7 Complements on Riemann Integrals

7.1 Riemann integrability on an interval

7.1.1 Setting

In the theory of Riemann–Stieltjes integrals that we saw in Section 5.2, both the integrand and the integrator needed to be defined and bounded on a segment. However, it is not always the case that the integrand is bounded, or the domain of integration is a compact subset of $\mathbb{R}$. In this chapter, we take the Riemann integrals as example, to see how to make sense of Riemann integrals, in the case that the domain of definition is a general interval, or the integrand is not a bounded function.

For the integrand, we are going to take them to be *piecewise continuous functions*; and for the domain of definition, we only consider the intervals of the following forms,

$$\begin{aligned} &[a,b) \quad \text{for} \; -\infty < a < b \leqslant +\infty, \\ &(a,b] \quad \text{for} \; -\infty \leqslant a < b < +\infty, \\ &(a,b) \quad \text{for} \; -\infty \leqslant a < b \leqslant +\infty. \end{aligned}$$

We note that we allow $a = -\infty$ if the interval is open on the left side; $b = +\infty$ if the interval is open on the right side. From the theory for the intervals of $[a, b)$ type, we deduce easily the theory for the intervals of $(a, b]$ type by symmetry; then, for the intervals of (a, b) type, we decompose them into $(a, c] \cup [c, b)$, where $c \in (a, b)$. Therefore, in what follows, to study general intervals, it is actually sufficient to study only the intervals of $[a, b)$ type.

We also recall that we are only interested in real-valued functions here, since for functions taking values in a finite-dimensional vector space, we follow the decomposition as in Remark 5.2.2 (6), and define the corresponding integral on a general interval by linearity.

Definition 7.1.1 :

- A function $f : [a, b] \to \mathbb{R}$ is said to be *piecewise continuous on the segment* $[a, b]$ if there exists a partition $P = (x_k)_{0 \leqslant k \leqslant n} \in \mathcal{P}([a, b])$ such that for every $1 \leqslant k \leqslant n$, the restriction of f on the open subinterval (x_{k-1}, x_k) can be extended to a continuous function on $[x_{k-1}, x_k]$.
- Let $I \subseteq \mathbb{R}$ be a subset. A function $f : I \to \mathbb{R}$ is said to be *piecewise continuous on* I if for any segment $J \subseteq I$, the restricted function $f_{|J}$ is piecewise continuous on J.
- For any subset $I \subseteq \mathbb{R}$, we write $\mathcal{PC}(I, \mathbb{R})$ for the set of functions that are piecewise continuous on I.
- For any normed vector space $(W, \|\cdot\|)$, we may define the space $\mathcal{PC}(I, V)$ of piecewise continuous functions with values in W in a similar way.

Example 7.1.2 :

(1) The function $x \mapsto \frac{1}{x}$ is piecewise continuous on $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.

(2) The function $x \mapsto \ln x$ is piecewise continuous on $\mathbb{R}_{>0} = (0, +\infty)$.

Proposition 7.1.3 : *Let $I = [a, b]$ be a segment of $\mathbb{R}$. Any piecewise continuous function $f : I \to \mathbb{R}$ on I is bounded and Riemann-integrable on I.*

Proof : Let $f \in \mathcal{PC}(I, \mathbb{R})$ with $I = [a, b]$ which is a segment. By definition, for every $1 \leqslant k \leqslant n$, there is a continuous function $g : [x_{k-1}, x_k] \to \mathbb{R}$ such that $f_{|(x_{k-1},x_k)} \equiv g_{|(x_{k-1},x_k)}$. Since g is integrable on $[x_{k-1}, x_k]$, so is f, see Corollary 5.3.22. And we apply Proposition 5.2.10 to conclude that f is integrable on $[a, b]$. □

7.1.2 Integrability on an interval

We saw in Theorem 6.1.16 that for a series with values in a Banach space $(W, \|\cdot\|)$, if it converges absolutely, then it converges. And if it does not converge absolutely, by rearranging the terms, we are able to get any value as limit, see Theorem 6.5.5. When it comes to the Riemann integration on a general interval, we encounter similar phenomena. Actually, for a piecewise continuous function on a general interval I, its integral on any subsegment can be defined thanks to Proposition 7.1.3, then by taking larger and larger subsegments to cover the whole interval, we have a chance to get a meaningful limit, that we want to define as the integral on I. This limit may not be defined uniquely if we do not have *absolute convergence*. To make things simpler, we start with absolutely convergent integrals, which bring us back to study non-negative integrands. Later in Section 7.2.2, we will discuss the situation without the absolute convergence.

Let I be an interval, and denote by $\mathcal{PC}_+(I) = \mathcal{PC}_+(I, \mathbb{R}) := \mathcal{PC}(I, \mathbb{R}_+)$ the set of non-negative piecewise continuous functions on I.

Definition 7.1.4 : Let $f \in \mathcal{PC}_+(I)$ be a non-negative piecewise continuous function on I. We say that f is *integrable* on I if there exists $M \geqslant 0$ such that $\int_J f \leqslant M$ for any segment $J \subseteq I$, and we write

$$\int_I f = \sup_{\substack{J \subseteq I \\ J \text{ is a segment}}} \int_J f. \tag{7.1}$$

Remark 7.1.5 : If $a = \inf I$ and $b = \sup I$, we may also rewrite the integral in Eq. (7.1) as follows,

$$\int_a^b f = \int_I f.$$

Note that in the case that the interval is a segment $I = [a, b]$, and the function f is a non-negative piecewise continuous, the definition of integrability in Eq. (7.1) coincides with the notion of integrability in Definition 5.2.1, in the sense that (RS) condition is satisfied with $\alpha(x) = x$.

Proposition 7.1.6 : *Let $f \in \mathcal{PC}_+(I)$ be a non-negative integrable function on I. Then, for any sequence $(J_n = [a_n, b_n])_{n\geqslant 1}$ of segments with*

$$\forall n \geqslant 1, \quad J_n \subseteq J_{n+1} \subseteq \cdots \subseteq I \quad \textit{and} \quad \bigcup_{n\geqslant 1} J_n = I, \tag{7.2}$$

we have

$$\int_I f = \sup_{n\geqslant 1} \int_{J_n} f = \lim_{n\to\infty} \int_{J_n} f = \lim_{n\to\infty} \int_{a_n}^{b_n} f(x)\,\mathrm{d}x.$$

Proof : Let us consider a sequence $(J_n)_{n\geqslant 1}$ of segments satisfying Eq. (7.2). We want to show that the limit of $\int_{J_n} f$ is equal to $\int_I f$ defined in Eq. (7.1).

