6 Sequences and series

In the first-year calculus, we have discussed sequences and series with values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We have seen different criteria to describe their convergence. In the first term of this year, we have studied the notion of convergence of sequences in more general spaces, such as metric spaces. But if we want to discuss series, since an addition operation is needed, to make a more general theory, we will restrict ourselves to normed vector spaces. In this chapter, the sequences and series are considered to take their values in a normed vector space $(W, \|\cdot\|_W)$.

6.1 Basic notions

6.1.1 Reminders for real sequences

Definition 6.1.1 : Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers.

- We say that $(a_n)_{n\geqslant 1}$ converges to $\ell \in \mathbb{R}$, denoted $a_n \xrightarrow[n\to\infty]{} \ell$, if for every $\varepsilon > 0$, there exists $N \geqslant 1$ such that
$$\forall n \geqslant N, \quad |a_n - \ell| < \varepsilon.$$
- (Cauchy's condition) The above definition is equivalent to the following: for every $\varepsilon > 0$, there exists $N \geqslant 1$ such that
$$\forall m, n \geqslant N, \quad |a_m - a_n| < \varepsilon.$$

Proposition 6.1.2 : *Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers.*

(1) If $(a_n)_{n\geqslant 1}$ is non-decreasing and is bounded from above by some $M < \infty$, then $(a_n)_{n\geqslant 1}$ converges to a limit $\ell \leqslant M$.

(2) If $(a_n)_{n\geqslant 1}$ is non-increasing and is bounded from below by some $M > -\infty$, then $(a_n)_{n\geqslant 1}$ converges to a limit $\ell \geqslant M$.

Definition 6.1.3 : Given two sequences $a = (a_n)_{n\geqslant 1}$ and $b = (b_n)_{n\geqslant 1}$ of real numbers. We say that they are *adjacent* (相伴序列) if one is increasing, the other one is decreasing, with $a_n - b_n \xrightarrow[n\to\infty]{} 0$.

Proposition 6.1.4 : *If the sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ are adjacent, then they converge to the same limit.*

Definition 6.1.5 : Given two sequences $a = (a_n)_{n\geqslant 1}$ and $b = (b_n)_{n\geqslant 1}$ of real numbers. We define the following asymptotic relations.

(1) We say taht a is *dominated* by b, denoted $a_n = \mathcal{O}(b_n)$, if there exists a bounded sequence $c = (c_n)_{n\geqslant 1}$ and $N \in \mathbb{N}$ such that $a_n = c_n b_n$ for all $n \geqslant N$.

(2) We say that a is *negligible* compared to b, denoted $a_n = o(b_n)$, if there exists a sequence $\varepsilon = (\varepsilon_n)_{n\geqslant 1}$ that converges to 0 and $N \in \mathbb{N}$ such that $a_n = \varepsilon_n b_n$ for all $n \geqslant N$.

(3) We say that a is *equivalent* to b, denoted $a_n \sim b_n$, if there exists a sequence $c = (c_n)_{n\geqslant 1}$ that converges to 1 and $N \in \mathbb{N}$ such that $a_n = c_n b_n$ for all $n \geqslant N$.

Remark 6.1.6 :

(1) When we write these relations between a and b, we may add the condition $n \to \infty$ to emphasize that in the asymptotic relation, we are taking n to infinity (not some other value), or if there are other variables that might bring confusion.

(2) It can be checked that the binary relation $\sim$ is an equivalence relation on the space of real-valued sequences $\mathbb{R}^{\mathbb{N}}$. However, the asymptotic notations $\mathcal{O}$ and o do not satisfy symmetry.

Example 6.1.7 :

(1) Let $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ be defined by

$$\forall n \geqslant 1, \quad a_n = \frac{1}{n} \quad \text{and} \quad b_n = \frac{1}{n} + \frac{1}{n^2}.$$

Then, $a_n = \mathcal{O}(b_n)$ and $a_n \sim b_n$.

(2) Let $(a_n)_{n\geqslant 1} = (0, 1, 1, \dots)$ and $(b_n)_{n\geqslant 1} = (1, 1, 1, \dots)$. Then, $a_n = \mathcal{O}(b_n)$ and $a_n \sim b_n$.

(3) Let $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ be defined by

$$\forall n \geqslant 1, \quad a_n = n^2 \quad \text{and} \quad b_n = 2^n.$$

Then, $a_n = o(b_n)$ and $a_n = \mathcal{O}(b_n)$.

6.1.2 Definitions

Let $(u_n)_{n\geqslant 1}$ be a sequence with values in a normed vector space $(W, \|\cdot\|)$.

Definition 6.1.8 :

- The *series* (級數) with general term u_n is given by the sequence $(S_n)_{n\geqslant 0}$, defined by

$$S_0 = 0, \quad S_n = u_1 + \cdots + u_n = \sum_{k=1}^{n} u_k, \quad \forall n \geqslant 1.$$

We may also denote this series by $\sum_{n \geqslant 1} u_n$ or $\sum u_n$.

- For each $n \geqslant 1$, u_n is called *the n-th term* of the series $\sum u_n$, S_n is called the *n-th partial sum* of the series $\sum u_n$.
- If the sequence $(S_n)_{n \geqslant 0}$ converges in $(W, \|\cdot\|)$, then we say that the series $\sum u_n$ converges. In this case, its limit is called the sum of the series, and is denoted by $\sum_{n=1}^{\infty} u_n$; for each $n \geqslant 1$, we denote by R_n the *n-th remainder*, defined by

$$R_n = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{n} u_k = \sum_{k=n+1}^{\infty} u_k.$$

Remark 6.1.9 : We note that by definition, the convergence of a sequence $(S_n)_{n \geqslant 0}$ is equivalent to the convergence of the series $\sum(S_{n+1} - S_n)$, which are related by the relation

$$\sum_{n=0}^{N-1} (S_{n+1} - S_n) = S_N - S_0 = S_N.$$

Such a summation is called a *telescoping* summation.

Proposition 6.1.10 : *We have the following two properties.*

(1) If the series $\sum u_n$ converges, then $(S_n)_{n \geqslant 0}$ is a Cauchy sequence.

(2) Additionally, if $(W, \|\cdot\|)$ is a Banach space, then the series $\sum u_n$ converges if and only if $(S_n)_{n \geqslant 0}$ is a Cauchy sequence.

Proof : This proposition follows directly from the definition.

(1) The series $\sum u_n$ converges means that the sequence $(S_n)_{n \geqslant 1}$ converges. And it follows from Proposition 2.4.6 that a convergent sequence is a Cauchy sequence.

(2) It remains to show the converse. Suppose that $(S_n)_{n \geqslant 0}$ is a Cauchy sequence, since it takes value in a Banach space, it converges, so the series $\sum u_n$ is convergent.

$\square$

Corollary 6.1.11 (Cauchy's condition) **:** *Suppose that $(W, \|\cdot\|)$ is a Banach space. The series $\sum u_n$ converges if and only if for every $\varepsilon > 0$, there exists $N \geqslant 1$ such that*

$$\forall n \geqslant N, \forall k \geqslant 1, \quad \|u_{n+1} + \cdots + u_{n+k}\| < \varepsilon. \tag{6.1}$$

This condition is called Cauchy's condition **(柯西條件)** .

Proof : It is a direct consequence of Proposition 6.1.10 (2). □

Corollary 6.1.12 : *If* $\sum u_n$ *is a convergent series, then* $\lim_{n\to\infty} u_n = 0$.

Proof : It is a direct consequence of Cauchy's condition by taking the special case $k = 1$ in Eq. (6.1). □

Remark 6.1.13 : We note that the convergence to zero of the general term is *necessary but not sufficient* for a series to converge. For example, the harmonic series $\sum \frac{1}{n}$ diverges, but its general term tends to 0.

Definition 6.1.14 : Suppose that $(W, \|\cdot\|)$ is a Banach space, and let $\sum u_n$ be a series with general terms in W.

- If the series $\sum \|u_n\|$ converges, we say that the series $\sum u_n$ *converges absolutely* (絕對收斂) .
- If the series $\sum u_n$ converges but does not converge absolutely, then we say that $\sum u_n$ *converges conditionally* (條件收斂) .

Example 6.1.15 : The series $\sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} = \ln 2$ is convergent but not absolutely convergent. We will have a more thorough study of such series, called *alternating series*, in Section 6.4.1.

Theorem 6.1.16 : *Suppose that* $(W, \|\cdot\|)$ *is a Banach space. A series* $\sum u_n$ *that converges absolutely in* W *also converges.*

Proof : For every $n, k \geqslant 1$, we have

$$\|u_{n+1} + \cdots + u_{n+k}\| \leqslant \|u_{n+1}\| + \cdots + \|u_{n+k}\| .$$

Thus, Cauchy's condition for $\sum \|u_n\|$ implies Cauchy's condition for $\sum u_n$. □

Remark 6.1.17 : From the above theorem, to show that a series converges in a Banach space, we may show that it converges absolutely, which reduces to the convergence of a series with non-negative terms. This is the reason why understanding the behavior of a series with non-negative terms is important. In the next subsection, we are going to study sufficient conditions for such series to converge.

6.2 Series with non-negative terms

6.2.1 Comparison between series

Proposition 6.2.1 : *Let $\sum u_n$ be a series wtih non-negative terms. It converges if and only if the sequence $(S_n)_{n\geqslant 0}$ of partial sums is bounded from above.*

Proof : It is a direct consequence of Proposition 6.1.2. □

Proposition 6.2.2 (Comparison test) **:** *We consider two non-negative series $\sum u_n$ and $\sum v_n$ satisfying*

$$\forall n \geqslant 1, \quad 0 \leqslant u_n \leqslant v_n.$$

(1) If $\sum v_n$ converges, then $\sum u_n$ converges.

(2) If $\sum u_n$ diverges, then $\sum v_n$ diverges.

