k \geq 1}$ be given by ordered prime numbers, that is $p_1 = 2, p_2 = 3, p_3 = 5$, etc.

Theorem 6.8.8 (Euler's product) : For $s > 1$, we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}}.$$

(2) 當 $(a_n)_{n \geq 1}$ 是個一般項嚴格為負的序列時，相同的敘述也會成立。

(3) 假設序列 $(a_n)_{n \geq 1}$ 的正負號不會變化是很重要的。例如，我們可以考慮 $a_n = \frac{(-1)^n}{\sqrt{n}}$ 對於 $n \geq 1$ 。

- $\sum a_n$ 是個交錯級數，因此根據定理 6.4.2，他會收斂。
- 對於每個 $n \geq 1$ ，我們有

$$(1 + a_{2n})(1 + a_{2n+1}) = \left(1 + \frac{1}{\sqrt{2n}}\right)\left(1 - \frac{1}{\sqrt{2n+1}}\right) = 1 - \frac{1}{2n} + o\left(\frac{1}{n}\right), \text{ 當 } n \rightarrow \infty.$$

由於 $\sum \frac{1}{n}$ 發散，無窮積 $\prod(1 + a_{2n})(1 + a_{2n+1})$ 也會發散。

定義 6.8.6 : 令 $(a_n)_{n \geq 1}$ 為複數非零序列。如果 $\prod(1 + |a_n|)$ 收斂，則我們說無窮積 $\prod(1 + a_n)$ 絕對收斂。

定理 6.8.7 : 令 $(a_n)_{n \geq 1}$ 為複數非零序列。如果無窮積 $\prod(1 + a_n)$ 絕對收斂，那麼他會收斂。

證明 : 讓我們來檢查命題 6.8.3 所提供的柯西條件。對於每個 $n, k \geq 1$ ，根據三角不等式，我們有

$$\left| \prod_{j=1}^k (1 + a_{n+j}) - 1 \right| \leq \prod_{j=1}^k (1 + |a_{n+j}|) - 1.$$

因此，如果無窮積 $\prod(1 + |a_n|)$ 滿足柯西條件， $\prod(1 + a_n)$ 也會。 $\square$

第二小節 在黎曼 ζ 函數上的應用

令 $(p_k)_{k \geq 1}$ 為由排序好的質數所定義出來的序列，也就是說 $p_1 = 2, p_2 = 3, p_3 = 5$ 等等。

定理 6.8.8 【尤拉積】 : 對於 $s > 1$ ，我們有

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{k=1}^{\infty} \frac{1}{1 - p_k^{-s}}.$$

Moreover, the above infinite product converges absolutely.

Proof : For $n \geq 1$, let us write the n -th partial product to be

$$P_n = \prod_{k=1}^n \frac{1}{1 - p_k^{-s}}. \quad (6.20)$$

Our goal is to show that $P_n \xrightarrow{n \rightarrow \infty} \zeta(s)$.

Let us fix an integer $n \geq 1$. We may expand each factor in the right-hand side of Eq. (6.20) into a series, that is

$$\forall k \geq 1, \quad \frac{1}{1 - p_k^{-s}} = \sum_{m=0}^{\infty} \frac{1}{p_k^{ms}} = 1 + \sum_{m=1}^{\infty} \frac{1}{p_k^{ms}} \quad (6.21)$$

and

$$P_n = \prod_{k=1}^n \sum_{m=0}^{\infty} \frac{1}{p_k^{ms}} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{1}{p_1^{m_1 s} \cdots p_n^{m_n s}}.$$

For every $n \geq 1$, let

$$A_n = \{N \in \mathbb{N} : N \text{ has all its prime factors among } p_1, \dots, p_n\}.$$

Due to the uniqueness of prime factorization, we know that

$$P_n = \sum_{N \in A_n} \frac{1}{N^s}.$$

Therefore,

$$|P_n - \zeta(s)| \leq \sum_{N \geq p_{n+1}} \frac{1}{N^s}, \quad (6.22)$$

because all the terms defining the series $\zeta(s)$ are all positive. Since $\sum \frac{1}{N^s}$ converges, its remainder goes to zero, so the right-hand side of Eq. (6.22) goes to zero, that is $P_n \xrightarrow{n \rightarrow \infty} \zeta(s)$.

For every $k \geq 1$, we may rewrite the series in Eq. (6.21) as $1 + a_k$. The series $\sum a_k$ converges absolutely because all its terms are positive, and is bounded from above by $\zeta(s)$. Then, it follows from Theorem 6.8.4 that the infinite product $\prod(1 + a_k)$ converges absolutely. $\square$

此外，上式中的無窮積會絕對收斂。

證明：對於 $n \geq 1$ ，我們把第 n 個部份積記作

$$P_n = \prod_{k=1}^n \frac{1}{1 - p_k^{-s}}. \quad (6.20)$$

我們的目的是證明 $P_n \xrightarrow{n \rightarrow \infty} \zeta(s)$ 。

讓我們固定整數 $n \geq 1$ 。我們可以把式 (6.20) 右式中的每一個項展開為級數，也就是說：

$$\forall k \geq 1, \quad \frac{1}{1 - p_k^{-s}} = \sum_{m=0}^{\infty} \frac{1}{p_k^{ms}} = 1 + \sum_{m=1}^{\infty} \frac{1}{p_k^{ms}} \quad (6.21)$$

還有

$$P_n = \prod_{k=1}^n \sum_{m=0}^{\infty} \frac{1}{p_k^{ms}} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{1}{p_1^{m_1 s} \cdots p_n^{m_n s}}.$$

對於每個 $n \geq 1$ ，令

$$A_n = \{N \in \mathbb{N} : N \text{ 所有的質因數都在 } p_1, \dots, p_n \text{ 當中}\}.$$

根據質數分解定理的唯一性，我們知道

$$P_n = \sum_{N \in A_n} \frac{1}{N^s}.$$

因此

$$|P_n - \zeta(s)| \leq \sum_{N \geq p_{n+1}} \frac{1}{N^s}, \quad (6.22)$$

這是因為定義級數 $\zeta(s)$ 當中的每一項皆是正的。由於 $\sum \frac{1}{N^s}$ 收斂，他的餘項會趨近於零，所以式 (6.22) 的右方趨近於零，也就是 $P_n \xrightarrow{n \rightarrow \infty} \zeta(s)$ 。

對於每個 $k \geq 1$ ，我們可以把式 (6.21) 中的級數改寫為 $1 + a_k$ 。級數 $\sum a_k$ 會絕對收斂，因為他所有的項皆為正，而且 $\zeta(s)$ 是他一個上界。最後，我們使用定理 6.8.4 來推得無窮積 $\prod(1 + a_k)$ 會絕對收斂。 $\square$