- For every $n \geqslant 1$, we have $J_n \subseteq I$, since f is non-negative, we have $\int_{J_n} f \leqslant \int_I f$. By taking $\limsup$ on n, we find

$$\limsup_{n\to\infty} \int_{J_n} f \leqslant \int_I f.$$

- Given $\varepsilon > 0$. By the characterization of supremum, Eq. (7.1) tells us there exists a segment $J = [a, b] \subseteq I$ such that $\int_J f + \varepsilon \geqslant \int_I f$. Since $a, b \in I = \bigcup_{n\geqslant 1} J_n$, there exists $N \geqslant 1$ such that $a, b \in J_n$ for all $n \geqslant N$. Therefore, for $n \geqslant N$, we have

$$\int_{J_n} f \geqslant \int_J f \geqslant \int_I f - \varepsilon.$$

  In other words,

$$\liminf_{n\to\infty} \int_{J_n} f \geqslant \int_I f - \varepsilon.$$

  Since $\varepsilon > 0$ can be made arbitrarily small, we find

$$\liminf_{n\to\infty} \int_{J_n} f \geqslant \int_I f.$$

In conclusion, we have

$$\int_I f \leqslant \liminf_{n\to\infty} \int_{J_n} f \leqslant \limsup_{n\to\infty} \int_{J_n} f \leqslant \int_I f,$$

that is $\lim_{n\to\infty} \int_{J_n} f = \int_I f$. □

Example 7.1.7 : Below we give examples of non-negative continuous integrable / non-integrable functions.

(1) For $\lambda > 0$, the function $t \mapsto e^{-\lambda t}$ is integrable on $\mathbb{R}_+ = [0, +\infty)$. To see this, let us fix $\lambda > 0$ and take $J_n = [0, n]$ for all $n \geqslant 1$. For every $n \geqslant 1$, we have

$$\int_{I_n} e^{-\lambda t}\,\mathrm{d}t = \int_0^n e^{-\lambda t}\,\mathrm{d}t = \left[-\frac{e^{-\lambda t}}{\lambda}\right]_0^n = \frac{1 - e^{-\lambda n}}{\lambda} \leqslant \frac{1}{\lambda} < \infty.$$

The condition in Definition 7.1.4 is indeed satisfied.

(2) The function $x \mapsto |\sin x|$ is not integrable. In fact, for every $k \in \mathbb{N}_0$, we have

$$\int_{k\pi}^{(k+1)\pi} |\sin x| \, \mathrm{d}x = \int_0^{\pi} \sin x \, \mathrm{d}x = 2.$$

Therefore,

$$\forall n \geqslant 0, \qquad \int_0^{n\pi} |\sin x| \, \mathrm{d}x = 2n,$$

which cannot be bounded uniformly in n.

Example 7.1.8 (Riemann's integrals) **:** We study the integrability of functions $f : t \mapsto t^{-\alpha}$ for $\alpha \in \mathbb{R}$.

(1) For any $a > 0$, the function $t \mapsto t^{-\alpha}$ is integrable on $[a, +\infty)$ if and only if $\alpha > 1$.

(2) For any $a > 0$, the function $t \mapsto t^{-\alpha}$ is integrable on $(0, a]$ if and only if $\alpha < 1$.

(3) For $a < b$, the function $t \mapsto (b - t)^{-\alpha}$ is integrable on $[a, b)$ if and only if $\alpha < 1$.

(4) For $a < b$, the function $t \mapsto (t - a)^{-\alpha}$ is integrable on $(a, b]$ if and only if $\alpha < 1$.

Example 7.1.9 (Bertrand's integrals) **:** We study the integrability of functions $t \mapsto t^{-\alpha}|\ln t|^{-\beta}$ for $\alpha, \beta \in \mathbb{R}$.

(1) For any $a > 1$, it is integrable on $[a, +\infty)$ if and only if (i) $\alpha > 1$ or (ii) $\alpha = 1$ and $\beta > 1$.

(2) For any $a \in (0, 1)$, it is integrable on $(0, a]$ if and only if (i) $\alpha < 1$ or (ii) $\alpha = 1$ and $\beta > 1$.

See Exercise 7.2 for more details.

Definition 7.1.10 : Let $I \subseteq \mathbb{R}$ be an interval and $(W, \|\cdot\|)$ be a finite-dimensional Banach space[1]. A piecewise continuous function $f : I \to W$ is said to be *integrable* on I if $\|f\|$ is integrable on I in the sense of Definition 7.1.4. Given a sequence $(J_n)_{n \geqslant 1}$ of segments in I satisfying Eq. (7.2), we may define

$$\int_I f := \lim_{n \to \infty} \int_{J_n} f \in W. \tag{7.3}$$

We denote by $L^1(I, W)$ the set of piecewise continuous functions from I to W that are integrable in the sense defined here, that is

$$L^1(I, W) := \Big\{ f : I \to W : \int_I \|f\| < +\infty \Big\}.$$

[1]Note that we have explained in Remark 5.2.2 how to construct the integral of $\int_J f$ in the case that J is a segment and f is a W-valued function. It is also possible to make sense of this integral if W is a general Banach space (without the finite-dimensional assumption). For example, f is continuous, we use the uniform continuity of f to appriximate it by a step function.

Remark 7.1.11 : In Definition 7.1.10, if we take $(W, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, we find the corresponding notion of *integrability* for real-valued functions.

Proposition 7.1.12 : *In Definition 7.1.10, the limit of the sequence* $(\int_{J_n} f)_{n\geqslant 1}$ *exists, and does not depend on the choice of* $(J_n)_{n\geqslant 1}$*, as long as* $(J_n)_{n\geqslant 1}$ *is chosen to satisfy Eq. (7.2).*

Remark 7.1.13 : As a direct consequence of Proposition 7.1.12,

- for an interval of type $[a, b)$ with $-\infty < a < b < +\infty$, we may consider $J_n = [a, b - \frac{1}{n}]$ for all $n \geqslant 1$;
- for an interval of type $[a, +\infty)$ with $-\infty < a < +\infty$, we may consider $J_n = [a, n]$ for all $n \geqslant 1$.

