

8

Sequences and series of functions

函數的數列與級數

Let A be a set, and (M, d) be a metric space. We denote by $\mathcal{F}(A, M)$ the space of functions from A to M , and by $\mathcal{B}(A, M)$ the space of bounded functions from A to M . Instead of a metric space, we may also consider a vector spaces W over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , so that we have the $+$ operation. This vector space is equipped with a norm that we denote by $\|\cdot\|$.

In this chapter, we are interested in sequences and series of functions, which can also be seen as sequences and series with terms in $\mathcal{F}(A, M)$ or $\mathcal{F}(A, W)$.

8.1 Notions of convergence

We discuss different notions of convergence for sequences of functions, then for series of functions.

8.1.1 Sequences of functions

For a sequence of functions, we have different notions of convergence. Below we are going to discuss the *pointwise convergence* (Definition 8.1.1), and a stronger notion of convergence, called *uniform convergence* (Definition 8.1.4).

Definition 8.1.1 : Let $(f_n)_{n \geq 1}$ be a sequence of functions from A to M , that is, they are elements of $\mathcal{F}(A, M)$.

- Let $f \in \mathcal{F}(A, M)$. We say that the sequence $(f_n)_{n \geq 1}$ *converges pointwise* (逐點收斂) to f if for every $x \in A$, we have $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ in (M, d) .
- We say that the sequence $(f_n)_{n \geq 1}$ converges pointwise if there exists $f \in \mathcal{F}(A, M)$ such that $(f_n)_{n \geq 1}$ converges pointwise to f .
- Let $B \subseteq A$ be a subset. We say that $(f_n)_{n \geq 1}$ converges pointwise on B if $((f_n)|_B)_{n \geq 1}$ converges pointwise.

令 A 為集合，且 (M, d) 為賦距空間。我們記 $\mathcal{F}(A, M)$ 為由 A 映射到 M 的函數所構成的空間，還有 $\mathcal{B}(A, M)$ 為由 A 映射至 M 的有界函數所構成的空間。我們也可以把賦距空間取做在 $\mathbb{K} = \mathbb{R}$ 或 \mathbb{C} 上的向量空間 W ，這樣我們就會有 $+$ 的運算。我們把這個向量空間所賦予的範數記作 $\|\cdot\|$ 。

在這一章中，我們有興趣的是函數的序列和級數，我們也可以把他們看做是項在 $\mathcal{F}(A, M)$ 或 $\mathcal{F}(A, W)$ 裡面的序列和級數。

第一節 收斂概念

我們討論對於函數序列以及函數級數的不同收斂概念。

第一小節 函數序列

對於函數序列來說，我們有不同收斂概念。我們接下來會討論逐點收斂（定義 8.1.1）以及比較強的收斂概念，稱作均勻收斂（定義 8.1.4）。

定義 8.1.1 : 令 $(f_n)_{n \geq 1}$ 為由 A 映射至 M 的函數所構成的序列，也就是說他們是 $\mathcal{F}(A, M)$ 中的元素。

- 令 $f \in \mathcal{F}(A, M)$ 。如果在 (M, d) 中，對於每個 $x \in A$ ，我們有 $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ ，則我們說序列 $(f_n)_{n \geq 1}$ 會逐點收斂 (pointwise convergence) 至 f 。
- 如果存在 $f \in \mathcal{F}(A, M)$ 使得 $(f_n)_{n \geq 1}$ 逐點收斂至 f ，則我們說序列 $(f_n)_{n \geq 1}$ 會逐點收斂。
- 令 $B \subseteq A$ 為子集合。如果 $((f_n)|_B)_{n \geq 1}$ 會逐點收斂，則我們說 $(f_n)_{n \geq 1}$ 在 B 上逐點收斂。

Example 8.1.2 : Let us consider the sequence of functions $(f_n)_{n \geq 1}$ defined by

$$\begin{aligned} \forall n \geq 1, \quad f_n : [0, 1] &\rightarrow \mathbb{R} \\ x &\mapsto x^n. \end{aligned}$$

The sequence of functions $(f_n)_{n \geq 1}$ converges pointwise to the indicator function $f = \mathbb{1}_{\{1\}}$ on $[0, 1]$.

Remark 8.1.3 :

- (1) If a sequence $(f_n)_{n \geq 1}$ converges pointwise, then its limit function f is unique.
- (2) Let $(f_n)_{n \geq 1}$ be a pointwise convergent sequence of functions. Suppose that these functions take values in a finite dimensional vector space $(W, \|\cdot\|)$, then the limit does not depend on the norm, because all the norms are equivalent in W .
- (3) Properties such as linearity, product, inequality, monotonicity, etc., are preserved for the pointwise convergence of functions.
- (4) We see that in Example 8.1.2, the continuity at 1 is not preserved in the limit. Indeed, for all $n \in \mathbb{N}$, the function f_n is continuous, but the limit function f is not continuous at 1. In other words, the following two iterated limits are different,

$$\lim_{x \rightarrow 1} \lim_{n \rightarrow \infty} f_n(x) = \lim_{x \rightarrow 1} f(x) = 0 \neq 1 = \lim_{n \rightarrow \infty} 1 = \lim_{n \rightarrow \infty} \lim_{x \rightarrow 1} f_n(x).$$

We have already encountered a similar example in Example 6.7.2.

- (5) Analytic properties such as continuity and differentiability are not preserved for the pointwise convergence. We will define the notion of uniform convergence below (Definition 8.1.4), and will see that analytic properties can be preserved if this convergence occurs (Proposition 8.2.1).

Definition 8.1.4 : Let $(f_n)_{n \geq 1}$ be a sequence of functions from A to M .

- Let $f \in \mathcal{F}(A, M)$. We say that the sequence $(f_n)_{n \geq 1}$ *converges uniformly* (均勻收斂) to f if

$$\forall \varepsilon > 0, \exists N \geq 1, \forall n \geq N, \forall x \in A, \quad d(f_n(x), f(x)) \leq \varepsilon. \quad (8.1)$$

- We say that the sequence $(f_n)_{n \geq 1}$ converges uniformly if there exists $f \in \mathcal{F}(A, M)$ such that $(f_n)_{n \geq 1}$ converges uniformly to f .