Proof : Let $(S_n)_{n\geqslant 0}$ be the partial sums of $\sum u_n$ and $(T_n)_{n\geqslant 0}$ be the partial sums of $\sum v_n$. Then, for every $n \geqslant 0$, we have $S_n \leqslant T_n$. We conclude by Proposition 6.1.2. □

Theorem 6.2.3 : *Let $\sum u_n$ and $\sum v_n$ be two series with non-negative terms.*

(1) If $v_n = \mathcal{O}(u_n)$ and $\sum u_n$ converges, then $\sum v_n$ converges.

(2) If $u_n \sim v_n$, then the series $\sum u_n$ and $\sum v_n$ are of the same behavior (i.e. both divergent or convergent).

Proof :

(1) Suppose that $v_n = \mathcal{O}(u_n)$. Let $M > 0$ and $N \geqslant 1$ such that $v_n \leqslant M u_n$ for all $n \geqslant N$. This means that for $n \geqslant N$, we have

$$\sum_{k=1}^{n} v_k = \sum_{k=1}^{N-1} v_k + \sum_{k=N}^{n} v_k \leqslant \sum_{k=1}^{N-1} v_k + M \sum_{k=N}^{n} u_k.$$

Since $\sum u_n$ converges, the sequence $(\sum_{k=N}^{n} u_k)_{n\geqslant N}$ is bounded from above. Therefore, the series $\sum v_n$ converges.

(2) If $u_n \sim v_n$, it means that $u_n = \mathcal{O}(v_n)$ and $v_n = \mathcal{O}(u_n)$. So (1) implies that $\sum u_n$ converges if and only if $\sum v_n$ converges.

□

Remark 6.2.4 : We note that to apply Theorem 6.2.3, the assumption on non-negative terms is essential.

(1) For $n \geqslant 1$, let $u_n = \frac{(-1)^n}{n}$ and $v_n = \frac{1}{n}$. It is clear that $v_n = \mathcal{O}(u_n)$. However, $\sum u_n$ converges by Theorem 6.4.2, but $\sum v_n$ diverges from Proposition 6.2.6.

(2) For $n \geqslant 1$, let $u_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}$ and $v_n = \frac{(-1)^n}{\sqrt{n}}$. It is clear that $u_n \sim v_n$. However, $\sum v_n$ converges by

Theorem 6.4.2, but $\sum u_n$ diverges because $\sum \frac{1}{n}$ diverges.

Example 6.2.5 : Let us study the behavior of the series $\sum \frac{1}{n^2}$. First, we note that

$$\forall k \geqslant 2, \quad \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \leqslant \frac{1}{k^2} \leqslant \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

Since the telescoping series $\sum(\frac{1}{n-1} - \frac{1}{n})$ converges, it follows from Proposition 6.2.2 (1) that the series $\sum \frac{1}{n^2}$ also converges. Moreover, for $n \geqslant 2$, the following relation holds for the n-th remainder,

$$\frac{1}{n+1} \leqslant R_n := \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leqslant \frac{1}{n}.$$

By Cauchy's condition, the series $\sum \frac{1}{n^2}$ converges, and its n-th remainder is equivalent to $\frac{1}{n}$.
Another way to show the convergence is to note that $\frac{1}{n^2} \sim \frac{1}{n-1} - \frac{1}{n}$ and apply Theorem 6.2.3 (2). But to find an asymptotic formula for the n-th remainder, we need to apply Theorem 6.2.8 that we will see at a later stage.

Proposition 6.2.6 (Riemann series) **:** *Let α be a real number. The Riemann series $\sum \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.*

Proof : For real numbers $\alpha > \beta$, we have $\frac{1}{n^\alpha} < \frac{1}{n^\beta}$ for all $n \geqslant 1$. By Proposition 6.2.2, it is sufficient to show that $\sum \frac{1}{n}$ diverges, and for any $\alpha > 1$, the series $\sum \frac{1}{n^\alpha}$ converges.

- Let $\alpha = 1$. For every $k \geqslant 1$, we have

$$\frac{1}{k} = \int_k^{k+1} \frac{\mathrm{d}t}{k} \geqslant \int_k^{k+1} \frac{\mathrm{d}t}{t} = \ln(k+1) - \ln k.$$

This implies that

$$\forall n \geqslant 1, \quad \sum_{k=1}^{n} \frac{1}{k} \geqslant \sum_{k=1}^{n} (\ln(k+1) - \ln k) = \ln(n+1).$$

Since $\ln(n+1) \xrightarrow[n\to\infty]{} +\infty$, we deduce from Proposition 6.2.2 (2) that the series $\sum \frac{1}{n}$ diverges.

- Let $\alpha > 1$. Similarly, For every $k \geqslant 2$, we have

$$\frac{1}{k^\alpha} = \int_{k-1}^{k} \frac{\mathrm{d}t}{k^\alpha} \leqslant \int_{k-1}^{k} \frac{\mathrm{d}t}{t^\alpha} = \frac{1}{1-\alpha}\left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}}\right]. \tag{6.2}$$

This gives that

$$\forall n \geqslant 2, \quad \sum_{k=2}^{n} \frac{1}{k^\alpha} \leqslant \frac{1}{1-\alpha} \sum_{k=2}^{n} \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}}\right] = \frac{1}{\alpha-1}\left[1 - \frac{1}{n^{\alpha-1}}\right] \leqslant \frac{1}{\alpha-1}.$$

So the series $\sum \frac{1}{n^\alpha}$ converges when $\alpha > 1$.

$\square$

Remark 6.2.7 : For a fixed $\alpha > 1$, the Riemann series $\sum \frac{1}{n^\alpha}$ is convergent by Proposition 6.2.6. We may find an asymptotic expression for its remainder. For $k \geqslant 1$, we have

$$\frac{1}{k^\alpha} \geqslant \int_k^{k+1} \frac{\mathrm{d}t}{t^\alpha} = \frac{1}{1-\alpha}\left[\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{k^{\alpha-1}}\right].$$

This gives that

$$\forall n \geqslant 1, \quad \sum_{k=n}^{\infty} \frac{1}{k^\alpha} \geqslant \frac{1}{1-\alpha} \sum_{k=n}^{\infty} \left[\frac{1}{(k+1)^{\alpha-1}} - \frac{1}{k^{\alpha-1}}\right] = \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}.$$

Similarly, from Eq. (6.2), we find,

$$\forall n \geqslant 1, \quad \sum_{k=n}^{\infty} \frac{1}{k^\alpha} \leqslant \frac{1}{n^\alpha} + \frac{1}{1-\alpha} \sum_{k=n+1}^{\infty} \left[\frac{1}{k^{\alpha-1}} - \frac{1}{(k-1)^{\alpha-1}}\right] = \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}} + \frac{1}{n^\alpha}.$$

The above two relations imply that

$$\sum_{k=n}^{\infty} \frac{1}{k^\alpha} = \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}} + \mathcal{O}\Big(\frac{1}{n^\alpha}\Big), \quad \text{when } n \to \infty. \tag{6.3}$$

6.2.2 Partial sums and remainders

Theorem 6.2.8 : *Let $\sum u_n$ and $\sum v_n$ are two series with non-negative terms such that $u_n \sim v_n$. Then, the following properties hold.*

(1) If $\sum u_n$ converges, then $\sum v_n$ converges and their remainders satisfy

$$\sum_{k=n}^{\infty} u_k \sim \sum_{k=n}^{\infty} v_k, \quad n \to \infty. \tag{6.4}$$

(2) If $\sum u_n$ diverges, $\sum v_n$ diverges and their partial sums satisfy

$$\sum_{k=1}^{n} u_k \sim \sum_{k=1}^{n} v_k, \quad n \to \infty. \tag{6.5}$$

Proof : By Theorem 6.2.3, we already know that $\sum u_n$ and $\sum v_n$ have the same behavior.

(1) Let $\varepsilon > 0$. It follows from the equivalence $u_n \sim v_n$ that we may find $N \geqslant 1$ such that

$$\forall k \geqslant N, \quad (1-\varepsilon)u_k \leqslant v_k \leqslant (1+\varepsilon)u_k. \tag{6.6}$$

By taking a summation over $k \geqslant n$ with $n \geqslant N$, we find

$$\forall n \geqslant N, \quad (1-\varepsilon)\sum_{k=n}^{\infty} u_k \leqslant \sum_{k=n}^{\infty} v_k \leqslant (1+\varepsilon)\sum_{k=n}^{\infty} u_k.$$

This is exactly what Eq. (6.4) means.

(2) Let $\varepsilon > 0$. As above, we may find $N \geqslant 1$ such that Eq. (6.6) holds. Then, for any $n \geqslant N$, we have

$$\sum_{k=1}^{N-1} v_k + (1-\varepsilon)\sum_{k=N}^{n} u_k \leqslant \sum_{k=1}^{n} v_k \leqslant \sum_{k=1}^{N-1} v_k + (1+\varepsilon)\sum_{k=N}^{n} u_k. \tag{6.7}$$

We use the fact that the general terms v_n are non-negative and the series $\sum u_k$ diverges, we may find $N' \geqslant N$ such that the two following inequalities hold,

$$\sum_{k=1}^{N-1} v_k \leqslant \varepsilon \sum_{k=N}^{n} u_k, \quad \forall n \geqslant N', \tag{6.8}$$

$$(1-2\varepsilon)\sum_{k=1}^{N-1} u_k - \sum_{k=1}^{N-1} v_k \leqslant \varepsilon \sum_{k=N}^{n} u_k, \quad \forall n \geqslant N'. \tag{6.9}$$

If we use Eq. (6.8) to the right side of Eq. (6.7), we find

$$\sum_{k=1}^{n} v_k \leqslant (1+2\varepsilon)\sum_{k=N}^{n} u_k \leqslant (1+2\varepsilon)\sum_{k=1}^{n} u_k,$$

where the last inequality follows because the general terms u_n are non-negative. Similarly, if we use Eq. (6.9) to the left side of Eq. (6.7), we find

$$\sum_{k=1}^{n} v_k \geqslant (1-2\varepsilon)\sum_{k=1}^{N-1} u_k - \varepsilon\sum_{k=N}^{n} u_k + (1-\varepsilon)\sum_{k=N}^{n} u_k = (1-2\varepsilon)\sum_{k=1}^{n} u_k.$$

This is exactly what we need to show for Eq. (6.5).