Proof : We need to check that the limit in Eq. (7.3) is well defined, and does not depend on the choice of $(J_n)_{n\geqslant 1}$.

- Let $(J_n)_{n\geqslant 1}$ be a sequence of segments satisfying Eq. (7.2). For every $n \geqslant 1$, write $J_n = [a_n, b_n]$, let $u_n = \int_{J_n} f$ and $U_n = \int_{J_n} \|f\|$. We want to show that $(u_n)_{n\geqslant 1}$ is a Cauchy sequence. Since $(W, \|\cdot\|)$ is a Banach space, $(u_n)_{n\geqslant}$ converges.

  Let $\varepsilon > 0$. Since $\|f\|$ is integrable on I, the sequence $(U_n)_{n\geqslant 1}$ converges so is a Cauchy sequence. Therefore, we may find $N \geqslant 1$ such that $|U_p - U_q| < \varepsilon$ for all $p, q \geqslant N$. This means that for $p > q \geqslant N$, we have

  $$\|u_p - u_q\| = \left\| \int_{[a_p,a_q]} f + \int_{[b_q,b_p]} f \right\| \leqslant \int_{[a_p,a_q]} \|f\| + \int_{[b_q,b_p]} \|f\| = U_p - U_q < \varepsilon.$$

  This shows that $(u_n)_{n\geqslant 1}$ is a Cauchy sequence, so converges.

- Let $(J_n)_{n\geqslant 1}$ and $(K_n)_{n\geqslant 1}$ be seuqneces of segments satisfying Eq. (7.2). From the first part of the proof, we know that the following limits exist,

  $$u_n := \int_{J_n} f \xrightarrow[n\to\infty]{} u \quad \text{and} \quad v_n := \int_{K_n} f \xrightarrow[n\to\infty]{} v.$$

  For every $n \geqslant 1$, let $L_n := J_n \cup K_n$, which is a union of two segments. We note that for small values of n, L_n might not be a segment, but for large enough n, L_n will always be a segment (non-empty intersection between J_n and K_n). Therefore, we may find $N \geqslant 1$ such that L_n is a segment for all $n \geqslant N$. Then, $(L_{n+N})_{n\geqslant 1}$ is also a sequence of segments satisfying Eq. (7.2). We may write

  $$w_n := \int_{L_n} f \xrightarrow[n\to\infty]{} w.$$

  As in the first part, let

  $$\forall n \geqslant 1, \quad U_n = \int_{J_n} \|f\| \quad \text{and} \quad W_n = \int_{L_n} \|f\|.$$

  We have

  $$\|w_n - u_n\| = \left\| \int_{L_n \setminus J_n} f \right\| \leqslant \int_{L_n \setminus J_n} \|f\| = W_n - U_n \xrightarrow[n\to\infty]{} 0,$$

where the last convergence comes from Proposition 7.1.6. This implies that $w = u$. Similarly, we also have $w = v$, so $u = v$. □

7.1.3 Properties

The integral on a general interval I defined in Definition 7.1.10 satisfies many properties that are also satisfied for integrals defined on segments. It can be understood by the fact that the procedure of taking the limit in Definition 7.1.10 preserves linearity. We can use the following identities safely if f is a integrable W-valued piecewise continuous function,

- Union relation: $\int_I f + \int_J f = \int_{I\cup J} f$ provided that $I \cap J = \varnothing$ and two among the three integrals are well defined.
- Triangle inequality: $\|\int_I f\| \leqslant \int_I \|f\|$.
- Integration by parts for $\mathcal{C}^1$ functions.
- Change of variables with respect to a $\mathcal{C}^1$ function. See Exercise A1.5 for an example where problems may arise if the change of variables is not $\mathcal{C}^1$.

Now, we are going to give a few criteria for the integrability on an interval. We start with an interval of the form $I = [a, b)$ and consider $f \in \mathcal{PC}(I, W)$, where W is a finite-dimensional Banach space. The following properties can be proven almost immediately without any technicalities, so we only state the properties, without giving any proofs.

Proposition 7.1.14 : *Let $f \in \mathcal{PC}(I, W)$ be a piecewise continuous function on I. The following properties are equivalent.*

(1) f is integrable on $[a, b)$.

(2) (Partial integral) $x \mapsto \int_a^x \|f(t)\| \,\mathrm{d}t$ is bounded on $[a, b)$.

(3) (Partial integral) $x \mapsto \int_a^x \|f(t)\| \,\mathrm{d}t$ has a limit when $x \to b-$.

(4) (Remainder integral) The limit of $x \mapsto \int_x^b \|f(t)\| \,\mathrm{d}t$ when $x \to b-$ is 0.

(5) (Cauchy's criterion) For $\varepsilon > 0$, there exists $A \in I$ such that

$$\forall x, y \in [A, b), x < y, \quad \int_x^y \|f(t)\| \,\mathrm{d}t < \varepsilon.$$

Proof : It is a direction consequence of Definition 7.1.10, Proposition 7.1.12, and Remark 7.1.13. □

Proposition 7.1.15 : *Let $f \in \mathcal{PC}(I, W)$ be a piecewise continuous function on I and $c \in \mathring{I}$. Write $I_- := I \cap (-\infty, c]$ and $I_+ := I \cap [c, +\infty)$. Then, the following properties are equivalent.*

(1) f is integrable on I.

(2) f is integrable on I_- and I_+.

And in this case, we have $\int_I f = \int_{I_-} f + \int_{I_+} f$.

Proof : It is a direct consequence of the union relation. □

Proposition 7.1.16 : *Let $f \in \mathcal{PC}(I, W)$ be a piecewise continuous function on I with values in a finite-dimensional Banach space W, and $\varphi \in \mathcal{PC}_+(I)$ be a non-negative piecewise continuous function on I.*

(1) If $\|f\| \leqslant \varphi$ on I and φ is integrable, then f is integrable and we have $\|\int_I f\| \leqslant \int_I \varphi$.

(2) If f takes values in $\mathbb{R}_+$ and is non-integrable with $f \leqslant \varphi$, then φ is non-integrable.

Proof :

(1) For $x \in [a, b)$, we have

$$\int_a^x \|f(t)\| \, \mathrm{d}t \leqslant \int_a^x \varphi(t) \, \mathrm{d}t \leqslant \int_a^b \varphi(t) \, \mathrm{d}t = \int_I \varphi.$$

The left side in the above formula is bounded, so we can conclude by Proposition 7.1.14. Moreover, by the triangle inequality, we have $\|\int_I f\| \leqslant \int_I \|f\|$.