範例 8.1.2 : 讓我們考慮定義如下的函數序列 $(f_n)_{n \geq 1}$:

$$\begin{aligned} \forall n \geq 1, \quad f_n : [0, 1] &\rightarrow \mathbb{R} \\ x &\mapsto x^n. \end{aligned}$$

函數序列 $(f_n)_{n \geq 1}$ 會在 $[0, 1]$ 上逐點收斂到指標函數 $f = \mathbb{1}_{\{1\}}$ 。

註解 8.1.3 :

- (1) 如果序列 $(f_n)_{n \geq 1}$ 逐點收斂，則他的極限函數 f 是唯一的。
- (2) 令 $(f_n)_{n \geq 1}$ 為會逐點收斂的函數序列。假設這些函數取值在有限維度的向量空間 $(W, \|\cdot\|)$ 中，那麼他的極限不會取決於他的範數，因為所有 W 中的範數都是等價的。
- (3) 函數的性質像是線性、乘積、不等式、單調性等等都會被逐點收斂所保存。
- (4) 我們在範例 8.1.2 中看到，在 1 的連續性在極限時不會被保存下來。的確，對於所有 $n \in \mathbb{N}$ ，函數 f_n 是連續的，但是極限函數 f 在 1 不連續。換句話說，下面這兩個迭代極限是不同的：

$$\lim_{x \rightarrow 1} \lim_{n \rightarrow \infty} f_n(x) = \lim_{x \rightarrow 1} f(x) = 0 \neq 1 = \lim_{n \rightarrow \infty} 1 = \lim_{n \rightarrow \infty} \lim_{x \rightarrow 1} f_n(x).$$

我們在範例 6.7.2 中有看過類似的現象。

- (5) 分析性質像是連續性和可微分性是無法被逐點收斂所保存的。稍後我們會定義均勻收斂的概念 (定義 8.1.4)，並且看到分析性質可以被這樣的收斂概念所保存 (命題 8.2.1)。

定義 8.1.4 : 令 $(f_n)_{n \geq 1}$ 為由 A 映射至 M 的函數所構成的序列。

- 令 $f \in \mathcal{F}(A, M)$ 。如果

$$\forall \varepsilon > 0, \exists N \geq 1, \forall n \geq N, \forall x \in A, \quad d(f_n(x), f(x)) \leq \varepsilon, \quad (8.1)$$

則我們說序列 $(f_n)_{n \geq 1}$ 會均勻收斂 (uniform convergence) 至 f 。

- Let $B \subseteq A$ be a subset. We say that $(f_n)_{n \geq 1}$ converges uniformly on B if $((f_n)|_B)_{n \geq 1}$ converges uniformly.

Remark 8.1.5 : We may rewrite the definition of pointwise convergence using quantifiers. We say that $(f_n)_{n \geq 1}$ converges pointwise to f if

$$\forall x \in A, \forall \varepsilon > 0, \exists N \geq 1, \forall n \geq N, \quad d(f_n(x), f(x)) \leq \varepsilon. \quad (8.2)$$

If we compare Eq. (8.1) and Eq. (8.2), we see that the choice of N depends on $x \in A$ in the case of pointwise convergence, but does not depend on $x \in A$ in the case of uniform convergence. This is the reason why the convergence characterized by the condition Eq. (8.1) is called *uniform* convergence. This remark easily leads to the following corollary.

Corollary 8.1.6 : *If the sequence of functions $(f_n)_{n \geq 1}$ converges uniformly to f , then it converges pointwise to f .*

Remark 8.1.7 : Due to the uniqueness of the pointwise limit (Remark 8.1.3), we deduce the uniqueness of the uniform limit of a sequence of functions. To show that a sequence of functions $(f_n)_{n \geq 1}$ converges uniformly, we may start by computing its pointwise limit f , then show that $(f_n)_{n \geq 1}$ converges uniformly to f .

Proposition 8.1.8 (Cauchy's criterion for uniform convergence) : *Suppose that (M, d) is a complete metric space. Let $(f_n)_{n \geq 1}$ be a sequence of functions in $\mathcal{F}(A, M)$. Then, $(f_n)_{n \geq 1}$ converges uniformly if and only if it satisfies the uniform Cauchy condition, that is*

$$\forall \varepsilon > 0, \exists N \geq 1, \forall m, n \geq N, \forall x \in A, \quad d(f_n(x), f_m(x)) \leq \varepsilon.$$

- 如果存在 $f \in \mathcal{F}(A, M)$ 使得 $(f_n)_{n \geq 1}$ 會均勻收斂至 f ，則我們說序列 $(f_n)_{n \geq 1}$ 會均勻收斂。
- 令 $B \subseteq A$ 為子集合。如果 $((f_n)|_B)_{n \geq 1}$ 會均勻收斂，則我們說 $(f_n)_{n \geq 1}$ 在 B 上均勻收斂。

註解 8.1.5 : 我們可以把逐點收斂的定義數學化。如果

$$\forall x \in A, \forall \varepsilon > 0, \exists N \geq 1, \forall n \geq N, \quad d(f_n(x), f(x)) \leq \varepsilon, \quad (8.2)$$

則我們說 $(f_n)_{n \geq 1}$ 會逐點收斂至 f 。如果我們比較式 (8.1) 和式 (8.2)，我們看到在逐點收斂的情況下， N 的選擇會取決於 $x \in A$ ；但在均勻收斂的情況下，則不會取決於 $x \in A$ 。這就是為什麼我們把式 (8.1) 條件所刻劃的收斂稱作均勻收斂。這個註解會讓我們得到下面這個系理。

系理 8.1.6 : 如果函數序列 $(f_n)_{n \geq 1}$ 會均勻收斂到 f ，那麼他會逐點收斂到 f 。

註解 8.1.7 : 透過逐點收斂的唯一性（註解 8.1.3），我們能推得函數序列均勻收斂極限的唯一性。要證明函數序列 $(f_n)_{n \geq 1}$ 均勻收斂，我們可以先從計算他逐點收斂的極限 f 開始，然後證明 $(f_n)_{n \geq 1}$ 會均勻收斂到 f 。

命題 8.1.8 【均勻收斂的柯西準則】 : 假設 (M, d) 是個完備賦距空間。令 $(f_n)_{n \geq 1}$ 為在 $\mathcal{F}(A, M)$ 中的序列。則 $(f_n)_{n \geq 1}$ 會均勻收斂，若且唯若他滿足均勻柯西準則，換句話說：

$$\forall \varepsilon > 0, \exists N \geq 1, \forall m, n \geq N, \forall x \in A, \quad d(f_n(x), f_m(x)) \leq \varepsilon.$$

Proof : Given $\varepsilon > 0$. Let $N \geq 1$ such that the uniform Cauchy condition holds, that is

$$\forall m, n \geq N, \forall x \in A, \quad d(f_n(x), f_m(x)) \leq \varepsilon. \quad (8.3)$$

For each $x \in A$, we see that $(f_n(x))_{n \geq 1}$ is a Cauchy sequence, so it converges to some limit that we denote by $f(x)$. By taking the limit $m \rightarrow \infty$ in Eq. (8.3), we find

$$\forall n \geq N, \forall x \in A, \quad d(f_n(x), f(x)) \leq \varepsilon,$$

which is the characterization of $(f_n)_{n \geq 1}$ uniformly converging to f from Eq. (8.1). \square

Definition 8.1.9 : The notion of uniform convergence can be described using a distance (or a norm).

- Let (M, d) be a metric space and $\mathcal{B}(A, M)$ be the set of bounded functions from A to M . We may equip $\mathcal{B}(A, M)$ with the following distance

$$\forall f, g \in \mathcal{B}(A, M), \quad d_\infty(f, g) = d_{\infty, A}(f, g) := \sup_{x \in A} d(f(x), g(x)), \quad (8.4)$$

called the *distance of uniform convergence*. A sequence of bounded functions $(f_n)_{n \geq 1}$ converges uniformly to f is equivalent to the convergence of $(f_n)_{n \geq 1}$ to f with respect to the distance d_∞ .