□

The above result has useful applications when it comes to asymptotic expansion of sequences or series. We are going to look at an important example below.

Example 6.2.9 : The sequence $(H_n)_{n \geqslant 1}$ of harmonic numbers is defined by

$$\forall n \geqslant 1, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(1) We first note that when $n \to \infty$, we have the following equivalence,

$$\frac{1}{n} \sim \ln\left(1 + \frac{1}{n}\right).$$

Since both series $\sum \frac{1}{n}$ and $\sum \ln(1 + \frac{1}{n})$ are with non-negative terms, it follows from

Theorem 6.2.3 (2) that they are of the same behavior. It is not hard to see that

$$\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n} \left[\ln(k+1) - \ln(k)\right] = \ln(n+1)$$

diverges when $n \to \infty$, so we deduce that $\sum \frac{1}{n}$ also diverges. Moreover, it follows from Theorem 6.2.8 (2) that their partial sums are equivalent. In other words, for $n \to \infty$, we have

$$H_n \sim \sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \ln(n+1) \sim \ln n.$$

This gives the first term in the asymptotic expansion of the harmonic numbers.

(2) To get the following terms in the asymptotic expansion of $(H_n)_{n\geqslant 1}$, let us consider the sequence $(A_n)_{n\geqslant 1}$ defined by $A_n = H_n - \ln n$ for $n \geqslant 1$. Then, for $n \geqslant 2$, we may write

$$A_n - A_{n-1} = \frac{1}{n} - \ln n + \ln(n-1) = \frac{1}{n} + \ln\left(1-\frac{1}{n}\right) \sim -\frac{1}{2n^2}. \tag{6.10}$$

By the result on the Riemann series in Proposition 6.2.6, we know that the series $\sum \frac{1}{n^2}$ converges, and again by Theorem 6.2.3 (2), we know that the series $\sum(A_n - A_{n-1})$ converges. Additionally, since

$$\forall n \geqslant 2, \quad \sum_{k=2}^{n}(A_k - A_{k-1}) = A_n - A_1,$$

we deduce that the sequence $(A_n)_{n\geqslant 1}$ converges. Let us define $\gamma := \lim_{n\to\infty} A_n$, called *Euler's constant*, and we have

$$H_n = \ln n + A_n = \ln n + \gamma + o(1), \quad \text{when } n \to \infty.$$

(3) Following the computations in (2), due to the equivalence in Eq. (6.10) and Theorem 6.2.8 (1), we know that the following equivalence holds,

$$\gamma - A_n = \sum_{k=n+1}^{\infty} (A_k - A_{k-1}) \sim -\frac{1}{2} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \sim -\frac{1}{2n}.$$

where the last equivalence comes from Example 6.2.5. This gives the asymptotic expansion of H_n below

$$H_n = \ln n + \gamma + \frac{1}{2n} + o\Big(\frac{1}{n}\Big), \quad \text{when } n \to \infty.$$

(4) We may go further in the above asymptotic expansion. Let us consider the sequence $(D_n)_{n\geqslant 1}$ defined by

$$D_n = H_n - \ln n - \gamma - \frac{1}{2n} = A_n - \gamma - \frac{1}{2n}, \quad \forall n \geqslant 1.$$

Then, when $n \to \infty$, we have

$$
\begin{aligned}
D_n - D_{n-1} &= \frac{1}{n} + \ln\left(1 - \frac{1}{n}\right) + \frac{1}{2(n-1)} - \frac{1}{2n} \\
&= \frac{1}{n} - \Big(\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + o\Big(\frac{1}{n^3}\Big)\Big) + \frac{1}{2n}\Big(1 + \frac{1}{n} + \frac{1}{n^2} + o\Big(\frac{1}{n^2}\Big)\Big) - \frac{1}{2n} \\
&= \frac{1}{6n^3} + o\Big(\frac{1}{n^3}\Big).
\end{aligned}
$$

Since the Riemann series $\sum \frac{1}{n^3}$ converges, we know that the series $\sum(D_n - D_{n-1})$ also converges, and when $n \to \infty$, we have

$$\sum_{k=n+1}^{\infty} (D_k - D_{k-1}) \sim \frac{1}{6} \sum_{k=n+1}^{\infty} \frac{1}{k^3}.$$

By applying Eq. (6.3), we deduce that $\sum_{k=n+1}^{\infty} \frac{1}{k^3} \sim \frac{1}{2n^2}$, and the left-hand side is equal to $\lim_{k\to\infty} D_k - D_n = -D_n$. This allows us to obtain that $D_n \sim -\frac{1}{12}\frac{1}{n^2}$ when $n \to \infty$. The asymptotic expansion of the harmonic numbers writes

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12}\frac{1}{n^2} + o\Big(\frac{1}{n^2}\Big), \quad \text{when } n \to \infty.$$

(5) You may repeat the above procedure to find the following asymptotic expansion

$$H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{k\geqslant 1}^{N} \frac{B_{2k}}{2kn^{2k}} + o\Big(\frac{1}{n^{2N}}\Big), \quad \text{when } n \to \infty,$$

where $(B_{2k})_{k\geqslant 1}$ are Bernoulli numbers whose first terms are given by $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, etc.

Remark 6.2.10 : In Exercise 6.12, you may apply the same method as in Example 6.2.9 to deduce an asymptotic formula for $n!$, called *Stirling's formula*.

6.2.3 Comparison between series and integrals

Integrals and series are closely related: a series is a discrete summation, whereas an integral is the limit of such discrete summations. When a series converges, it is usually not trivial to get an exact formula or value for its limit; however, there are a lot of functions who have a nice primitive that we can compute (Appendix 1). In this subsection, we introduce a method allowing us to study the behavior of a series with the help of integrals. The idea behind is similar to what we have done in the proof of Proposition 6.2.6, but here we give a more general setting and a more precise result for such a method.

Proposition 6.2.11 : *Let $f : [1, +\infty) \to \mathbb{R}_+$ be a non-increasing function with $\lim_{x\to\infty} f(x) = 0$. For* *every integer $n \geqslant 1$, let us define*

$$S_n = \sum_{k=1}^{n} f(k), \quad I_n = \int_1^n f(t)\,\mathrm{d}t, \quad D_n = S_n - I_n.$$

Then, the following properties hold.

(1) For $n \geqslant 1$, we have $0 \leqslant f(n+1) \leqslant D_{n+1} \leqslant D_n \leqslant f(1)$.

(2) The sequence $(D_n)_{n\geqslant 1}$ converges, and denote $D := \lim_{n\to\infty} D_n$.

(3) The series $\sum f(n)$ and the integral $\int_1^\infty f(t)\,\mathrm{d}t := \lim_{x\to\infty} \int_1^x f(t)\,\mathrm{d}t$ have the same behavior, that is both are either convergent or divergent.

(4) For $n \geqslant 1$, we have $0 \leqslant D_n - D \leqslant f(n)$.

Proof :

(1) Since f is non-increasing, we find

$$\forall k \geqslant 0, \quad f(k+1) \leqslant \int_k^{k+1} f(t)\,\mathrm{d}t \leqslant f(k).$$

Let us fix $n \geqslant 1$. We have

$$I_{n+1} = \int_1^{n+1} f(t)\,\mathrm{d}t = \sum_{k=1}^{n} \int_k^{k+1} f(t)\,\mathrm{d}t \leqslant \sum_{k=1}^{n} f(k) = S_n$$

Therefore, $f(n+1) = S_{n+1} - S_n \leqslant S_{n+1} - I_{n+1} = D_{n+1}$, which shows the first part of the inequality.

Next, we write

$$D_n - D_{n+1} = \int_n^{n+1} f(t)\,\mathrm{d}t - f(n+1) \geqslant 0,$$

This shows that $D_{n+1} \leqslant D_n$. To conclude, we note that $D_1 = f(1)$, and by induction, we find $D_n \leqslant D_1 = f(1)$ for all $n \geqslant 1$.

(2) From (1), we know that $(D_n)_{n\geqslant 1}$ is a non-negative sequence bounded from below by 0, so it converges.

(3) From (2), we know that $\lim_{n\to\infty}(S_n - I_n)$ exists. Therefore, both sequences $(S_n)_{n\geqslant 1}$ and $(I_n)_{n\geqslant 1}$ have the same behavior.

(4) For $n \geqslant 1$, we write the telescoping summation

$$D_n - D = \lim_{N\to\infty}(D_n - D_N) = \lim_{N\to\infty} \sum_{k=n}^{N-1} (D_k - D_{k+1}).$$

From (1), we know that $D_k - D_{k+1} \geqslant 0$ for all $k \geqslant 1$, so $D_n - D \geqslant 0$. For every $k \geqslant 1$, we also have

$$D_k - D_{k+1} = \int_k^{k+1} f(t)\,\mathrm{d}t - f(k+1) \leqslant f(k) - f(k+1),$$

so

$$D_n - D \leqslant \lim_{N\to\infty} \sum_{k=n}^{N-1} (f(k) - f(k+1)) = \lim_{N\to\infty} (f(n) - f(N)) = f(n).$$

□

Remark 6.2.12 :

(1) In Proposition 6.2.11, if f is non-increasing on $[M, +\infty)$ for some $M > 0$, then the qualitative statements such as (2) and (3) still hold, whereas the bounds in (1) and (4) need to be adjusted.

(2) From Proposition 6.2.11 (4), we also know that

$$0 \leqslant \sum_{k=1}^{n} f(k) - \int_1^n f(t)\,\mathrm{d}t - D \leqslant f(n).$$

In other words, we have an asymptotic expansion

$$\sum_{k=1}^{n} f(k) = \int_1^n f(t)\,\mathrm{d}t + D + \mathcal{O}(f(n)), \quad \text{when } n \to \infty. \tag{6.11}$$

If we apply this to the function $f(x) = \frac{1}{x}$, then we find

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + D + \mathcal{O}\Big(\frac{1}{n}\Big), \quad \text{when } n \to \infty,$$

which is similar to the result obtained in Example 6.2.9, stronger than (2), but weaker than (3).