(2) By contradiction, if φ were integrable, then by (1), f would also be integrable. □

Example 7.1.17 : Check that the non-negative function $f : t \mapsto \frac{1}{\sqrt{t(1-t)}}$ is integrable on $(0, 1)$. Let us take $c = \frac{1}{2}$, $I_- = (0, \frac{1}{2}]$ and $I_+ = [\frac{1}{2}, 1)$.

- For $t \in I_-$, we have $f(t) \leqslant \frac{2}{\sqrt{t}}$. The function $t \mapsto \frac{1}{\sqrt{t}}$ is integrable on $(0, \frac{1}{2}]$, so is f.
- For $t \in I_+$, we have $f(t) \leqslant \frac{2}{\sqrt{1-t}}$. The function $t \mapsto \frac{1}{\sqrt{1-t}}$ is integrable on $[\frac{1}{2}, 1)$, so is f.

7.1.4 Comparison of integrals

We consider a finite-dimensional Banach space $(W, \|\cdot\|)$. We are going to give some comparison results for non-negative integrable and non-integrable functions. These results are analogous to those for series, see Section 6.2.1. Note that the function that we compare to needs to be *non-negative*.

Definition 7.1.18 : Let $f : [a, b) \to W$ and $g : [a, b) \to \mathbb{R}$ be two piecewise continuous functions.

- We write $f \underset{b}{=} \mathcal{O}(g)$ or $f(x) = \mathcal{O}(g(x))$ when $x \to b$ if there exists $M > 0$ and $\delta > 0$ such that

$$\forall x \in [a, b) \cap B(b, \delta), \quad \|f(x)\| \leqslant M|g(x)|.$$

- We write $f \underset{b}{=} o(g)$ or $f(x) = g(x)$ when $x \to b$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x \in [a, b) \cap B(b, \delta), \quad \|f(x)\| \leqslant \varepsilon|g(x)|.$$

- If $W = \mathbb{R}$, we write $f \underset{b}{\sim} g$ or $f(x) \sim g(x)$ when $x \to b$ if $f - g \underset{b}{=} o(g)$.

We recall that for convergent series with asymptotic relations for their non-negative general terms, we may compare their remainders, see Theorem 6.2.8. When it comes to non-negative integrable functions, we may compare their *remainder integrals*, as stated in the following proposition.

Proposition 7.1.19 (Comparison for integrable functions) **:** *Let $f : [a, b) \to W$ be a piecewise continuous function, and $g : [a, b) \to \mathbb{R}_+$ be a non-negative integrable function. Then, the following properties hold.*

(1) If $f \underset{b}{=} \mathcal{O}(g)$, then f is integrable on $[a, b)$ and $\int_x^b f \underset{x \to b}{=} \mathcal{O}(\int_x^b g)$.

(2) If $f \underset{b}{=} o(g)$, then f is integrable on $[a, b)$ and $\int_x^b f \underset{x \to b}{=} o(\int_x^b g)$.

(3) If $W = \mathbb{R}$ and $f \underset{b}{\sim} g$, then f is integrable on $[a, b)$ and $\int_x^b f \underset{x \to b}{\sim} \int_x^b g$.

Remark 7.1.20 :

(1) We insist again that these comparison results are only valid for a non-negative function g. It is possible to have $f \sim_b g$ but the integrals $\int_a^b f$ and $\int_a^b g$ have different behaviors. The same phenomenon also occurs for series, we recall the result from Remark 6.2.4. For the integrals, we will give a corresponding counterexample later, see Example 7.2.16.

(2) We note that these comparison relations are "preserved" by taking a primitive. However, we do not have a similar result for derivatives. For example, $t \mapsto t^{3/2} \sin\left(\frac{1}{t}\right)$ is integrable on $(0, 1]$ and is $o(t)$ when $t \to 0+$. From Proposition 7.1.19 (2), we know that

$$\int_0^x t^{3/2} \sin\left(\frac{1}{t}\right) \mathrm{d}t \underset{x \to 0+}{=} o(x^2).$$

But the following derivative is clearly not $o(1)$, because is not convergent when $t \to 0+$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{3/2} \sin\left(\frac{1}{t}\right)\right) = \frac{3}{2} t^{1/2} \sin\left(\frac{1}{t}\right) - t^{-1/2} \cos\left(\frac{1}{t}\right).$$

Proof :

(1) By assumption, there exists $M > 0$ and $\delta > 0$ such that

$$\forall t \in [b - \delta, b), \quad \|f(t)\| \leqslant M g(t).$$

Therefore, for any $x \in [b - \delta, b)$, we have

$$\left\|\int_x^b f\right\| \leqslant \int_x^b \|f\| \leqslant M \int_x^b g,$$

which is what we want to show.

(2) Let $\varepsilon > 0$. By assumption, there exists $\delta > 0$ such that

$$\forall t \in [b-\delta, b), \quad \|f(t)\| \leqslant \varepsilon g(t).$$

Therefore, for any $x \in [b-\delta, b)$, we have

$$\left\| \int_x^b f \right\| \leqslant \int_x^b \|f\| \leqslant \varepsilon \int_x^b g,$$

which is what we want to show.

(3) Suppose $f \underset{b}{\sim} g$. This means that $f - g \underset{b}{=} o(g)$. Then we apply the result from (2) to conclude that

$$\int_x^b (f-g) \underset{x\to b}{=} o\Big(\int_x^b g\Big) \Leftrightarrow \int_x^b f \underset{x\to b}{\sim} \int_x^b g.$$

□

Example 7.1.21 : The Gamma function Γ is defined by

$$\forall x > 0, \quad \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, \mathrm{d}t.$$

The function $f : t \mapsto t^{x-1}e^{-t}$ is continuous on $\mathbb{R}_+^* = (0, +\infty)$.

- Around 0. When $t \to 0$, we have $f(t) \sim t^{x-1}$. Since $x - 1 > -1$, by Riemann's integral (Example 7.1.8) and the comparison for integrable functions (Proposition 7.1.19), we deduce that f is integrable around 0.
- Around $+\infty$. We have

$$t^{x-1}e^{-t} = \mathcal{O}\Big(\frac{1}{t^2}\Big) \quad \text{when } t \to \infty.$$

Since the function $t \mapsto \frac{1}{t^2}$ is integrable around $+\infty$, by the comparison for integrable functions (Proposition 7.1.19), we deduce that f is integrable around $+\infty$.