- Let $(W, \|\cdot\|)$ be a normed vector space and $\mathcal{B}(A, W)$ be the set of bounded functions from A to W . We may equip $\mathcal{B}(A, W)$ with the following norm

$$\forall f \in \mathcal{B}(A, W), \quad \|f\|_\infty = \|f\|_{\infty, A} := \sup_{x \in A} \|f(x)\|, \quad (8.5)$$

called the *norm of uniform convergence*. A sequence of bounded functions $(f_n)_{n \geq 1}$ converges uniformly to f is equivalent to the convergence of $(f_n)_{n \geq 1}$ to f with respect to the norm $\|\cdot\|_\infty$.

Proposition 8.1.10 : Let $(W, \|\cdot\|)$ be a Banach space. Then, the following properties hold.

- (1) The space of bounded functions $\mathcal{B}(A, W)$ equipped with the norm $\|\cdot\|_\infty$, defined in Eq. (8.5), is a Banach space.

證明 : 給定 $\varepsilon > 0$ 。令 $N \geq 1$ 使得均勻柯西準則成立，也就是說：

$$\forall m, n \geq N, \forall x \in A, \quad d(f_n(x), f_m(x)) \leq \varepsilon. \quad (8.3)$$

對於每個 $x \in A$ ，我們看到 $(f_n(x))_{n \geq 1}$ 是個柯西序列，所以會收斂到某個極限，我們記作 $f(x)$ 。藉由在式 (8.3) 中對 $m \rightarrow \infty$ 取極限，我們得到

$$\forall n \geq N, \forall x \in A, \quad d(f_n(x), f(x)) \leq \varepsilon,$$

這剛好就是式 (8.1) 中，刻劃 $(f_n)_{n \geq 1}$ 均勻收斂到 f 的方式。 \square

定義 8.1.9 : 均勻收斂的概念可以使用距離（或是範數）來描述。

- 令 (M, d) 為賦距空間以及 $\mathcal{B}(A, M)$ 為由 A 映射到 M 有界函數所構成的集合。我們可以賦予 $\mathcal{B}(A, M)$ 下面這個距離：

$$\forall f, g \in \mathcal{B}(A, M), \quad d_\infty(f, g) = d_{\infty, A}(f, g) := \sup_{x \in A} d(f(x), g(x)), \quad (8.4)$$

稱作均勻收斂距離。對有界函數序列 $(f_n)_{n \geq 1}$ 來說，他會均勻收斂到 f 與在距離 d_∞ 之下， $(f_n)_{n \geq 1}$ 會收斂到 f 等價。

- 令 $(W, \|\cdot\|)$ 為賦範向量空間以及 $\mathcal{B}(A, W)$ 為由 A 映射到 W 有界函數所構成的集合。我們可以賦予 $\mathcal{B}(A, W)$ 下面這個範數：

$$\forall f \in \mathcal{B}(A, W), \quad \|f\|_\infty = \|f\|_{\infty, A} := \sup_{x \in A} \|f(x)\|, \quad (8.5)$$

稱作均勻收斂範數。對有界函數序列 $(f_n)_{n \geq 1}$ 來說，他會均勻收斂到 f 與在範數 $\|\cdot\|_\infty$ 之下， $(f_n)_{n \geq 1}$ 會收斂到 f 等價。

命題 8.1.10 : 令 $(W, \|\cdot\|)$ 為 Banach 空間。那麼下列性質成立。

- (1) 有界函數空間 $\mathcal{B}(A, W)$ 在賦予式 (8.5) 中定義的範數 $\|\cdot\|_\infty$ 時，是個 Banach 空間。

(2) A sequence $(f_n)_{n \geq 1}$ of $\mathcal{B}(A, W)$ converges uniformly to $f \in \mathcal{B}(A, W)$ if and only if $(f_n)_{n \geq 1}$ converges to f under the norm $\|\cdot\|_\infty$ given in Eq. (8.5), that is $\|f_n - f\|_\infty \xrightarrow{n \rightarrow \infty} 0$.

Proof :

(1) It is not hard to check that $\|\cdot\|_\infty$ defines a norm on the vector space $\mathcal{B}(A, W)$. To check that it is complete, let us be given a sequence $(f_n)_{n \geq 1}$ in $\mathcal{B}(A, W)$, which is Cauchy with respect to the norm $\|\cdot\|_\infty$. For every $x \in A$, we know that $(f_n(x))_{n \geq 1}$ is a Cauchy sequence in the Banach space $(W, \|\cdot\|)$, so it converges to some limit $f(x) := \lim_{n \rightarrow \infty} f_n(x)$. Since $(f_n)_{n \geq 1}$ is Cauchy in $(\mathcal{B}(A, W), \|\cdot\|_\infty)$, there exists $M > 0$ such that $\|f_n\|_\infty \leq M$ for all $n \geq 1$. Therefore, for every $x \in A$, we have $\|f(x)\| = \lim_{n \rightarrow \infty} \|f_n(x)\| \leq M$, so $\|f\|_\infty \leq M$, that is $f \in \mathcal{B}(A, W)$. In the end, it is not hard to check that $\|f_n - f\|_\infty \xrightarrow{n \rightarrow \infty} 0$, so we conclude that $(\mathcal{B}(A, W), \|\cdot\|_\infty)$ is complete.

(2) It is exactly a rewriting of Eq. (8.1) in the normed vector space $(W, \|\cdot\|)$ with help of the new norm defined in Eq. (8.5). \square

Example 8.1.11 : Consider the sequence of functions $(f_n)_{n \geq 1}$ defined by

$$\forall n \in \mathbb{N}, \quad \forall x \in [0, 1], \quad f_n(x) = x^n(1 - x).$$

It is not hard to see that $(f_n)_{n \geq 1}$ converges pointwise to the zero function. For every $n \in \mathbb{N}$, the function f_n is of class \mathcal{C}^∞ , so we may take its derivative to find its extrema on $[0, 1]$. We have

$$\forall x \in [0, 1], \quad f'_n(x) = nx^{n-1}\left(1 - \frac{n+1}{n}x\right).$$

Therefore, the function f_n is increasing on $[0, \frac{n}{n+1}]$ and decreasing on $[\frac{n}{n+1}, 1]$ with maximum at $x_n = \frac{n}{n+1}$, that is

$$\forall x \in [0, 1], \quad f_n(x) \leq f_n(x_n) = \frac{1}{n+1} \left(\frac{n}{n+1}\right)^n \leq \frac{1}{n+1} \xrightarrow{n \rightarrow \infty} 0.$$

Therefore, the sequence $(f_n)_{n \geq 1}$ converges uniformly to the zero function on $[0, 1]$.