Example 6.2.13 : Take $s \in \mathbb{R}$ and $f(x) = x^{-s}$ in Proposition 6.2.11. Along with Proposition 6.2.6, we know that $\sum n^{-s}$ converges if $s > 1$ and diverges if $s \leqslant 1$. For $s > 1$, this series is called the *Riemann zeta function*,

$$\zeta(s) = \sum_{n\geqslant 1} \frac{1}{n^s}.$$

For $s \in (0,1)$, Eq. (6.11) gives us

$$\sum_{k=1}^{n} \frac{1}{k^s} = \frac{n^{1-s} - 1}{1-s} + C(s) + \mathcal{O}\Big(\frac{1}{n^s}\Big), \quad \text{when } n \to \infty,$$

where $C(s)$ is a constant depending on s. The constant term in the above summation is equal to $\zeta(s)$, so the remainder of the series writes,

$$\sum_{k=n+1}^{\infty} \frac{1}{k^s} = \frac{n^{1-s}}{s-1} + \mathcal{O}\Big(\frac{1}{n^s}\Big) \quad \text{when } n \to \infty,$$

which is the result we found in Eq. (6.2).

Proposition 6.2.14 (Bertrand's series) **:** *For $\alpha, \beta \in \mathbb{R}$, the corresponding Bertrand series is given by*

$$\sum_{n \geqslant 2} \frac{1}{n^\alpha (\ln n)^\beta}.$$

(1) When $\alpha > 1$, the Bertrand series converges.

(2) When $\alpha = 1$ and $\beta > 1$, the Bertrand series converges.

(3) Otherwise, the Bertrand series diverges.

Proof : We are going to apply the comparison test (Proposition 6.2.2) and Proposition 6.2.11 to show these properties.

(1) Let $\alpha > 1$. Note that we have the compraison

$$\frac{1}{n^\alpha (\ln n)^\beta} = o\left(\frac{1}{n^{(1+\alpha)/2}}\right) \quad \text{when } n \to \infty.$$

We know that the Riemann series $\sum \frac{1}{n^{(1+\alpha)/2}}$ converges, because $\frac{1+\alpha}{2} > 1$. By comparison (Proposition 6.2.2), we deduce that the series $\sum \frac{1}{n^\alpha (\ln n)^\beta}$ is convergent.

(2) Let $\alpha = 1$ and $\beta > 1$, and let us apply Proposition 6.2.11 to the non-increasing function $f(x) = \frac{1}{x(\ln x)^\beta}$. The integral of f writes as below,

$$\int_2^n f(t)\,\mathrm{d}t = \int_2^n \frac{1}{t(\ln t)^\beta}\,\mathrm{d}t = \int_{\ln 2}^{\ln n} \frac{1}{s^\beta}\,\mathrm{d}s = \frac{1}{1-\beta}\left[\frac{1}{(\ln n)^{\beta-1}} - \frac{1}{(\ln 2)^{\beta-1}}\right].$$

The right hand side of the above integral converges to some finite limit, so the series $\sum \frac{1}{n(\ln n)^\beta}$ is convergent.

(3) Let us first deal with the case $\alpha = \beta = 1$. We apply Proposition 6.2.11 to the non-increasing function $f(x) = \frac{1}{x \ln x}$. The integral of f writes as below,

$$\int_2^n f(t)\,\mathrm{d}t = \int_2^n \frac{1}{t \ln t}\,\mathrm{d}t = \int_{\ln 2}^{\ln n} \frac{1}{s}\,\mathrm{d}s = \ln\ln n - \ln\ln 2.$$

The right hand side of the above integral diverges, so the series $\sum \frac{1}{n \ln n}$ is divergent.

When $\alpha = 1$ and $\beta < 1$, we note that we have

$$\frac{1}{n \ln n} \leqslant \frac{1}{n(\ln n)^\beta}.$$

We conclude by comparison (Proposition 6.2.2) that the series $\sum \frac{1}{n(\ln n)^\beta}$ diverges.

When $\alpha < 1$, we have the relation

$$\frac{1}{n^{(1+\alpha)/2}} = o\left(\frac{1}{n^\alpha (\ln n)^\beta}\right) \quad \text{when } n \to \infty,$$

and the result follows from the divergence of the Riemann series $\sum \frac{1}{n^{(1+\alpha)/2}}$ with $\frac{1+\alpha}{2} < 1$. $\square$

6.3 Tests of convergence

Theorem 6.3.1 (D'Alembert's criterion, ratio test) **:** *Let $(u_n)_{n\geqslant 1}$ be a sequence of real numbers. Suppose that it is strictly positive from a certain index. Additionally, assume that the following limit exists*

$$\ell := \lim_{n\to\infty} \frac{u_{n+1}}{u_n} \in [0, +\infty].$$

Then, the following statements hold.

(1) If $\ell < 1$, then the series $\sum u_n$ is convergent.

(2) If $\ell > 1$, then $u_n \xrightarrow[n\to\infty]{} +\infty$ and the series $\sum u_n$ is divergent.

(3) If $\ell = 1$ and the ratio $\frac{u_{n+1}}{u_n}$ stays above 1 for all large enough n, then the series $\sum u_n$ is divergent.

Remark 6.3.2 : When $\ell = 1$, even if the ratio $\frac{u_{n+1}}{u_n}$ always stays below 1 for all n, d'Alembert's criterion does not allow us to conclude. For example, we may consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. In both cases, the limit is $\ell = 1$, and the ratio $\frac{u_{n+1}}{u_n}$ is smaller than 1 for all $n \geqslant 1$. However, the former series is divergent, and the latter series is convergent.

Proof :

(1) Suppose that $\ell < 1$. Let $N \geqslant 1$ such that for $n \geqslant N$, we have $u_n > 0$ and $\frac{u_{n+1}}{u_n} \leqslant \frac{1+\ell}{2} =: r < 1$. Then, for $n \geqslant N$, we find $u_n \leqslant r^{n-N} u_N$, and

$$\sum_{k=1}^{n} u_k = \sum_{k=1}^{N-1} u_k + \sum_{k=N}^{n} u_k \leqslant \sum_{k=1}^{N-1} u_k + \sum_{k=N}^{n} r^{k-N} u_N.$$

In the above summation, the first term is a constant, and the second term is a geometric series with ratio $r < 1$, so converges.

(2) Suppose that $\ell > 1$. We proceed in a similar way as above, with inequalities reversed. Let $N \geqslant 1$ such that for $n \geqslant N$, we have $u_n > 0$ and $\frac{u_{n+1}}{u_n} \geqslant \frac{1+\ell}{2} =: r > 1$. Then, for $n \geqslant N$, we find $u_n \geqslant r^{n-N} u_N$, and

$$\sum_{k=1}^{n} u_k = \sum_{k=1}^{N-1} u_k + \sum_{k=N}^{n} u_k \geqslant \sum_{k=1}^{N-1} u_k + \sum_{k=N}^{n} r^{k-N} u_N.$$

In the above summation, the first term is a constant, and the second term is a geometric series with ratio $r > 1$, so diverges.

(3) Suppose that $\ell = 1$ and let $N \geqslant 1$ such that $u_n > 0$ and the ratio $\frac{u_{n+1}}{u_n} \geqslant 1$ for all $n \geqslant N$. Then, for all $n \geqslant N$, we have $u_n \geqslant u_N > 0$. Clearly, this implies that the series $\sum u_n$ is divergent. $\square$

Example 6.3.3 : For a given $z \in \mathbb{C}^*$, let us look at the series $\sum \frac{z^n}{n!}$. The ratio of two consecutive terms writes

$$\frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|}{n+1} \xrightarrow[n\to\infty]{} 0 < 1.$$

By the d'Alembert's criterion, the series $\sum \frac{z^n}{n!}$ converges absolutely, so it also converges. Therefore, this series converges for all $z \in \mathbb{C}$. This is a direct consequence of Theorem 6.1.16 if we see $\mathbb{C}$ as a two-dimensional vector space over $\mathbb{R}$.

A simple generalization of Theorem 6.3.1 is stated in the following corollary. To get the absolute convergence of a complex-valued series, we may look at the $\liminf$ and $\limsup$ of the ratio.

Corollary 6.3.4 : *Let $\sum u_n$ be a series with nonzero terms in a Banach space $(W, \|\cdot\|)$. Let*

$$r = \liminf_{n\to\infty} \frac{\|u_{n+1}\|}{\|u_n\|} \quad \textit{and} \quad R = \limsup_{n\to\infty} \frac{\|u_{n+1}\|}{\|u_n\|}.$$

(1) If $R < 1$, then the series $\sum u_n$ converges absolutely.

(2) If $r > 1$, then the series $\sum u_n$ diverges.

(3) If $r \leqslant 1 \leqslant R$, then we cannot conclude.

Remark 6.3.5 : Do not forget that this corollary is useful especially when $(W, \|\cdot\|) = (\mathbb{C}, |\cdot|)$.

Proof : The proof is very similar to that of Theorem 6.3.1. Let us prove it for (1) as an example.

Let $\sum u_n$ be a series with nonzero terms in $(W, \cdot\cdot)$ such that $R < 1$. By the definition of $\limsup$, there exists $N \geqslant 1$ such that

$$\forall n \geqslant N, \quad \frac{\|u_{n+1}\|}{\|u_n\|} \leqslant \frac{1+R}{2} =: x < 1.$$

Then, we may follow the same argument to conclude. □

Theorem 6.3.6 (Cauchy's criterion, root test) **:** *Let $(u_n)_{n\geqslant 1}$ be a sequence of real numbers. Suppose that it is non-negative from a certain index. Additionally, assume that the following limit exists,*

$$\lambda := \lim_{n\to\infty} (u_n)^{1/n} \in [0, +\infty].$$

Then, the following properties hold.

(1) If $\lambda < 1$, then the series $\sum u_n$ is convergent.

(2) If $\lambda > 1$, then the series $\sum u_n$ is divergent.