In conclusion, the Gamma function $\Gamma(x)$ is well defined for all $x > 0$. We may check some values taken by the Gamma function: $\Gamma(1) = 1$ (direct computation), $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (Gaussian integral up to a change of variables, see Exercise A1.6). Additionally, by an integration by parts, we may show that

$$\Gamma(x+1) = x\Gamma(x), \quad \forall x > 0. \tag{7.4}$$

You may find more properties and a characterization of the Gamma function in Exercise 7.9.

Example 7.1.22 : Our goal is to find an asymptotic expression of $\arccos x$ around $x = 1$. First, note that

$$\forall x \in [0,1], \quad \int_x^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \arccos x.$$

Then, we note that the following equivalent relation,

$$\frac{1}{\sqrt{1-t^2}} = \frac{1}{\sqrt{1+t}\sqrt{1-t}} \sim \frac{1}{\sqrt{2}\sqrt{1-t}}, \quad \text{when } t \to 1.$$

Since $t \mapsto \frac{1}{\sqrt{1-t}}$ is integrable on $[0,1)$, by Proposition 7.1.19 (3), when $x \to 1$, we find that

$$\arccos x = \int_x^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}} \sim \int_x^1 \frac{\mathrm{d}t}{\sqrt{2}\sqrt{1-t}} = \sqrt{2(1-x)}, \quad \text{when } x \to 1.$$

Example 7.1.23 : The following is the Gaussian integral, whose value was computed in Exercise A1.6,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\,\mathrm{d}t = 1.$$

Let us estimate the tail of the above integral

$$F(x) := \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}}\,\mathrm{d}t, \quad \text{when } x \to \infty.$$

(1) First, we have the following asymptotic comparison,

$$e^{-\frac{t^2}{2}} = o(te^{-\frac{t^2}{2}}), \quad \text{when } t \to \infty.$$

It follows from Proposition 7.1.19 (2) that, when $x \to \infty$, we have

$$F(x) = o\Big(\int_x^{\infty} te^{-\frac{t^2}{2}}\,\mathrm{d}t\Big) = o\Big(\Big[-e^{-\frac{t^2}{2}}\Big]_x^{\infty}\Big) = o\Big(e^{-\frac{x^2}{2}}\Big).$$

(2) To get a more precise asymptotic formula for $F(x)$ when $x \to \infty$, we may start with an integration by parts. We write

$$\sqrt{2\pi}F(x) = \int_x^{\infty} \frac{-te^{-\frac{t^2}{2}}}{-t}\,\mathrm{d}t = \left[\frac{e^{-\frac{t^2}{2}}}{-t}\right]_{t=x}^{\infty} - \int_x^{\infty} \frac{e^{-\frac{t^2}{2}}}{t^2}\,\mathrm{d}t = \frac{e^{-\frac{x^2}{2}}}{x} - \int_x^{\infty} \frac{e^{-\frac{t^2}{2}}}{t^2}\,\mathrm{d}t.$$

Moreover, we have

$$\int_x^{\infty} \frac{e^{-\frac{t^2}{2}}}{t^2}\,\mathrm{d}t = o\left(\int_x^{\infty} e^{-\frac{t^2}{2}}\,\mathrm{d}t\right) = o(F(x)), \quad \text{when } x \to \infty,$$

we deduce that

$$F(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{x^2}{2}}}{x}(1 + o(1)), \quad \text{when } x \to \infty.$$

By induction, you may show that for any $n \geqslant 0$,

$$\sqrt{2\pi}F(x) = e^{-\frac{x^2}{2}} \left(\frac{1}{x} + \sum_{k=1}^{n} (-1)^k \frac{(2k-1)!!}{x^{2k+1}}\right) + (-1)^{n+1}(2n+1)!! \int_x^{\infty} \frac{e^{-\frac{t^2}{2}}}{t^{2n+2}}\,\mathrm{d}t,$$

which implies that

$$F(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\left(\frac{1}{x}+\sum_{k=1}^{n}(-1)^k\frac{(2k-1)!!}{x^{2k+1}}\right)(1+o(1)),\quad \text{when } x\to\infty.$$

We recall that for divergent series with asymptotic relations for their non-negative general terms, we may compare their partial sums, see Theorem 6.2.8. When it comes to non-negative non-integrable functions, we may compare their *partial integrals*, as stated in the following proposition.

Proposition 7.1.24 (Comparison for non-integrable functions) **:** *Let $f:[a,b)\to W$ and $g:[a,b)\to \mathbb{R}_+$ be a non-negative non-integrable function. Then, the following properties hold.*

(1) If $f \underset{b}{=} \mathcal{O}(g)$, then $\int_a^x f \underset{x\to b}{=} \mathcal{O}(\int_a^x g)$.

(2) If $f \underset{b}{=} o(g)$, then $\int_a^x f \underset{x\to b}{=} o(\int_a^x g)$.

(3) If $W=\mathbb{R}$ and $f \underset{b}{\sim} g$, then f is non-integrable on $[a,b)$ and $\int_a^x f \underset{x\to b}{\sim} \int_a^x g$.

Proof : Since g is non-negative and not integrable on $[a,b)$, we have

$$\lim_{x\to b}\int_a^x g=+\infty. \tag{7.5}$$

(1) By assumption, there exists $M>0$ and $\delta>0$ such that

$$\forall t\in[b-\delta,b),\quad \|f(t)\|\leqslant Mg(t).$$

Therefore, for any $x\in[b-\delta,b)$, we have

$$\left\|\int_a^x f\right\|\leqslant\int_a^x\|f\|\leqslant\int_a^{b-\delta}\|f\|+M\int_{b-\delta}^x g\leqslant\int_a^{b-\delta}\|f\|+M\int_a^x g.$$

Additionally, from Eq. (7.5), we deduce that there exists $\delta'\in(0,\delta)$ such that

$$\forall x\in[b-\delta',b),\quad \int_a^{b-\delta}\|f\|\leqslant M\int_a^x g.$$

Putting the two above relations together, we get

$$\forall x\in[b-\delta',b),\quad \left\|\int_a^x f\right\|\leqslant 2M\int_a^x g,$$

which is what we want to show.