(2) 在 $\mathcal{B}(A, W)$ 中的序列 $(f_n)_{n \geq 1}$ 會均勻收斂到 $f \in \mathcal{B}(A, W)$ 若且唯若 $(f_n)_{n \geq 1}$ 在式 (8.5) 中定義的範數 $\|\cdot\|_\infty$ 之下會收斂到 f ，換句話說 $\|f_n - f\|_\infty \xrightarrow{n \rightarrow \infty} 0$ 。

證明 :

(1) 不難檢查 $\|\cdot\|_\infty$ 定義在向量空間 $\mathcal{B}(A, W)$ 上的範數。接著我們檢查這是個完備空間。我們給定 $\mathcal{B}(A, W)$ 中的序列 $(f_n)_{n \geq 1}$ ，使得他對於範數 $\|\cdot\|_\infty$ 來說是個柯西序列。對於每個 $x \in A$ ，我們知道 $(f_n(x))_{n \geq 1}$ 是個在 Banach 空間 $(W, \|\cdot\|)$ 中的柯西序列，所以他會收斂到某個極限 $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ 。由於 $(f_n)_{n \geq 1}$ 是個在 $(\mathcal{B}(A, W), \|\cdot\|_\infty)$ 中的柯西序列，存在 $M > 0$ 使得 $\|f_n\|_\infty \leq M$ 對於所有 $n \geq 1$ 。因此，對於每個 $x \in A$ ，我們有 $\|f(x)\| = \lim_{n \rightarrow \infty} \|f_n(x)\| \leq M$ ，所以 $\|f\|_\infty \leq M$ ，也就是說 $f \in \mathcal{B}(A, W)$ 。最後，我們不難檢查 $\|f_n - f\|_\infty \xrightarrow{n \rightarrow \infty} 0$ ，所以我們可以總結 $(\mathcal{B}(A, W), \|\cdot\|_\infty)$ 是完備的。

(2) 把式 (8.5) 中定義的範數用上來，這剛好就只是把式 (8.1) 在賦範向量空間 $(W, \|\cdot\|)$ 中改寫所得到的。 \square

範例 8.1.11 : 考慮函數序列 $(f_n)_{n \geq 1}$ 定義如下：

$$\forall n \in \mathbb{N}, \quad \forall x \in [0, 1], \quad f_n(x) = x^n(1 - x).$$

不難檢查 $(f_n)_{n \geq 1}$ 會逐點收斂到零函數。對於每個 $n \in \mathbb{N}$ ，函數 f_n 是 \mathcal{C}^∞ 類的，所以我們可以透過計算他的微分，來找出在 $[0, 1]$ 上的極值。我們有

$$\forall x \in [0, 1], \quad f'_n(x) = nx^{n-1}\left(1 - \frac{n+1}{n}x\right).$$

因此，函數 f_n 在 $[0, \frac{n}{n+1}]$ 上會遞增，在 $[\frac{n}{n+1}, 1]$ 上會遞減，所以在 $x_n = \frac{n}{n+1}$ 時有最大值在，也就是說：

$$\forall x \in [0, 1], \quad f_n(x) \leq f_n(x_n) = \frac{1}{n+1} \left(\frac{n}{n+1}\right)^n \leq \frac{1}{n+1} \xrightarrow{n \rightarrow \infty} 0.$$

因此，序列 $(f_n)_{n \geq 1}$ 會在 $[0, 1]$ 上均勻收斂到零函數。

Remark 8.1.12 : If a sequence of functions $(f_n)_{n \geq 1}$ converges pointwise to f , in order to show that this convergence is not uniform, we may look at the negation of Eq. (8.1), which writes

$$\exists \varepsilon > 0, \forall N \geq 1, \exists n \geq N \exists x \in A \quad d(f_n(x), f(x)) > \varepsilon.$$

In other words, we need to find a sequence $(x_n)_{n \geq 1}$ with values in A and an extraction $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that the sequence $(d(f_{\varphi(n)}(x_n), f(x_n)))_{n \geq 1}$ is bounded away from 0.

Example 8.1.13 : Let us consider the following sequence of functions,

$$\forall n \in \mathbb{N}, \quad \forall x \geq 0, \quad f_n(x) = \frac{x + \sqrt{n}}{x + n}.$$

It is easy to see that the sequence of functions $(f_n)_{n \geq 1}$ converges pointwise to the zero function. To show that it does not converge uniformly, we follow Remark 8.1.12. Let $x_n = n$ for $n \geq 1$. Then, we have

$$\forall n \in \mathbb{N}, \quad f_n(x_n) - 0 = \frac{n + \sqrt{n}}{n + n} \xrightarrow{n \rightarrow \infty} \frac{1}{2} \neq 0.$$

We conclude that the convergence $f_n \xrightarrow{n \rightarrow \infty} f$ is pointwise but not uniform.

The following theorem tells us which additional assumptions we may add to upgrade a pointwise convergence to a uniform convergence.

Theorem 8.1.14 (Dini's theorem) : Let (K, d) be a compact space, and $(f_n)_{n \geq 1}$ be a sequence of continuous functions from K to \mathbb{R} . Suppose that

- (i) the sequence is increasing, that is for every $x \in K$ and $n \in \mathbb{N}$, we have $f_n(x) \leq f_{n+1}(x)$;
- (ii) the sequence $(f_n)_{n \geq 1}$ converges pointwise to a continuous function $f : K \rightarrow \mathbb{R}$.

Then, the sequence $(f_n)_{n \geq 1}$ converges uniformly to f .

註解 8.1.12 : 如果函數序列 $(f_n)_{n \geq 1}$ 逐點收斂到 f ，如果想要證明這個收斂不是均勻的，我們可以考慮式 (8.1) 的否命題，寫做：

$$\exists \varepsilon > 0, \forall N \geq 1, \exists n \geq N \exists x \in A \quad d(f_n(x), f(x)) > \varepsilon.$$

換句話說，我們需要找到取值在 A 中的序列 $(x_n)_{n \geq 1}$ 以及萃取函數 $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ 使得序列 $(d(f_{\varphi(n)}(x_n), f(x_n)))_{n \geq 1}$ 有個嚴格為正的均勻下界。

範例 8.1.13 : 讓我們考慮下面這個函數序列：

$$\forall n \in \mathbb{N}, \quad \forall x \geq 0, \quad f_n(x) = \frac{x + \sqrt{n}}{x + n}.$$

不難看出來函數序列 $(f_n)_{n \geq 1}$ 會逐點收斂到零函數。如果想要證明他不會均勻收斂，我們使用註解 8.1.12 中提到的方法。對於 $n \geq 1$ ，令 $x_n = n$ 。那麼我們有

$$\forall n \in \mathbb{N}, \quad f_n(x_n) - 0 = \frac{n + \sqrt{n}}{n + n} \xrightarrow{n \rightarrow \infty} \frac{1}{2} \neq 0.$$

我們總結 $f_n \xrightarrow{n \rightarrow \infty} f$ 會逐點收斂但不會均勻收斂。

下面的定理告訴我們，可以加上什麼樣的假設，讓我們可以把逐點收斂升級為均勻收斂。

定理 8.1.14 【Dini 定理】：令 (K, d) 為緊緻空間，且 $(f_n)_{n \geq 1}$ 為由 K 映射至 \mathbb{R} 的連續函數所構成的序列。假設

- (i) 序列是遞增的，也就是說對於每個 $x \in K$ 還有 $n \in \mathbb{N}$ ，我們有 $f_n(x) \leq f_{n+1}(x)$ ；
- (ii) 序列 $(f_n)_{n \geq 1}$ 會逐點收斂到一個連續函數 $f : K \rightarrow \mathbb{R}$ 。