(3) If $\lambda = 1$, and $(u_n)^{1/n}$ stays above 1 for all large enough n, then the series $\sum u_n$ is divergent.

Proof :

(1) Suppose that $\lambda < 1$. Let $\mu = \frac{1+\lambda}{2} \in (\lambda, 1)$. Take $N \geqslant 1$ such that $u_n > 0$ and $(u_n)^{1/n} \leqslant \mu$ for all $n \geqslant N$. This shows that for any $n \geqslant N$, we have

$$\sum_{k=N}^{n} u_k \leqslant \sum_{k=N}^{n} \mu^k \leqslant \frac{\mu^N}{1-\mu} < \infty.$$

Therefore, the series $\sum_{k \geqslant N} u_k$ converges, and so does the series $\sum_{n \geqslant 1} u_n$.

(2) Suppose that $\lambda > 1$. Let $\mu = \frac{1+\lambda}{2} \in (1, \lambda)$. Take $N \geqslant 1$ such that $(u_n)^{1/n} \geqslant \mu$ for all $n \geqslant N$. This shows that for any $n \geqslant N$, we have

$$\sum_{k=N}^{n} u_k \geqslant \sum_{k=N}^{n} \mu^k \geqslant \mu^n \xrightarrow[n\to\infty]{} +\infty.$$

Clearly, the series $\sum u_n$ is divergent.

(3) Suppose that $\lambda = 1$ and there exists $N \geqslant 1$ such that $(u_n)^{1/n} \geqslant 1$ for all $n \geqslant N$. Let us fix such an $N \geqslant 1$. Then, for all $n \geqslant N$, we also have $u_n \geqslant 1$, so the series $\sum u_n$ is divergent. $\square$

Remark 6.3.7 : Similar to Remark 6.3.2, when $\lambda = 1$ and $(u_n)^{1/n}$ always stays below 1 for all n, Cauchy's criterion does not allow us to conclude. We may again take the same series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ as examples.

Corollary 6.3.8 : *Let $(u_n)_{n \geqslant 1}$ be a sequence in a Banach space $(W, \|\cdot\|)$ and*

$$\lambda := \limsup_{n\to\infty} \|u_n\|^{1/n} \in [0, +\infty].$$

Then, the following properties hold.

(1) If $\lambda < 1$, then the series $\sum u_n$ is absolutely convergent.

(2) If $\lambda > 1$, then the series $\sum u_n$ is divergent.

(3) If $\lambda = 1$, then we cannot conclude.

Proof : The proof is similar to that of Theorem 6.3.6, which is left as an exercise, see Exercise 6.16. $\square$

Remark 6.3.9 : In Exercise 2.31, we saw the following inequality for a sequence $(a_n)_{n \geqslant 1}$ with general strictly positive terms,

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leqslant \liminf_{n\to\infty} (a_n)^{1/n} \leqslant \limsup_{n\to\infty} (a_n)^{1/n} \leqslant \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

This means that the root test is stronger than the ratio test. For example, we may look at the sequence $(a_n)_{n \geqslant 1}$

defined by

$$\forall n \geqslant 1, \quad a_n = (1 + (-1)^n)2^n + 1 = \begin{cases} 2^{n+1} + 1 & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}. \end{cases}.$$

Then, we have

$$0 = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} < \liminf_{n\to\infty} (a_n)^{1/n} = 1 < \limsup_{n\to\infty} (a_n)^{1/n} = 2 < \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty.$$

If we want to apply the ratio test (Corollary 6.3.4), we see that we are in the third scenario; whereas if we apply the root test (Corollary 6.3.8), we are in the second scenario, that is the series $\sum a_n$ is divergent. However, we may note that by applying the ratio test to $\sum a_{2n}$, we find the divergence of the series $\sum a_{2n}$, leading to the divergence of $\sum a_n$.

6.4 Conditionally convergent series

In this section, we still consider series with terms in a Banach space $(W, \|\cdot\|)$, which converge conditionally. We recall that if a series converges but does not converge absolutely, then we say that it converges conditionally, see Definition 6.1.14. We note that the special cases $W = \mathbb{R}$ or $\mathbb{C}$ are useful in practice.

6.4.1 Alternating series

Definition 6.4.1 : Let $\sum u_n$ be a series with terms in $\mathbb{R}$. We say that it is an *alternating series* (交錯級數) if $(-1)^n u_n$ has the same sign for all $n \geqslant 1$. Up to a global sign change, we may rewrite the series $\sum u_n$ as $\sum (-1)^n a_n$, where $a_n \geqslant 0$ for all $n \geqslant 1$.

Theorem 6.4.2 : *Let $(a_n)_{n\geqslant 1}$ be a non-negative sequence. Suppose that it is non-increasing and tends to 0. Then, the alternating series $\sum (-1)^n a_n$ converges, and its remainder satisfies*

$$\forall n \geqslant 1, \quad |R_n| \leqslant a_{n+1}, \quad \textit{where } R_n = \sum_{k=n+1}^{\infty} (-1)^k a_k.$$

Remark 6.4.3 : In Exercise 6.22, you can see that under some additional mild assumptions, we get a finer estimation on the remainder of an alternating series. In particular, you may apply this result to the alternating series $\sum \frac{(-1)^{n+1}}{n}$.

Proof : Since (a_n) is non-increasing, we find, for all $n \geqslant 1$,

$$S_{2n+2} - S_{2n} = a_{2n+2} - a_{2n+1} \leqslant 0 \quad \text{and} \quad S_{2n+1} - S_{2n-1} = a_{2n} - a_{2n+1} \geqslant 0.$$

In other words, the sequence $(S_{2n})_{n\geqslant 1}$ is non-increasing, and the sequence $(S_{2n-1})_{n\geqslant 1}$ is non-decreasing. Since $S_{2n} - S_{2n-1} = a_{2n} \xrightarrow[n\to\infty]{} 0$, the sequences $(S_{2n})_{n\geqslant 1}$ and $(S_{2n-1})_{n\geqslant 1}$ are adjacent. Then, it follows from Proposition 6.1.4 that they converge to the same limit, denoted S. Therefore,

$S_n \xrightarrow[n\to\infty]{} S$ and

$$\forall n \geqslant 1, \quad S_{2n-1} \leqslant S_{2n+1} \leqslant S \leqslant S_{2n}$$

This implies that

$$\forall n \geqslant 1, \quad |R_{2n}| = |S - S_{2n}| \leqslant S_{2n} - S_{2n+1} = a_{2n+1}.$$

Similarly,

$$\forall n \geqslant 1, \quad |R_{2n-1}| = |S - S_{2n-1}| \leqslant S_{2n} - S_{2n-1} = a_{2n}.$$

□

Example 6.4.4 : By Theorem 6.4.2, the series $\sum \frac{(-1)^{n+1}}{n}$ is convergent. Let us compute its sum. By Example 6.2.9, we know that the harmonic numbers have the following asymptotic behavior,

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1), \quad \text{when } n \to \infty.$$

Let us denote the partial sums of the alternating series as below,

$$\forall n \geqslant 1, \quad S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}.$$

Then, for every $n \geqslant 1$, we find

$$H_{2n} - S_{2n} = \sum_{k=1}^{n} \frac{2}{2k} = H_n.$$

In other words, we have the following asymptotic behavior for S_{2n},

$$S_{2n} = H_{2n} - H_n = (\ln(2n) + \gamma + o(1)) - (\ln n + \gamma + o(1)) = \ln 2 + o(1), \quad \text{when } n \to \infty.$$

This means that the series $\sum \frac{(-1)^{n+1}}{n}$ converges to $\ln 2$.

6.4.2 Dirichlet's test

Let us consider a series $\sum u_n$ whose general term can be rewritten as $u_n = a_n b_n$ for $n \geqslant 1$. We write $S_n = \sum_{k=1}^{n} b_k$ for $n \geqslant 1$ and $S_0 = 0$.

Proposition 6.4.5 (Abel's transform) **:** *For every $n \geqslant 0$, we have*

$$\sum_{k=1}^{n} u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n. \tag{6.12}$$

Proof : For every $n \geqslant 0$, we have

$$\begin{aligned}\sum_{k=1}^{n} u_k &= \sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k (S_k - S_{k-1}) \\ &= \sum_{k=1}^{n} a_k S_k - \sum_{k=0}^{n-1} a_{k+1} S_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n.\end{aligned}$$

□

Remark 6.4.6 : This is exactly the integration by parts for the Riemann–Stieltjes integral when the integrator is given by the Gauss function $\lfloor\cdot\rfloor$, see Corollary 5.2.24, and try to compare the two formulas.

Theorem 6.4.7 (Dirichlet's test) **:** *Let $\sum u_n$ be a series with general terms in a Banach space $(W, \|\cdot\|)$. Suppose that its general term u_n writes $u_n = a_n b_n$ with $a_n \in \mathbb{R}$ and $b_n \in W$ for all $n \geqslant 1$, and satisfies*

(i) the sequence $(a_n)_{n\geqslant 1}$ is non-negative, non-increasing, and tends to 0;

(ii) the series $\sum b_n$ is bounded.

Then, the series $\sum u_n$ is convergent.

Remark 6.4.8 : We can make the same observation as in Remark 6.4.6. We have already seen an application of Corollary 5.2.24 to show the convergence of series in Exercise 5.21. This theorem is based on the Abel's transform (Proposition 6.4.5), and the result is exactly the same as in Exercise 5.21.