(2) Let $\varepsilon>0$. By assumption, there exists $\delta>0$ such that

$$\forall t\in[b-\delta,b),\quad \|f(t)\|\leqslant\varepsilon g(t).$$

Therefore, for any $x \in [b-\delta, b)$, we have

$$\left\| \int_a^x f \right\| \leqslant \int_a^{b-\delta} \|f\| + \varepsilon \int_{b-\delta}^x g \leqslant \int_a^{b-\delta} \|f\| + \varepsilon \int_a^x g.$$

Additionally, from Eq. (7.5), we deduce that there exists $\delta' \in (0, \delta)$ such that

$$\forall x \in [b-\delta', b), \qquad \int_a^{b-\delta} \|f\| \leqslant \varepsilon \int_a^x g.$$

Putting the two above relations together, we get

$$\forall x \in [b-\delta', b), \qquad \left\| \int_a^x f \right\| \leqslant 2\varepsilon \int_a^x g,$$

which is what we want to show.

(3) Suppose $f \underset{b}{\sim} g$. This means that $f - g \underset{b}{=} o(g)$. Then we apply the result from (2) to conclude that

$$\int_a^x (f-g) \underset{x\to b}{=} o\Big(\int_a^x g\Big) \Leftrightarrow \int_a^x f \underset{x\to b}{\sim} \int_a^x g.$$

□

Example 7.1.25 : The following integral diverges when $x \to \infty$ (Example 7.1.9),

$$\int_2^x \frac{\mathrm{d}t}{\ln t}.$$

We want to find an equivalent expression of it when $x \to \infty$. By an integration by parts, we find

$$\int_2^x \frac{\mathrm{d}t}{\ln t} = \left[\frac{t}{\ln t}\right]_{t=2}^x + \int_2^x \frac{\mathrm{d}t}{(\ln t)^2}.$$

We also have the following asymptotic comparison,

$$\frac{1}{(\ln t)^2} = o\left(\frac{1}{\ln t}\right), \qquad \text{when } t \to \infty,$$

which leads to

$$\int_2^x \frac{\mathrm{d}t}{(\ln t)^2} = o\left(\int_2^x \frac{\mathrm{d}t}{\ln t}\right), \qquad \text{when } x \to \infty.$$

Putting all the above relations together, we deduce

$$\int_2^x \frac{\mathrm{d}t}{\ln t} \sim \frac{x}{\ln x}, \qquad \text{when } x \to \infty.$$

7.2 Improper integrals

As we mentioned in Section 7.1.1, to study the integrals of functions on general intervals, it is enough to consider the case $I = [a, b)$ where $-\infty < a < b \leqslant +\infty$. The integrands that we are going to consider below are not necessarily non-negative. If the interval of integration writes $I = (a, b)$, where $-\infty < a < b < +\infty$, then as in Proposition 7.1.15, we need to take any $c \in (a, b)$ and divide the interval into $I_- = (a, c]$ and $I_+ = [c, b)$, and deal with them independently, see Definition 7.2.9.

7.2.1 Definition and properties

Definition 7.2.1 : Let $f : [a, b) \to \mathbb{R}$ be a piecewise continuous function.

(1) We say that the integral $\int_{[a,b)} f = \int_{[a,b)} f(t)\,\mathrm{d}t$ converges if the function

$$x \mapsto \int_{[a,x]} f(t)\,\mathrm{d}t := \int_a^x f(t)\,\mathrm{d}t$$

is well defined and has a finite limit when $x \to b-$. In this case, the limit is denoted by $\int_{[a,b)} f$ or $\int_{[a,b)} f(t)\,\mathrm{d}t$. Such an integral is called an improper integral (瑕積分).

(2) If the above limit does not exist, then we say that the integral $\int_{[a,b)} f$ diverges.

Remark 7.2.2 : In the case that f is non-negative, then the convergence defined in Definition 7.2.1 coincides with the notion of integrability defined in Definition 7.1.4.

Proposition 7.2.3 (Cauchy's criterion) **:** *Let $f : [a, b) \to \mathbb{R}$ be a piecewise continuous function. The following properties are equivalent.*

(1) The integral $\int_{[a,b)} f$ converges.

(2) For any $\varepsilon > 0$, there exists $c \in [a, b)$ such that for any $x, y \in [c, b)$ with $x < y$, we have

$$\Big|\int_x^y f(t)\,\mathrm{d}t\Big| < \varepsilon.$$

Proof : Since $(\mathbb{R}, |\cdot|)$ is complete, the convergence and the Cauchy's property are equivalent. □

Proposition 7.2.4 : *Let $f : [a, b) \to \mathbb{R}$ be a piecewise continuous function. Let $c \in [a, b)$. Then, the following properties hold.*

(1) Both integrals $\int_{[a,b)} f$ and $\int_{[c,b)} f$ have the same behavior.

(2) If they both converge, we have

$$\int_{[a,b)} f = \int_{[a,c]} f + \int_{[c,b)} f.$$

Proof : Both properties are direct consequences of the cyclic relation on segments (Proposition 5.2.10),

$$\forall c \in [a,b), \forall x \in [c,b), \quad \int_a^x f = \int_a^c f + \int_c^x f.$$

□

Corollary 7.2.5 : *Let $f : [a,b) \to \mathbb{R}$ be a piecewise continuous function. If the integral $\int_{[a,b)} f$ converges, then when $x \to b-$, we have*

$$\int_{[x,b)} f \to 0.$$

In such a case, the integral $\int_{[a,x]} f$ is called partial integral, *and the integral $\int_{[x,b)} f$ is called the* remainder integral *of the integral $\int_{[a,b)} f$.*

Proof : Suppose that the limit when $x \to b-$ of $\int_{[a,x]} f$ is finite, and is denoted by $\int_{[a,b)} f$. This means that $\int_{[x,b)} f = \int_{[a,b)} f - \int_{[a,x]} f$ tends to 0 when $x \to b-$. □

Proposition 7.2.6 : *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Assume that $f \in R(x;a,b)$, then the improper integral $\int_{[a,b)} f$ converges and we have*

$$\int_{[a,b)} f = \int_a^b f.$$

Remark 7.2.7 : This proposition shows that, if a function $f : [a,b] \to \mathbb{R}$ is Riemann-integrable, then its integral and the improper integral on $[a,b)$ coincide. In other words, the definition of the improper integral in Definition 7.2.1 generalizes the notion we saw previously on segments in Chapter 5. Therefore, we may also denote the improper integral using the classical notation,

$$\int_a^b f := \int_{[a,b)} f,$$

whenever $f : [a,b) \to \mathbb{R}$ is a function that is piecewise continuous on $[a,b)$ and such that the integral $\int_{[a,b)} f$ converges.