那麼，序列 $(f_n)_{n \geq 1}$ 會均勻收斂到 f 。

Proof : For every $n \in \mathbb{N}$, let us define the continuous function $g_n = f - f_n \geq 0$. By the assumption (i), the sequence of functions $(g_n)_{n \geq 1}$ is decreasing. Given $\varepsilon > 0$, we define $E_n = \{x \in K : g_n(x) < \varepsilon\}$ for $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, since g_n is continuous, the set E_n is open; since the sequence $(g_n)_{n \geq 1}$ is decreasing, the sequence $(E_n)_{n \geq 1}$ is increasing. Due to the assumption (ii), we find that $\bigcup_{n \geq 1} E_n = K$. Since K is compact, by the Borel–Lebesgue property (Definition 3.1.3), there exists $N \geq 1$ such that $E_N = \bigcup_{n=1}^N E_n = K$. This means that for any $n \geq N$ and $x \in K$, we have $|f_n(x) - f(x)| < \varepsilon$. \square

證明：對於每個 $n \in \mathbb{N}$ ，我們定義連續函數 $g_n = f - f_n \geq 0$ 。根據假設 (i)，函數序列 $(g_n)_{n \geq 1}$ 是遞減的。給定 $\varepsilon > 0$ ，對於 $n \in \mathbb{N}$ ，我們定義 $E_n = \{x \in K : g_n(x) < \varepsilon\}$ 。對於每個 $n \in \mathbb{N}$ ，由於 g_n 是連續的，集合 E_n 是個開集；由於序列 $(g_n)_{n \geq 1}$ 是遞減的，序列 $(E_n)_{n \geq 1}$ 是遞增的。假設 (ii) 會告訴我們 $\bigcup_{n \geq 1} E_n = K$ 。由於 K 是緊緻的，根據 Borel–Lebesgue 性質（定義 3.1.3），存在 $N \geq 1$ 使得 $E_N = \bigcup_{n=1}^N E_n = K$ 。這代表著對於任意 $n \geq N$ 以及 $x \in K$ ，我們有 $|f_n(x) - f(x)| < \varepsilon$ 。 \square

Remark 8.1.15 : There is another version of Dini’s theorem, stated as below. Let $I = [a, b]$ be a segment and $(f_n)_{n \geq 1}$ be a sequence of (not necessarily continuous) functions from I to \mathbb{R} . Suppose that

- (i) for each $n \geq 1$, the function f_n is increasing on I ;
- (ii) the sequence $(f_n)_{n \geq 1}$ converges pointwise to a continuous function $f : I \rightarrow \mathbb{R}$.

Then, the sequence $(f_n)_{n \geq 1}$ converges uniformly to f . See Exercise 8.7 for a proof.

註解 8.1.15 : 我們下面給出另一個版本 Dini 定理的敘述。令 $I = [a, b]$ 為線段且 $(f_n)_{n \geq 1}$ 為由 I 映射至 \mathbb{R} （未必連續）的函數序列。假設

- (i) 對於每個 $n \geq 1$ ，函數 f_n 在 I 上是遞增的；
- (ii) 序列 $(f_n)_{n \geq 1}$ 會逐點收斂到連續函數 $f : I \rightarrow \mathbb{R}$ 。

那麼，序列 $(f_n)_{n \geq 1}$ 會均勻收斂至 f 。證明請見習題 8.7。

8.1.2 Series of functions

In this section, let $(u_n)_{n \geq 1}$ be a sequence of functions from A to W , where $(W, \|\cdot\|)$ is a Banach space.

Definition 8.1.16 :

- We say that the series of functions $\sum u_n$ converges pointwise if for every $x \in A$, the series $\sum u_n(x)$ converges. We write

$$\begin{aligned} \sum_{n \geq 1} u_n : A &\rightarrow W \\ x &\mapsto \sum_{n \geq 1} u_n(x). \end{aligned}$$

- The function defined by $S_n(x) = \sum_{k=1}^n u_k(x)$ for $x \in A$ is called the n -th partial sum of the series of functions $\sum u_n$.
- If the series of functions $\sum u_n$ converges pointwise, then the n -th remainder is given by $R_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$ for $x \in A$.

第二小節 函數級數

在這個章節，令 $(u_n)_{n \geq 1}$ 為由 A 映射至 W 的函數序列，其中 $(W, \|\cdot\|)$ 是個 Banach 空間。

定義 8.1.16 :

- 如果對於每個 $x \in A$ ，級數 $\sum u_n(x)$ 會收斂，則我們說函數級數 $\sum u_n$ 逐點收斂。我們記

$$\begin{aligned} \sum_{n \geq 1} u_n : A &\rightarrow W \\ x &\mapsto \sum_{n \geq 1} u_n(x). \end{aligned}$$

- 對於 $x \in A$ ，我們定義函數 $S_n(x) = \sum_{k=1}^n u_k(x)$ ，稱作函數級數 $\sum u_n$ 的第 n 個部份和。
- 如果函數級數 $\sum u_n$ 會逐點收斂，那麼第 n 個餘項定義做 $R_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$ 對於 $x \in A$ 。

- We say that the series of functions $\sum u_n$ converges uniformly if the partial sums $(S_n)_{n \geq 0}$ converges uniformly.

Proposition 8.1.17 : The series of functions $\sum u_n$ converges uniformly if and only if

- (i) the series $\sum u_n$ converges pointwise, and
- (ii) the sequence of remainders $(R_n)_{n \geq 0}$ converges uniformly to the zero function.

Proof : Let $\sum u_n$ be a series of functions, $(S_n)_{n \geq 0}$ be its partial sums, and $(R_n)_{n \geq 0}$ be its remainders.

- Suppose that $\sum u_n$ converges uniformly to u , which means that $(S_n)_{n \geq 0}$ converges uniformly to u , and it follows from Corollary 8.1.6 that this convergence takes place pointwise. The uniform convergence means that $\|S_n - u\|_\infty \xrightarrow{n \rightarrow \infty} 0$, since $u - S_n = R_n$, we see that it is equivalent to $\|R_n\|_\infty \xrightarrow{n \rightarrow \infty} 0$.
- Suppose that (i) and (ii) holds, and denote by u the pointwise limit of $\sum u_n$. Since $R_n = u - S_n$, from its uniform convergence to zero, we find $\|S_n - u\|_\infty \xrightarrow{n \rightarrow \infty} 0$, which is the uniform convergence of $(S_n)_{n \geq 0}$ to u . \square

Example 8.1.18 : Let us consider the series of functions $\sum \frac{(-1)^n}{n} x^n$ where each term is a function defined on $[0, 1]$. We are going to show that this series of functions converges uniformly. For every $x \in [0, 1]$, the sequence $(\frac{x^n}{n})_{n \geq 1}$ is non-increasing with limit zero. It follows from Theorem 6.4.2 that the series $\sum \frac{(-1)^n}{n} x^n$ converges, and the remainder $R_n(x)$ satisfies

$$\forall x \in [0, 1], \quad |R_n(x)| \leq \frac{x^{n+1}}{n+1} \leq \frac{1}{n+1},$$

which does not depend on $x \in [0, 1]$. This implies that the convergence of the series of functions is uniform.

Remark 8.1.19 : We note that saying that a sequence of functions $(f_n)_{n \geq 1}$ converges uniformly is equivalent to saying that the series of functions $\sum (f_{n+1} - f_n)$ converges uniformly.