Proof : Let us apply Abel transform Eq. (6.12) to the series $\sum u_n$. For every $n \geqslant 0$, we have

$$\sum_{k=1}^{n} u_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) S_k + a_n S_n,$$

where S_n is the n-th partial sum of the series $\sum b_n$. Let $M > 0$ such that $|S_n| = |\sum_{k=1}^{n} b_k| \leqslant M$ for all $n \geqslant 1$. Then, we have $|a_n S_n| \leqslant |a_n| M \xrightarrow[n\to\infty]{} 0$, so the series $\sum u_n$ and $\sum (a_n - a_{n+1}) S_n$ have the same behavior. Moreover, for every $k \geqslant 0$, we have

$$|(a_k - a_{k+1}) S_k| \leqslant (a_k - a_{k+1}) M,$$

since $(a_k)_{k\geqslant 1}$ is non-increasing. Thus, for every $n \geqslant 0$, we have

$$\sum_{k=1}^{n} |(a_k - a_{k+1}) S_k| \leqslant \sum_{k=1}^{n} (a_k - a_{k+1}) M = (a_1 - a_{n+1}) M \leqslant a_1 M.$$

This shows that the series $\sum (a_n - a_{n+1}) S_n$ is absolutely convergent, so convergent. □

Example 6.4.9 : Applying Theorem 6.4.7, we obtain the convergence of the following series.

(1) Let $(a_n)_{n\geqslant 0}$ be a non-increasing sequence that tends to 0. The alternating series $\sum(-1)^n a_n$ is convergent because the following partial sum is bounded,

$$\forall n \geqslant 1, \quad |(-1)^1 + (-1)^2 + \cdots + (-1)^n| \leqslant 1.$$

This is exactly the result in the first part of Theorem 6.4.2 for alternating series. However, the Dirichlet's test does not give us any estimate on the remainders of the series.

(2) Let $(a_n)_{n\geqslant 0}$ be a non-increasing sequence that tends to 0. Let $\theta \in \mathbb{R}\backslash 2\pi\mathbb{Z}$. Consider the series $\sum a_n e^{\mathrm{i}\,n\theta}$. For all $n \geqslant 0$, we have

$$\forall n \geqslant 0, \quad |1 + e^{\mathrm{i}\,\theta} + \cdots + e^{\mathrm{i}\,n\theta}| = \left|\frac{1 - e^{\mathrm{i}(n+1)\theta}}{1 - e^{\mathrm{i}\,\theta}}\right| = \left|\frac{\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\right| \leqslant \frac{1}{\left|\sin\left(\frac{\theta}{2}\right)\right|}.$$

Therefore, the series $\sum a_n e^{\mathrm{i}\,n\theta}$ converges if $\theta \in \mathbb{R}\backslash 2\pi\mathbb{Z}$.

6.5 Rearrangement of series

Let $(W, \|\cdot\|)$ be a Banach space and $\sum u_n$ be a series with general terms in W.

Definition 6.5.1 : We say that the series $\sum v_n$ is a *rearrangement* (重新排列) of $\sum u_n$ if there exists a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$v_n = u_{\varphi(n)}, \quad \forall n \in \mathbb{N}.$$

Theorem 6.5.2 : *Suppose that the series $\sum u_n$ is absolutely convergent with sum s. Then, any rearrangement of $\sum u_n$ is also absolutely convergent with sum s.*

Proof : Let $\sum v_n$ be a rearrangement of $\sum u_n$ defined by $v_n = u_{\varphi(n)}$ for all $n \in \mathbb{N}$, where $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijection. We note that for every $n \geqslant 1$, we have

$$\sum_{k=1}^{n} \|v_k\| = \sum_{k=1}^{n} \left\|u_{\varphi(k)}\right\| \leqslant \sum_{k=1}^{\infty} \|u_k\|.$$

The series $\sum \|v_k\|$ has non-negative terms and bounded from above, so it converges, that is $\sum v_k$ converges absolutely.

Let $\varepsilon > 0$. Since $\sum u_n$ converges absolutely, we may find $N \geqslant 1$ such that

$$\sum_{k=N+1}^{\infty} \|u_k\| \leqslant \varepsilon.$$

We write $(S_n)_{n\geqslant 0}$ for the partial sums of $\sum u_n$ and $(T_n)_{n\geqslant 0}$ for the partial sums of $\sum v_n$. Let $M \geqslant 1$ be such that

$$\{1, \ldots, N\} \subseteq \{\varphi(1), \ldots, \varphi(M)\}, \tag{6.13}$$

then for any $n \geqslant M+1$, we have $\varphi(n) \geqslant N+1$. Take $n \geqslant M+1$, we have

$$\|T_n - S_N\| = \left\|\sum_{k=1}^{n} v_k - \sum_{k=1}^{N} u_k\right\| = \left\|\sum_{k=1}^{n} u_{\varphi(k)} - \sum_{k=1}^{N} u_k\right\| \leqslant \sum_{k=N+1}^{\infty} \|u_k\| \leqslant \varepsilon,$$

where in the last equality, we use the inclusion from Eq. (6.13). To conclude, we write

$$\|T_n - s\| \leqslant \|T_n - S_N\| + \|S_N - s\| \leqslant \varepsilon + \varepsilon = 2\varepsilon.$$

Thus, the series $\sum v_n$ also converges to s. □

Remark 6.5.3 : In Theorem 6.5.2, it is important to assume that the series converges absolutely. Below, we provide a counterexample in Example 6.5.4 and give a general result in Theorem 6.5.5.

Example 6.5.4 : We already know that the following series converges (Example 6.4.4)

$$\sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

If we rearrange the terms in the following way, we get a different sum,

$$\begin{aligned}
&\left(1-\frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3}-\frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5}-\frac{1}{10}\right) - \frac{1}{12} + \ldots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\ln 2.
\end{aligned}$$

Theorem 6.5.5 (Riemann series theorem) **:** *Let $\sum u_n$ be a real-valued series. Suppose that it converges conditionally. Let $-\infty \leqslant x \leqslant y \leqslant +\infty$. Then, there exists a rearrangement $\sum v_n$ of $\sum u_n$ such that*

$$\liminf_{n\to\infty} T_n = x \quad \text{and} \quad \limsup_{n\to\infty} T_n = y,$$

where for every $n \geqslant 1$, $T_n = v_1 + \cdots + v_n$ is the n-th partial sum of $\sum v_n$.

Remark 6.5.6 : In particular, if we take $x = y$ in Theorem 6.5.5, then the theorem says that we can find a rearrangement whose sum is equal to $x = y$.

Proof : This statement can be shown by construction. We do not give the details here. □

6.6 Cauchy series

Definition 6.6.1 : Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $(\mathcal{A}, \|\cdot\|)$ be a normed vector space over $\mathbb{K}$. Consider a binary operator $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

(1) We say that $(\mathcal{A}, \cdot)$ is an *algebra* if $\cdot$ is bilinear, that is, the following are satisfied.
   (a) (Right distributivity) For $x, y, z \in \mathcal{A}$, we have $(x+y)\cdot z = x\cdot z + y\cdot z$.
   (b) (Left distributivity) For $x, y, z \in \mathcal{A}$, we have $z\cdot(x+y) = z\cdot x + z\cdot y$.
   (c) (Scalar multiplication) For $x, y \in \mathcal{A}$ and $a, b \in \mathbb{K}$, we have $(ax)\cdot(by) = (ab)(x\cdot y)$.

(2) We say that $(\mathcal{A}, \cdot, \|\cdot\|)$ is a *normed algebra* (賦範代數) if $(\mathcal{A}, \cdot)$ is an algebra, and the norm $\|\cdot\|$ is submultiplicative, i.e.,

$$\forall x, y \in \mathcal{A}, \quad \|xy\| \leqslant \|x\| \, \|y\| \, .$$

Example 6.6.2 :

(1) The simplest examples of normed algebras are $(\mathbb{R}, \times, |\cdot|)$ and $(\mathbb{C}, \times, |\cdot|)$.

(2) We have seen in Remark 3.2.15 that $\mathcal{L}_c(U)$ equipped with the oprator norm $|||\cdot|||$ is a normed algebra for any normed vector space U. In particular, if U is a finite-dimensional normed vector space, then $\mathcal{L}(U)$ equipped with the oprator norm $|||\cdot|||$ is a normed algebra.

(3) Equivalently, for an integer $n \geqslant 1$, the space of $n \times n$ matrices $\mathcal{M}_{n\times n}(\mathbb{K})$ equipped with the matrix norm $|||\cdot|||$ is also a normed algebra.

Theorem 6.6.3 (Cauchy product) **:** *Let $\sum_{n\geqslant 0} a_n$ and $\sum_{n\geqslant 0} b_n$ be two absolutely convergent series with terms in a complete normed algebra $(\mathcal{A}, ;\|\cdot\|)$. We define their Cauchy product to be the series $\sum_{n\geqslant 0} c_n$ given by*

$$\forall n \in \mathbb{N}_0, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

The series $\sum c_n$ is absolutely convergent, and its sum equals

$$\sum_{n\geqslant 0} c_n = \left(\sum_{p\geqslant 0} a_p\right)\left(\sum_{q\geqslant 0} b_q\right). \tag{6.14}$$

Proof : Let us denote the following sums,

$$A := \sum_{p=0}^{\infty} \|a_p\| \quad \text{and} \quad B := \sum_{q=0}^{\infty} \|b_q\| \, .$$

Let $n \geqslant 0$. The n-th partial sum of $\sum c_n$ writes

$$\begin{aligned}\sum_{k=0}^{n} \|c_k\| \leqslant \sum_{k=0}^{n} \left(\sum_{\substack{p+q=k \\ p,q \geqslant 0}} \|a_p\| \cdot \|b_q\| \right) &\leqslant \sum_{0 \leqslant p,q \leqslant n} \|a_p\| \cdot \|b_q\| \\ &= \left(\sum_{p=0}^{n} \|a_p\| \right) \left(\sum_{q=0}^{n} \|b_q\| \right) \leqslant AB.\end{aligned}$$

Thus, the series $\sum c_n$ is absolutely convergent.