Proposition 7.2.8 : *Let $f : [a,b) \to \mathbb{R}$ be bounded and F be a primitive of f. The following properties are equivalent,*

(1) The integral $\int_a^b f$ converges,

(2) F has a finite limit at $b-$.

In this case, we have

$$\int_a^b f = \lim_{x \to b-} F(x) - F(a),$$

and the function

$$x \mapsto \int_x^b f$$

is well defined on $[a, b)$, differentiable on (a, b), with derivative $-f$.

Proof : The proof follows directly from what has been said above and is left as an exercise. □

Definition 7.2.9 : Let $-\infty < a < b < +\infty$ and $f : (a, b) \to \mathbb{R}$ be a piecewise continuous function. Fix $c \in (a, b)$. We say that the *improper integral*

$$\int_{(a,b)} f := \int_a^b f$$

is well defined, if both $\int_{(a,c]} f$ and $\int_{[c,b)} f$ are well defined.

Remark 7.2.10 : In Definition 7.2.9, we note that the choice of $c \in (a, b)$ is irrelevant. In fact, for any $c_1, c_2 \in (a, b)$ with $c_1 < c_2$, we have

- $\int_{(a,c_1]} f$ converges if and only if $\int_{(a,c_2]} f$ converges;
- $\int_{[c_1,b)} f$ converges if and only if $\int_{[c_2,b)} f$ converges.

Example 7.2.11 : Let us consider the function $f : (0, 1) \to \mathbb{R}$, defined by

$$\forall x \in (0, 1), \quad f(x) = \frac{1}{x} - \frac{1}{1-x}.$$

(1) If we consider $I_n = [\frac{1}{n}, 1 - \frac{1}{n}]$, then we have, for any $n \geqslant 1$,

$$\int_{I_n} f = \big[\ln x + \ln(1-x)\big]_{x=\frac{1}{n}}^{1-\frac{1}{n}} = 0.$$

(2) If we consider $J_n = [\frac{1}{n}, 1 - \frac{2}{n}]$, then we have, for any $n \geqslant 1$,

$$\int_{J_n} f = \big[\ln x + \ln(1-x)\big]_{x=\frac{1}{n}}^{1-\frac{2}{n}} = \ln\left(1 - \frac{2}{n}\right) - \ln\left(1 - \frac{1}{n}\right) + \ln 2,$$

which implies that

$$\int_{J_n} f \xrightarrow[n\to\infty]{} \ln 2.$$

(3) However, the integral of f on $(0, \frac{1}{2}]$ does not converge. For $n \geqslant 1$, consider $K_n = (\frac{1}{n}, \frac{1}{2}]$, then

$$\int_{K_n} f = \big[\ln x + \ln(1-x)\big]_{x=\frac{1}{n}}^{\frac{1}{2}} = 2\ln\frac{1}{2} - \ln\frac{1}{n} - \ln\left(1 - \frac{1}{n}\right) \xrightarrow[n\to\infty]{} +\infty.$$

The existence of the improper integral of f on $(0, 1)$ needs to be checked as in (3), see Definition 7.2.9.

In conclusion, (1) and (2) give different finite limits because f is *not integrable* in the sense of Definition 7.1.10.

7.2.2 Conditional convergence

We saw that there are series that converge conditionally, i.e. they converge but do not converge absolutely, see Section 6.4. The way we define the integral on a general interval in Definition 7.2.1 is similar to the definition of a series, so we also have integrals that converge, but do not converge absolutely, and we say that such integrals *covnerge conditionally*. For such integrals, we also have Abel's transform, and the corresponding Dirichlet's test for convergent integrals, see Theorem 7.2.14.

Definition 7.2.12 : Given a piecewise continuous function $f : I \to \mathbb{R}$, we say that its integral $\int_I f$ is *conditionally convergent* if its integral converges (in the sense of an improper integral, see Definition 7.2.1 and Definition 7.2.9) but does not converge absolutely (or f is not integrable on I, see Definition 7.1.10).

Example 7.2.13 : The following integral is conditionally convergent,

$$\int_\pi^\infty \frac{\sin x}{x}\,\mathrm{d}x. \tag{7.6}$$

- Let us show that this integral does not converge absolutely. For every $k \in \mathbb{N}$, we have

  $$\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| \mathrm{d}x \geqslant \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{(k+1)\pi}\,\mathrm{d}x = \frac{2}{(k+1)\pi}.$$

  Since the series $\sum_{n\geqslant 1} \frac{1}{n}$ diverges, we deduce that

  $$\int_\pi^\infty \left|\frac{\sin x}{x}\right| \mathrm{d}x$$

  also diverges. Therefore, Eq. (7.6) does not converge absolutely.

- Let $t > \pi$. We write

  $$\int_\pi^t \frac{\sin x}{x}\,\mathrm{d}x = \left[-\frac{\cos x}{x}\right]_{x=\pi}^t - \int_\pi^t \frac{\cos x}{x^2}\,\mathrm{d}x = \frac{1}{\pi} - \frac{\cos t}{t} - \int_\pi^t \frac{\cos x}{x^2}\,\mathrm{d}x.$$

  When $t \to \infty$, we have $\frac{\cos t}{t} \to 0$. Additionally, since the function $x \mapsto \frac{\cos x}{x^2}$ is integrable on $[\pi, \infty)$, we know that the integral

  $$\int_\pi^t \frac{\cos x}{x^2}\,\mathrm{d}x$$

  converges when $t \to \infty$.

Theorem 7.2.14 (Abel's rule) **:** *Let $f : [a, b) \to \mathbb{R}$ be of class $\mathcal{C}^1$ and $g : [a, b) \to \mathbb{R}$ be continuous. Suppose that*

(i) f is decreasing with $\lim_{x\to b} f(x) = 0$;

(ii) There exists $M > 0$ such that for any $x \in [a, b)$, we have $|\int_a^x g(t)\,\mathrm{d}t| \leqslant M$.

Then, the integral $\int_a^b f(t)g(t)\,\mathrm{d}t$ is convergent.