- 如果部份和 $(S_n)_{n \geq 0}$ 會均勻收斂，則我們說函數級數 $\sum u_n$ 均勻收斂。

命題 8.1.17 : 函數級數 $\sum u_n$ 會均勻收斂若且唯若

- (i) 函數級數 $\sum u_n$ 會逐點收斂，而且
- (ii) 餘項序列 $(R_n)_{n \geq 0}$ 會均勻收斂到零函數。

證明 : 令 $\sum u_n$ 為函數級數， $(S_n)_{n \geq 0}$ 為他的部份和，還有 $(R_n)_{n \geq 0}$ 為他的餘項。

- 假設 $\sum u_n$ 均勻收斂至 u ，這代表著 $(S_n)_{n \geq 0}$ 均勻收斂至 u ，從系理 8.1.6 我們得知這個收斂也會是逐點的。均勻收斂代表 $\|S_n - u\|_\infty \xrightarrow{n \rightarrow \infty} 0$ ，由於 $u - S_n = R_n$ ，我們得知這和 $\|R_n\|_\infty \xrightarrow{n \rightarrow \infty} 0$ 等價。
- 假設 (i) 和 (ii) 成立，並把 $\sum u_n$ 逐點收斂的極限記作 u 。由於 $R_n = u - S_n$ ，且會均勻收斂至零，我們得到 $\|S_n - u\|_\infty \xrightarrow{n \rightarrow \infty} 0$ ，我們會得到 $(S_n)_{n \geq 0}$ 均勻收斂至 u 。 \square

範例 8.1.18 : 讓我們考慮函數級數 $\sum \frac{(-1)^n}{n} x^n$ 其中每項都是定義在 $[0, 1]$ 上的函數。我們這裡要證明這個函數級數會均勻收斂。對於每個 $x \in [0, 1]$ ，序列 $(\frac{x^n}{n})_{n \geq 1}$ 非遞增且極限為零。從定理 6.4.2 我們得知級數 $\sum \frac{(-1)^n}{n} x^n$ 會收斂，而且餘項 $R_n(x)$ 滿足

$$\forall x \in [0, 1], \quad |R_n(x)| \leq \frac{x^{n+1}}{n+1} \leq \frac{1}{n+1},$$

這不會取決於 $x \in [0, 1]$ 。這代表這個函數級數的收斂是均勻的。

註解 8.1.19 : 我們注意到函數序列 $(f_n)_{n \geq 1}$ 均勻收斂與函數級數 $\sum (f_{n+1} - f_n)$ 均勻收斂是等價的。

Proposition 8.1.20 (Cauchy's condition) : A series of functions $\sum u_n$ converges uniformly if and only if for every $\varepsilon > 0$, there exists $N \geq 1$ such that

$$\forall n \geq N, \forall k \geq 1, \quad \|u_{n+1} + \cdots + u_{n+k}\|_\infty < \varepsilon.$$

This is the Cauchy's condition in the case of a series of functions.

Proof : This is very similar to Corollary 6.1.11. From Proposition 8.1.10 (1), we know that $(\mathcal{B}(A, W), \|\cdot\|_\infty)$ is a Banach space, in which a sequence converges if and only if it is Cauchy. \square

Definition 8.1.21 : Let $u_n \in \mathcal{B}(A, W)$ for every $n \geq 1$. We say that the series of functions $\sum u_n$ converges normally (正規收斂) on A if the series $\sum \|u_n\|_{\infty, A}$ converges.

Proposition 8.1.22 : Suppose that $(W, \|\cdot\|)$ is a Banach space. Let $\sum u_n$ be a series of bounded functions from A to W that converges normally on A . Then, the following properties hold.

- (1) For every $a \in A$, the series $\sum u_n(a)$ converges absolutely.
- (2) The series of functions $\sum u_n$ converges uniformly.

Proof :

- (1) Let $a \in A$. For every $n \geq 1$, we have $\|u_n(a)\| \leq \|u_n\|_\infty$. Since $\sum \|u_n\|_\infty$ is convergent, we deduce that $\sum u_n(a)$ converges absolutely.
- (2) For every $n, k \geq 1$ and $x \in A$, we have

$$\|u_n(x) + \cdots + u_{n+k}(x)\| \leq \|u_n(x)\| + \cdots + \|u_{n+k}(x)\| \leq \|u_n\|_\infty + \cdots + \|u_{n+k}\|_\infty.$$

Therefore, the Cauchy's condition for the series $\sum \|u_n\|_\infty$ implies the Cauchy's condition for the series $\sum u_n(x)$, uniformly for all $x \in A$. This means that the series of functions $\sum u_n$ converges uniformly. \square

命題 8.1.20 【柯西條件】 : 函數級數 $\sum u_n$ 會均勻收斂若且唯若對於每個 $\varepsilon > 0$, 存在 $N \geq 1$ 使得

$$\forall n \geq N, \forall k \geq 1, \quad \|u_{n+1} + \cdots + u_{n+k}\|_\infty < \varepsilon.$$

這個是在函數級數情況下的柯西條件。

證明 : 這個與系理 6.1.11 非常相同。從命題 8.1.10 (1), 我們知道 $(\mathcal{B}(A, W), \|\cdot\|_\infty)$ 是個 Banach 空間, 而在這個空間中, 序列收斂若且唯若他是個柯西序列。 \square

定義 8.1.21 : 對於每個 $n \geq 1$, 令 $u_n \in \mathcal{B}(A, W)$ 。如果級數 $\sum \|u_n\|_{\infty, A}$ 收斂, 則我們說函數級數 $\sum u_n$ 會在 A 上正規收斂 (normal convergence)。

命題 8.1.22 : 假設 $(W, \|\cdot\|)$ 是個 Banach 空間。令 $\sum u_n$ 為由 A 映射到 W 的有界函數構成的級數, 且會在 A 上正規收斂。那麼下列性質會成立。

- (1) 對於每個 $a \in A$, 級數 $\sum u_n(a)$ 會絕對收斂。
- (2) 函數級數 $\sum u_n$ 會均勻收斂。

證明 :

- (1) 令 $a \in A$ 。對於每個 $n \geq 1$, 我們有 $\|u_n(a)\| \leq \|u_n\|_\infty$ 。由於 $\sum \|u_n\|_\infty$ 會收斂, 我們推得 $\sum u_n(a)$ 會絕對收斂。
- (2) 對於每個 $n, k \geq 1$ 以及 $x \in A$, 我們有

$$\|u_n(x) + \cdots + u_{n+k}(x)\| \leq \|u_n(x)\| + \cdots + \|u_{n+k}(x)\| \leq \|u_n\|_\infty + \cdots + \|u_{n+k}\|_\infty.$$

因此, 級數 $\sum \|u_n\|_\infty$ 的柯西條件蘊含級數 $\sum u_n(x)$ 的柯西條件, 而且對於所有 $x \in A$ 是均勻的。這代表著函數級數 $\sum u_n$ 會均勻收斂。 \square

Remark 8.1.23 : Let us assume that $(W, \|\cdot\|)$ is a Banach space, and $u_n \in \mathcal{B}(A, W)$ for all $n \geq 1$. A series of functions $\sum u_n$ can also be seen as a series with terms in the Banach space $(\mathcal{B}(A, W), \|\cdot\|_\infty)$, meaning that the normal convergence of the series of functions $\sum u_n$ is the same as the absolute convergence of the series $\sum u_n$ with terms $u_n \in \mathcal{B}(A, W)$. This allows us to find an alternative proof to (2), by noting that from Theorem 6.1.16, we deduce that the series $\sum u_n$ converges in $\mathcal{B}(A, W)$, that is the series of functions $\sum u_n$ converges uniformly.