To compute the sum of the series $\sum c_n$, let us define the following quantities,

$$\forall n \geqslant 0, \quad \Delta_n = \sum_{k=0}^{2n} c_k - \left(\sum_{p=0}^{n} a_p \right) \left(\sum_{q=0}^{n} b_q \right).$$

Then, for any $n \geqslant 0$, we have

$$\Delta_n = \sum_{\substack{p+q \leqslant 2n \\ p,q \geqslant 0}} a_p b_q - \sum_{0 \leqslant p,q \leqslant n} a_p b_q = \sum_{\substack{p \geqslant n+1, q \geqslant 0 \\ p+q \leqslant 2n}} a_p b_q + \sum_{\substack{q \geqslant n+1, p \geqslant 0 \\ p+q \leqslant 2n}} a_p b_q.$$

Therefore, for any $n \geqslant 0$, the triangle inequality gives,

$$\begin{aligned}\|\Delta_n\| &\leqslant \sum_{\substack{p \geqslant n+1, q \geqslant 0 \\ p+q \leqslant 2n}} \|a_p\| \, \|b_q\| + \sum_{\substack{q \geqslant n+1, p \geqslant 0 \\ p+q \leqslant 2n}} \|a_p\| \, \|b_q\| \\ &\leqslant \sum_{p \geqslant n+1, q \geqslant 0} \|a_p\| \, \|b_q\| + \sum_{q \geqslant n+1, p \geqslant 0} \|a_p\| \, \|b_q\| \\ &= B \cdot \sum_{p=n+1}^{\infty} \|a_p\| + A \cdot \sum_{q=n+1}^{\infty} \|b_q\| \xrightarrow[n \to \infty]{} 0.\end{aligned}$$

This proves Eq. (6.14). □

6.7 Double sequences, double series

6.7.1 Double sequences and double limits

Let $(W, \|\cdot\|)$ be a Banach space. A sequence $(u_{m,n})_{m,n \geqslant 1}$, taking values in W with two indices, is called a *double sequence* (雙下標序列).

Definition 6.7.1 : Let $(u_{m,n})_{m,n \geqslant 1}$ be a double sequence. Let $\ell \in W$. We say that the double sequence $(u_{m,n})_{m,n \geqslant 1}$ converges to ℓ, denoted

$$\lim_{m,n \to \infty} u_{m,n} = \ell,$$

if for every $\varepsilon > 0$, there exists $N \geqslant 1$ such that

$$\forall m, n \geqslant N, \quad \|u_{m,n} - \ell\| < \varepsilon. \tag{6.15}$$

We call ℓ the *limit* or the *double limit* of the double sequence $(u_{m,n})_{m,n \geqslant 1}$.

Example 6.7.2 : Let $(u_{m,n})_{m,n\geqslant 1}$ be a real-valued double sequence defined by

$$\forall m,n \geqslant 1, \quad u_{m,n} = \mathbb{1}_{m\geqslant n}.$$

Then, we have,

$$\forall n \geqslant 1, \quad \lim_{m\to\infty} u_{m,n} = 1 \quad \text{and} \quad \lim_{n\to\infty}\lim_{m\to\infty} u_{m,n} = 1, \tag{6.16}$$

and

$$\forall m \geqslant 1, \quad \lim_{n\to\infty} u_{m,n} = 0 \quad \text{and} \quad \lim_{m\to\infty}\lim_{n\to\infty} u_{m,n} = 0. \tag{6.17}$$

However, this double sequence does not converge in the sense of Definition 6.7.1, where the uniformity in both indices m and n is required. We call the limits in Eq. (6.16) and Eq. (6.17) *iterated limits* of the double sequence $(u_{m,n})_{m,n\geqslant 1}$. This shows that when taking an *iterated limit* in a double sequence, the order in which the limits are taken is important.

Theorem 6.7.3 : *Let $(u_{m,n})_{m,n\geqslant 1}$ be a double sequence. Suppose that*

(i) the limit $\lim_{m,n\to\infty} u_{m,n}$ exists and equals $\ell \in W$;

(ii) for every $m \geqslant 1$, the limit $\lim_{n\to\infty} u_{m,n}$ exists.

Then, the following iterated limit exists and satisfies

$$\lim_{m\to\infty}\lim_{n\to\infty} u_{m,n} = \ell.$$

Proof : From the assumption (ii), we may define $\ell_m := \lim_{n\to\infty} u_{m,n}$ for every $m \geqslant 1$. Let $\varepsilon > 0$. From the assumption (i), we may find $N \geqslant 0$ such that

$$\|u_{m,n} - \ell\| \leqslant \varepsilon, \quad \forall m,n \geqslant N.$$

Fix $m \geqslant N$, by the definition of ℓ_m just above, we may find $N' = N'(m) \geqslant 1$ such that

$$\|\ell_m - u_{m,n}\| \leqslant \varepsilon, \quad \forall n \geqslant N'.$$

Then, for any $n \geqslant \max(N, N')$, we have

$$\|\ell - \ell_m\| \leqslant \|\ell - u_{m,n}\| + \|u_{m,n} - \ell_m\| \leqslant \varepsilon + \varepsilon = 2\varepsilon.$$

This shows that $\lim_{m\to\infty} \ell_m = \ell$. □

6.7.2 Double series

Due to the Riemann series theorem (Theorem 6.5.5), when we want to discuss the order of summations in double series, we are only interested in absolutely convergent ones.

Theorem 6.7.4 : *Let $(u_{m,n})_{m,n\geqslant 1}$ be a double sequence with general terms in a Banach space. Then, the two following properties are equivalent.*

(1) For every $n \geqslant 1$, the series $\sum_m u_{m,n}$ is absolutely convergent, and the series $\sum_n(\sum_m \|u_{m,n}\|)$ converges.

(2) For every $m \geqslant 1$, the series $\sum_n u_{m,n}$ is absolutely convergent, and the series $\sum_m(\sum_n \|u_{m,n}\|)$ converges.

Moreover, when one of the above properties holds, we have

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} u_{m,n}\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} u_{m,n}\right). \tag{6.18}$$

Proof : By symmetry, it is enough to show that (1) $\Rightarrow$ (2). Assume that (1) holds. For every $n \geqslant 1$, let $A_n = \sum_{m\geqslant 1}\|u_{m,n}\|$. Then, the preoperty (1) states that $\sum A_n$ converges. Let us fix $m \geqslant 1$, then $\|u_{m,n}\| \leqslant A_n$ for every $n \geqslant 1$, so $\sum_{n\geqslant 1} u_{m,n}$ converges absolutely. Let $B_m := \sum_{n\geqslant 1}\|u_{m,n}\|$ for every $m \geqslant 1$. Then, for any $M \geqslant 1$, we have

$$\sum_{m=1}^{M} B_m = \sum_{m=1}^{M}\sum_{n\geqslant 1}\|u_{m,n}\| = \sum_{n\geqslant 1}\left(\sum_{m=1}^{M}\|u_{m,n}\|\right) \leqslant \sum_{n\geqslant 1} A_n,$$

where in the second equality, we use the linearity on convergent series. On the left-hand side, we have a series with non-negative terms, and on the right-hand side, we have an upper bound that is independent of M, so the series $\sum B_m$ converges. This allows us to conclude that (2) holds.

We note that when (1) is satisfied, the right-hand side of Eq. (6.18) is well defined, because for every $n \geqslant 1$, the absolute convergence of $\sum_m u_{m,n}$ implies that $\sum_m u_{m,n}$ converges and satisfies

$$\left\|\sum_m u_{m,n}\right\| \leqslant \sum_m \|u_{m,n}\|.$$

Then, the convergence of $\sum_n\sum_m\|u_{m,n}\|$ implies that of $\sum_n\|\sum_m u_{m,n}\|$, which implies the convergence of $\sum_n(\sum_m u_{m,n})$. Since we have shown that (1) and (2) are equivalent, the left-hand side of Eq. (6.18) is also well defined.

Now, we are going to show Eq. (6.18). Define

$$\forall n \geqslant 1, \quad S_n = \sum_{p=1}^{n}\sum_{q=1}^{n} u_{p,q},$$

and let us show that S_n converges to the right-hand side of Eq. (6.18), then we conclude by symmetry. Let

$$\forall m,q \geqslant 1, \quad a_{m,q} = \sum_{p=1}^{m} u_{p,q} \quad \text{and} \quad \forall q \geqslant 1, \quad a_q = \sum_{p\geqslant 1} u_{p,q}.$$

Let $\varepsilon > 0$ and $Q \geqslant 1$ such that $\sum_{q \geqslant Q} A_q \leqslant \varepsilon$. Then, for $n \geqslant Q$, we have

$$\sum_{q=1}^{\infty} a_q - S_n = \sum_{q=1}^{\infty} a_q - \sum_{q=1}^{n} a_{n,q} = \sum_{q=1}^{Q} (a_q - a_{n,q}) + \sum_{q=Q+1}^{n} (a_q - a_{n,q}) + \sum_{q=n+1}^{\infty} a_q.$$

We note that for $q \geqslant 1$, we have $\|a_q\| \leqslant A_q$ and for $q \geqslant Q$, we have $\|a_q - a_{n,q}\| = \left\|\sum_{p \geqslant n+1} u_{p,q}\right\| \leqslant A_q$. Thus, the above inequality gives

$$\left\|\sum_{q=1}^{\infty} a_q - S_n\right\| \leqslant \left\|\sum_{q=1}^{Q} (a_q - a_{n,q})\right\| + \sum_{q=Q+1}^{\infty} A_q \leqslant \left\|\sum_{q=1}^{Q} (a_q - a_{n,q})\right\| + \varepsilon.$$

Since $a_{n,q} \xrightarrow[n \to \infty]{} a_q$ for every $q \geqslant 1$, by taking $\limsup$ when $n \to \infty$, we find

$$\limsup_{n \to \infty} \left\|\sum_{q=1}^{\infty} a_q - S_n\right\| \leqslant \varepsilon.$$

The choice of $\varepsilon > 0$ is arbitrary, so we find

$$\limsup_{n \to \infty} \left\|\sum_{q=1}^{\infty} a_q - S_n\right\| = 0 \quad \Rightarrow \quad \lim_{n \to \infty} \left\|\sum_{q=1}^{\infty} a_q - S_n\right\| = 0,$$

that is, $S_n \xrightarrow[n \to \infty]{} \sum_{q=1}^{\infty} a_q$. $\square$

6.8 Infinite products

6.8.1 Convergence and divergence

Let $(u_n)_{n \geqslant 1}$ be a sequence with values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We may define the sequence $(P_n)_{n \geqslant 0}$ as below

$$P_0 = 1, \quad P_n = \prod_{k=1}^{n} u_k, \quad \forall n \geqslant 1.$$

For each $n \geqslant 1$, we call u_n the n-th factor of the infinite product $\prod u_n$, and P_n the n-th partial product of the infinite product $\prod u_n$.