Proof : Let $\varepsilon > 0$. Due to the assumption (i), we may find $A \in [a, b)$ such that

$$\forall x \in [A, b), \quad 0 \leqslant f(x) \leqslant \varepsilon.$$

We may also define

$$\forall x \in [a, b), \quad G(x) = \int_a^x g(t)\,\mathrm{d}t,$$

and it follows from (ii) that $|G(x)| \leqslant M$ for all $x \in [a, b)$.

For any $x, y \in [a, b)$ with $y \geqslant x \geqslant A$, an integration by parts gives us

$$\int_x^y f(t)g(t)\,\mathrm{d}t = \Big[f(t)G(t)\Big]_{t=x}^{y} - \int_x^y f'(t)G(t)\,\mathrm{d}t.$$

Let us control each of the terms on the right side of the above formula. We have

$$|f(y)G(y) - f(x)G(x)| \leqslant 2\varepsilon M,$$

and

$$\Big|\int_x^y f'(t)G(t)\,\mathrm{d}t\Big| \leqslant \int_x^y (-f'(t))M\,\mathrm{d}t = [f(x) - f(y)]M \leqslant \varepsilon M.$$

This means that

$$\Big|\int_x^y f(t)g(t)\,\mathrm{d}t\Big| \leqslant 3\varepsilon M,$$

so the Cauchy's criterion (Proposition 7.2.3) is satisfied, which implies the convergence of $\int_a^b f(t)g(t)\,\mathrm{d}t$. □

Example 7.2.15 : When $\alpha > 0$, the following integrals converge,

$$\int_1^\infty \frac{\sin x}{x^\alpha}\,\mathrm{d}x, \quad \int_1^\infty \frac{\cos x}{x^\alpha}\,\mathrm{d}x, \quad \text{and} \quad \int_1^\infty \frac{e^{\mathrm{i}\,x}}{x^\alpha}\,\mathrm{d}x.$$

This is a direct consequence of Theorem 7.2.14, or by an integration by parts as in Example 7.2.13.

Example 7.2.16 : Let us consider the two following functions defined on $[1, +\infty)$,

$$\forall x \in [1, +\infty), \quad f(x) = \frac{e^{\mathrm{i}\,x}}{\sqrt{x}} \quad \text{and} \quad g(x) = \frac{e^{\mathrm{i}\,x}}{\sqrt{x}} + \frac{1}{x}.$$

These two functions are equivalent when $x \to +\infty$.

- It follows from Example 7.2.15 that $\int_1^\infty f(x)\,\mathrm{d}x$ converges.
- $\int_1^\infty g(x)\,\mathrm{d}x$ cannot converge, because otherwise, $\int_1^\infty (g(x)-f(x))\,\mathrm{d}x = \int_1^\infty \frac{1}{x}\,\mathrm{d}x$ would converge, which is false.

7.3 Laplace's method

To conclude the chapter, we introduce the Laplace's method, which is very useful when it comes to finding an asymptotic expression.

Theorem 7.3.1 (Laplace's method) **:** *Let $-\infty \leqslant a < b \leqslant +\infty$, and two functions $g, h : (a,b) \to \mathbb{R}$ be of class C^2. Suppose that*

(i) The function $x \mapsto g(x)e^{h(x)}$ is integrable on (a,b);

(ii) There exists $c \in (a,b)$ such that

(a) h is increasing on (a,c) and decreasing on (c,b), with $h''(c) < 0$;

(b) $g(c) \neq 0$.

Then, when $\lambda \to +\infty$, we have

$$\int_a^b g(x)e^{\lambda h(x)}\,\mathrm{d}x \sim \sqrt{\frac{2\pi}{-\lambda h''(c)}} \cdot g(c)e^{\lambda h(c)}. \tag{7.7}$$

Proof : The rigorous proof of this theorem is more involved, and only give a sketch below to illustrate the ideas. Additionally, let us take the function g be a constant function $g \equiv 1$. We write the Taylor expansion of h around c,

$$h(c+x) = h(c) + \underbrace{h'(c)}_{=0}x + h''(c)\frac{x^2}{2} + o(x^2) \quad \text{when } x \to 0.$$

We have the following approximations, which need to be justified carefully,

$$\begin{aligned}\int_a^b e^{\lambda h(x)}\,\mathrm{d}x &\approx \int_{-\varepsilon}^{\varepsilon} e^{\lambda h(c+x)}\,\mathrm{d}x \approx \int_{-\varepsilon}^{\varepsilon} e^{\lambda h(c) + \frac{\lambda h''(c)}{2}x^2}\,\mathrm{d}x \\ &\approx e^{\lambda h(c)} \int_{-\infty}^{\infty} e^{\frac{\lambda h''(c)}{2}x^2}\,\mathrm{d}x = \sqrt{\frac{2\pi}{-\lambda h''(c)}} \cdot e^{\lambda h(x)},\end{aligned}$$

where the last equality follows from the Gaussian integral, see Exercise A1.6. □

Example 7.3.2 : Let us consider the Gamma function defined in Example 7.1.21,

$$\forall x > 0, \quad \Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,\mathrm{d}t.$$

We recall the recurrence relation $\Gamma(x+1) = x\Gamma(x)$ satisfied by all $x > 0$. In particular, for any integer $n \geqslant 1$, we have $\Gamma(n+1) = n!$. We may apply Laplace's method to the Gamma function to find an asymptotic expression for $n!$, called Stirling's formula, also see Exercise 6.12.

We have

$$n! = \Gamma(n+1) = \int_0^{+\infty} t^n e^{-t} \, \mathrm{d}t = \int_0^{+\infty} e^{n \ln t - t} \, \mathrm{d}t.$$

To make the integrand in the above formula in the form as in Eq. (7.7), we make the change of variables $t = nx$, and we have

$$n! = \int_0^{+\infty} n e^{n \ln(nx) - nx} \, \mathrm{d}x = n^{n+1} \int_0^{+\infty} e^{n(\ln x - x)} \, \mathrm{d}x.$$

Let us consider the function $h : [0, +\infty) \to \mathbb{R}$, $x \mapsto \ln x - x$. We have

$$\forall x > 0, \quad h'(x) = \frac{1}{x} - 1 \quad \text{and} \quad h''(x) = -\frac{1}{x^2} < 0.$$

Therefore, we may take $c = 1$ and check that h is increasing on $(0, 1)$ and decreasing on $(1, +\infty)$. By applying Laplace's method, we find

$$n! \sim n^{n+1} \sqrt{\frac{2\pi}{n}} e^{-n} = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$