Example 8.1.24 : Let us define a sequence of functions $(f_n)_{n \geq 1}$ on $[0, 1]$ as below,

$$f_1 \equiv 1 \quad \text{and} \quad \forall n \geq 1, \forall x \in [0, 1], \quad f_{n+1}(x) = 1 + \frac{1}{2} \int_0^x f_n(t) dt.$$

For any $n \geq 1$ and $x \in [0, 1]$, we have

$$\begin{aligned} |f_{n+2}(x) - f_{n+1}(x)| &= \frac{1}{2} \left| \int_0^x (f_{n+1}(t) - f_n(t)) dt \right| \\ &\leq \frac{1}{2} \int_0^x \|f_{n+1} - f_n\|_\infty dt \\ &\leq \frac{1}{2} \|f_{n+1} - f_n\|_\infty, \end{aligned}$$

implying $\|f_{n+2} - f_{n+1}\|_\infty \leq \frac{1}{2} \|f_{n+1} - f_n\|_\infty$. Therefore, by induction, we find

$$\forall n \geq 1, \quad \|f_{n+1} - f_n\|_\infty \leq \frac{1}{2^{n-1}} \|f_2 - f_1\|_\infty.$$

It follows that the series $\sum (f_{n+1} - f_n)$ converges normally, so uniformly, and the sequence $(f_n)_{n \geq 1}$ converges also uniformly.

Example 8.1.25 : Let us consider the series of functions $\sum \frac{(-1)^n}{n} x^n$ defined on $[0, 1]$. We have seen that this series of functions converges uniformly on $[0, 1]$ (Example 8.1.18).

- However, it does not converge normally on $[0, 1]$, because $\|u_n\|_\infty = \frac{1}{n}$ for $n \geq 1$, and the series $\sum \frac{1}{n}$ diverges.
- It does converge normally on $[0, a]$ for any $a \in [0, 1)$, because $\|(u_n)_{|[0, a]}\|_\infty = \frac{a^n}{n}$ for $n \geq 1$, and the series $\sum \frac{a^n}{n}$ converges.

註解 8.1.23 : 讓我們假設 $(W, \|\cdot\|)$ 是個 Banach 空間，而且對於所有 $n \geq 1$ ，我們有 $u_n \in \mathcal{B}(A, W)$ 。函數級數 $\sum u_n$ 也可以被看作取值在 Banach 空間 $(\mathcal{B}(A, W), \|\cdot\|_\infty)$ 的級數，也就是說，函數級數 $\sum u_n$ 的正規收斂性與當我們把級數中的項看作 $u_n \in \mathcal{B}(A, W)$ 時， $\sum u_n$ 的絕對收斂性等價。這讓我們得到另一個方法來證明 (2)：我們可以注意到從定理 6.1.16，我們能推得級數 $\sum u_n$ 在 $\mathcal{B}(A, W)$ 中收斂，這意味著函數級數 $\sum u_n$ 會均勻收斂。

範例 8.1.24 : 讓我們考慮定義在 $[0, 1]$ 上的函數序列 $(f_n)_{n \geq 1}$ 如下：

$$f_1 \equiv 1 \quad \text{以及} \quad \forall n \geq 1, \forall x \in [0, 1], \quad f_{n+1}(x) = 1 + \frac{1}{2} \int_0^x f_n(t) dt.$$

對於任意 $n \geq 1$ 以及 $x \in [0, 1]$ ，我們有

$$\begin{aligned} |f_{n+2}(x) - f_{n+1}(x)| &= \frac{1}{2} \left| \int_0^x (f_{n+1}(t) - f_n(t)) dt \right| \\ &\leq \frac{1}{2} \int_0^x \|f_{n+1} - f_n\|_\infty dt \\ &\leq \frac{1}{2} \|f_{n+1} - f_n\|_\infty, \end{aligned}$$

這蘊含 $\|f_{n+2} - f_{n+1}\|_\infty \leq \frac{1}{2} \|f_{n+1} - f_n\|_\infty$ 。因此，透過數學歸納法，我們得到

$$\forall n \geq 1, \quad \|f_{n+1} - f_n\|_\infty \leq \frac{1}{2^{n-1}} \|f_2 - f_1\|_\infty.$$

因此這告訴我們級數 $\sum (f_{n+1} - f_n)$ 會正規收斂，所以也會均勻收斂，而且序列 $(f_n)_{n \geq 1}$ 也會均勻收斂。

範例 8.1.25 : 讓我們考慮定義在 $[0, 1]$ 上的函數級數 $\sum \frac{(-1)^n}{n} x^n$ 。我們有看過這個函數級數會在 $[0, 1]$ 上均勻收斂（範例 8.1.18）。

- 然而，他不會在 $[0, 1]$ 上正規收斂，因為對於 $n \geq 1$ ，我們有 $\|u_n\|_\infty = \frac{1}{n}$ ，而且級數 $\sum \frac{1}{n}$ 會發散。
- 對於任意 $a \in [0, 1)$ ，他會在 $[0, a]$ 上正規收斂，因為對於 $n \geq 1$ ，我們有 $\|(u_n)_{|[0, a]}\|_\infty =$

8.2 Properties of the uniform limit

In this section, we are going to discuss some analytic properties of the limit of a convergent sequence of functions. We are going to consider metric spaces (X, d_X) and (M, d_M) , and a sequence of functions $(f_n)_{n \geq 1}$ in $\mathcal{B}(X, M)$.

8.2.1 Continuity

Proposition 8.2.1 : Suppose that $(f_n)_{n \geq 1}$ is a sequence of functions from X to M and converges uniformly to f . If f_n is continuous at a for every $n \geq 1$, then f is continuous at a .

Proof : Let $\varepsilon > 0$. Due to the uniform convergence of $(f_n)_{n \geq 1}$ to f , we may find $N \geq 1$ such that

$$\forall n \geq N, \forall x \in X, \quad d_M(f_n(x), f(x)) \leq \varepsilon.$$

Since f_N is continuous at a , we may find $\delta > 0$ such that

$$\forall y \in X, \quad d_X(x, y) < \delta \Rightarrow d_M(f_N(x), f_N(y)) \leq \varepsilon.$$

Therefore, for any $y \in X$ such that $d_X(x, y) < \delta$, we have

$$d_M(f(x), f(y)) \leq d_M(f(x), f_N(x)) + d_M(f_N(x), f_N(y)) + d_M(f_N(y), f(y)) \leq 3\varepsilon.$$

This shows that f is continuous at a . \square

Corollary 8.2.2 : Let $(f_n)_{n \geq 1}$ be a sequence of continuous functions from X to M . If $(f_n)_{n \geq 1}$ converges uniformly to f on X , then f is continuous on X .