Definition 6.8.1 : The notion of convergence and divergence of an infinite product $\prod u_n$ is given below.

(1) If there are infinitely many factors u_n that are zero, we say that the infinite product $\prod u_n$ diverges to zero.

(2) If $u_n \neq 0$ for all $n \geqslant 1$, we say that

  (a) the infinite product converges to $P \neq 0$ if $P_n \xrightarrow[n \to \infty]{} P$, and we write $P = \prod_{n=1}^{\infty} u_n$;

  (b) the infinite product diverges to 0 if $P_n \xrightarrow[n \to \infty]{} 0$;

  (c) the infinite series diverge otherwise.

(3) If there exists $N \geqslant 1$ such that $u_n \neq 0$ for all $n \geqslant N$, let us define

$$\forall n \geqslant 1, \quad v_n = u_{n+N-1},$$

and its corresponding partial products $(P'_n)_{n\geqslant 0}$ given by

$$P'_0 = 1, \quad P'_n = \prod_{k=1}^{n} v_k = \prod_{k=N}^{N+n-1} u_k, \quad \forall n \geqslant 1.$$

(a) If the infinite product $\prod v_n$ converges to $P \neq 0$, then we say that the infinite product $\prod u_n$ converges to $u_1 \dots u_{N-1} P$ and write its limit as

$$\prod_{n\geqslant 1} u_n := u_1 \dots u_{N-1} \prod_{n\geqslant N} u_n = u_1 \dots u_{N-1} \prod_{n\geqslant 1} v_n;$$

(b) If the infinite product $\prod v_n$ diverges to 0, we say that the infinite product $\prod u_n$ diverges to 0;

(c) Otherwise, we say that the infinite product $\prod u_n$ diverges.

Remark 6.8.2 : From Definition 6.8.1, we know that by adding or removing finitely many zeros to an infinite product, we do not change its convergent or divergent behavior.

Proposition 6.8.3 (Cauchy's condition) **:** *The infinite product $\prod u_n$ converges if and only if for every $\varepsilon > 0$, there exists $N \geqslant 1$ such that*

$$\forall n \geqslant N, \forall k \geqslant 1, \quad |u_{n+1} \dots u_{n+k} - 1| < \varepsilon. \tag{6.19}$$

Proof : Since the notion of convergence of $\prod u_n$ and the condition Eq. (6.19) are not changed if we remove finitely zero terms from $(u_n)_{n\geqslant 1}$, we may assume that $u_n \neq 0$ for all $n \geqslant 1$.

- Suppose that $\prod u_n$ converges. Let

$$P = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \prod_{k=1}^{n} u_k \neq 0.$$

This means that the partial products $(P_n)_{n\geqslant 0}$ of $\prod u_n$ are bounded from below by some constant $M > 0$. Let $\varepsilon > 0$, by the Cauchy's condition for sequences from Corollary 6.1.11, we may find $N \geqslant 1$ such that

$$\forall n \geqslant N, \forall k \geqslant 1, \quad |P_{n+k} - P_n| < \varepsilon M,$$

which, by dividing by $|P_n|$, implies

$$\forall n \geqslant N, \forall k \geqslant 1, \quad \left| \frac{P_{n+k}}{P_n} - 1 \right| < \varepsilon,$$

which is exactly Eq. (6.19).

- Suppose that for every $\varepsilon > 0$, there exists $N \geqslant 1$ such that the condition Eq. (6.19) holds. Let $\varepsilon = \frac{1}{2}$, and take $N \geqslant 1$ such that Eq. (6.19) holds. This shows that for every $n \geqslant N$, we have $u_n \neq 0$. For $n \geqslant N$, let $Q_n = \prod_{k=N+1}^{n} u_k$. From Eq. (6.19), we also deduce that $\frac{1}{2} < |Q_n| < \frac{3}{2}$ for all $n \geqslant N$. Additionally, for every $n \geqslant N$ and $k \geqslant 1$, we also have

$$\left|\frac{Q_{n+k}}{Q_n} - 1\right| < \varepsilon \quad \Rightarrow \quad |Q_{n+k} - Q_n| < \varepsilon |Q_n| < \frac{3}{2}\varepsilon.$$

This means that the sequence $(Q_n)_{n \geqslant N}$ satisfies Cauchy's condition, so converges. This also means that the product $\prod u_n$ converges.

□

Theorem 6.8.4 : *Let $(a_n)_{n\geqslant 1}$ be a sequence with strictly positive general terms. Then, the infinite product $\prod(1 + a_n)$ converges if and only if the series $\sum a_n$ converges.*

Proof : The convergence of the infinite product $\prod(1 + a_n)$ is equivalent to the convergence of the series $\sum \ln(1 + a_n)$.

- If $\sum \ln(1 + a_n)$ converges, it means that $\ln(1 + a_n) \xrightarrow[n\to\infty]{} 0$, so $a_n \xrightarrow[n\to\infty]{} 0$. Thus, we have the equivalence relation $\ln(1 + a_n) \sim a_n$ when $n \to \infty$. We apply Theorem 6.2.8 and we find that the series $\sum a_n$ converges.
- If the series $\sum a_n$ converges, then $a_n \xrightarrow[n\to\infty]{} 0$. Then, we conclude in a similar way.

□

Remark 6.8.5 :

(1) If some of the terms in $(a_n)_{n\geqslant 1}$ are zero, both the values of $\sum a_n$ and $\prod(1 + a_n)$ are not changed. Therefore, it is reasonable to assume that the sequence $(a_n)_{n\geqslant 1}$ does not contain any zero.

(2) In the case that $(a_n)_{n\geqslant 1}$ is a sequence with strictly negative general terms, the same statement also holds.

(3) It is important to assume that the sequence $(a_n)_{n\geqslant 1}$ has a constant sign. We may consider the example $a_n = \frac{(-1)^n}{\sqrt{n}}$ for $n \geqslant 1$.
   - The series $\sum a_n$ is an alternating series, and by Theorem 6.4.2, it converges.
   - For every $n \geqslant 1$, we have

   $$(1 + a_{2n})(1 + a_{2n+1}) = \Big(1 + \frac{1}{\sqrt{2n}}\Big)\Big(1 - \frac{1}{\sqrt{2n+1}}\Big) = 1 - \frac{1}{2n} + o\Big(\frac{1}{n}\Big), \quad \text{when } n \to \infty.$$

   Since $\sum \frac{1}{n}$ diverges, the infinite product $\prod(1 + a_{2n})(1 + a_{2n+1})$ also diverges.

Definition 6.8.6 : Let $(a_n)_{n\geqslant 1}$ be a complex-valued nonzero sequence. We say that the infinite product $\prod(1+a_n)$ converges absolutely if $\prod(1+|a_n|)$ converges.

Theorem 6.8.7 : *Let $(a_n)_{n\geqslant 1}$ be a complex-valued nonzero sequence. If the infinite product $\prod(1+a_n)$ converges absolutely, then it converges.*

Proof : Let us check Cauchy's condition provided in Proposition 6.8.3. For every $n, k \geqslant 1$, by the triangle inequality, we have

$$\left|\prod_{j=1}^{k}(1+a_{n+j})-1\right| \leqslant \prod_{j=1}^{k}(1+|a_{n+j}|)-1.$$

Therefore, if the infinite product $\prod(1+|a_n|)$ satisfies the Cauchy's condition, so does $\prod(1+a_n)$. □

6.8.2 Application to the Riemann zeta function

Let the sequence $(p_k)_{k\geqslant 1}$ be given by ordered prime numbers, that is $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.

Theorem 6.8.8 (Euler's product) **:** *For $s > 1$, we have*

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{k=1}^{\infty}\frac{1}{1-p_k^{-s}}.$$

Moreover, the above infinite product converges absolutely.

Proof : For $n \geqslant 1$, let us write the n-th partial product to be

$$P_n = \prod_{k=1}^{n}\frac{1}{1-p_k^{-s}}. \tag{6.20}$$

Our goal is to show that $P_n \xrightarrow[n\to\infty]{} \zeta(s)$.

Let us fix an integer $n \geqslant 1$. We may expand each factor in the right-hand side of Eq. (6.20) into a series, that is

$$\forall k \geqslant 1, \quad \frac{1}{1-p_k^{-s}} = \sum_{m=0}^{\infty}\frac{1}{p_k^{ms}} = 1 + \sum_{m=1}^{\infty}\frac{1}{p_k^{ms}} \tag{6.21}$$

and

$$P_n = \prod_{k=1}^{n}\sum_{m=0}^{\infty}\frac{1}{p_k^{ms}} = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{1}{p_1^{m_1 s}\dots p_n^{m_n s}}.$$

For every $n \geqslant 1$, let

$$A_n = \{N \in \mathbb{N} : N \text{ has all its prime factors among } p_1, \dots, p_n\}.$$

Due to the uniqueness of prime factorization, we know that

$$P_n = \sum_{N \in A_n} \frac{1}{N^s}.$$

Therefore,

$$|P_n - \zeta(s)| \leqslant \sum_{N \geqslant p_{n+1}} \frac{1}{N^s}, \tag{6.22}$$

because all the terms defining the series $\zeta(s)$ are all positive. Since $\sum \frac{1}{N^s}$ converges, its remainder goes to zero, so the right-hand side of Eq. (6.22) goes to zero, that is $P_n \xrightarrow[n\to\infty]{} \zeta(s)$.

For every $k \geqslant 1$, we may rewrite the series in Eq. (6.21) as $1 + a_k$. The series $\sum a_k$ converges absolutely because all its terms are positive, and is bounded from above by $\zeta(s)$. Then, it follows from Theorem 6.8.4 that the infinite product $\prod(1 + a_k)$ converges absolutely. □