$\frac{a^n}{n}$, 而且級數 $\sum \frac{a^n}{n}$ 會收斂。

第二節 均勻極限的性質

在這個章節中，我們會討論收斂函數序列極限的分析性質。我們會考慮賦距空間 (X, d_X) 和 (M, d_M) ，以及在 $\mathcal{B}(X, M)$ 中的函數序列 $(f_n)_{n \geq 1}$ 。

第一小節 連續性

命題 8.2.1 : 假設 $(f_n)_{n \geq 1}$ 是個由 X 映射到 M 的函數序列，而且會均勻收斂到 f 。如果對於所有 $n \geq 1$ ， f_n 在 a 連續，那麼 f 在 a 連續。

證明 : 令 $\varepsilon > 0$ 。由於 $(f_n)_{n \geq 1}$ 會均勻收斂到 f ，我們能找到 $N \geq 1$ 使得

$$\forall n \geq N, \forall x \in X, \quad d_M(f_n(x), f(x)) \leq \varepsilon.$$

由於 f_N 在 a 連續，我們能找到 $\delta > 0$ 使得

$$\forall y \in X, \quad d_X(x, y) < \delta \Rightarrow d_M(f_N(x), f_N(y)) \leq \varepsilon.$$

因此，對於任意 $y \in X$ 滿足 $d_X(x, y) < \delta$ ，我們有

$$d_M(f(x), f(y)) \leq d_M(f(x), f_N(x)) + d_M(f_N(x), f_N(y)) + d_M(f_N(y), f(y)) \leq 3\varepsilon.$$

這證明了 f 在 a 連續。 \square

系理 8.2.2 : 令 $(f_n)_{n \geq 1}$ 為由 X 映射到 M 的連續函數所構成的序列。如果 $(f_n)_{n \geq 1}$ 會在 X 上均勻收斂到 f ，那麼 f 在 X 上連續。

Proof : It is a direct consequence of Proposition 8.2.1. \square

Corollary 8.2.3 : Let $\sum u_n$ be a series of continuous functions from $[a, b]$ to a Banach space $(W, \|\cdot\|)$. If the series $\sum u_n$ converges uniformly on $[a, b]$, then the limit function $\sum u_n$ is continuous on $[a, b]$.

Proof : It is a direct consequence of Corollary 8.2.2 by taking $(X, d_X) = ([a, b], |\cdot|)$ and $(M, d_M) = (W, \|\cdot\|)$. \square

Example 8.2.4 : Let us consider the series of functions $\sum_{n \geq 0} u_n$ defined on \mathbb{R}_+ as below,

$$\forall x \geq 0, \quad u_n(x) = \frac{x^n}{n!}.$$

- For each $x \geq 0$, the series $\sum_{n \geq 0} u_n(x)$ converges, and we denote the limit by $u(x)$.
- The convergence of the series $\sum_{n \geq 0} u_n$ to u is not uniform. In fact, for every $N \geq 1$, we have

$$\left| \sum_{n \geq 0} u_n(x) - \sum_{n=0}^{N-1} u_n(x) \right| \geq \frac{x^N}{N!} \xrightarrow{x \rightarrow \infty} +\infty.$$

- For any $M > 0$, the convergence of the series $\sum_{n \geq 0} u_n$ to u on $[0, M]$ is uniform. To see this, we write, for any $x \in [0, M]$,

$$\left| \sum_{n \geq 0} u_n(x) - \sum_{n=0}^{N-1} u_n(x) \right| = \left| \sum_{n \geq N} u_n(x) \right| \leq \sum_{n \geq N} \frac{M^n}{n!} \xrightarrow{N \rightarrow \infty} 0,$$

which gives us a uniform upper bound of the remainder which does not depend on x .

- In consequence, the limit function u is continuous on $[0, M]$ for every $M > 0$, so it is also continuous on \mathbb{R}_+ .

This examples illustrates that to get the continuity of the limit function, we do not necessarily need the uniform convergence on the whole domain of definition. Since the continuity is a *local* regularity, it is sufficient to show the uniform convergence on, for example, all the segments.

證明 : 這是命題 8.2.1 的直接結果。 \square

系理 8.2.3 : 令 $\sum u_n$ 為由 $[a, b]$ 映射至 Banach 空間 $(W, \|\cdot\|)$ 的連續函數所構成的級數。如果級數 $\sum u_n$ 會在 $[a, b]$ 上均勻收斂，那麼極限函數 $\sum u_n$ 會在 $[a, b]$ 上連續。

證明 : 我們可以取 $(X, d_X) = ([a, b], |\cdot|)$ 還有 $(M, d_M) = (W, \|\cdot\|)$ ，那麼這會是系理 8.2.2 的直接結果。 \square

範例 8.2.4 : 讓我們考慮定義在 \mathbb{R}_+ 上的函數級數 $\sum_{n \geq 0} u_n$ 如下：

$$\forall x \geq 0, \quad u_n(x) = \frac{x^n}{n!}.$$

- 對於每個 $x \geq 0$ ，級數 $\sum_{n \geq 0} u_n(x)$ 收斂，我們把他的極限記作 $u(x)$ 。
- 級數 $\sum_{n \geq 0} u_n$ 不會均勻收斂到 u 。事實上，對於每個 $N \geq 1$ ，我們有

$$\left| \sum_{n \geq 0} u_n(x) - \sum_{n=0}^{N-1} u_n(x) \right| \geq \frac{x^N}{N!} \xrightarrow{x \rightarrow \infty} +\infty.$$

- 對於任意 $M > 0$ ，級數 $\sum_{n \geq 0} u_n$ 會在 $[0, M]$ 上均勻收斂到 u 。要看出這個，對於任意 $x \in [0, M]$ ，我們記

$$\left| \sum_{n \geq 0} u_n(x) - \sum_{n=0}^{N-1} u_n(x) \right| = \left| \sum_{n \geq N} u_n(x) \right| \leq \sum_{n \geq N} \frac{M^n}{n!} \xrightarrow{N \rightarrow \infty} 0,$$

這給我們餘項不取決於 x 的均勻上界。

- 因此，對於每個 $M > 0$ ，極限函數 u 在 $[0, M]$ 上會連續，所以也會在 \mathbb{R}_+ 上連續。

這個範例告訴我們如果要得到極限函數的連續性，我們不一定需要在整個定義域上面的均勻收斂。由於連續性是個局部的規律性，我們只需要證明，例如在所有線段上，會有均勻收斂即可。

8.2.2 Integration

Let $I \subseteq \mathbb{R}$ be an interval such that $\dot{I} \neq \emptyset$. Consider a sequence $(f_n)_{n \geq 1}$ of functions from I to a Banach space $(W, \|\cdot\|)$.

Proposition 8.2.5 : Let $(f_n)_{n \geq 1}$ be a sequence of continuous functions that converges uniformly to f on every segment of I . Let $a \in I$, and define the following primitives,

$$\varphi(x) = \int_a^x f(t) dt \quad \text{and} \quad \varphi_n(x) = \int_a^x f_n(t) dt, \quad \forall n \geq 1.$$

Then, the sequence $(\varphi_n)_{n \geq 1}$ converges uniformly to φ on every segment of I .

Remark 8.2.6 : The conclusion of Proposition 8.2.5 means that we may interchange the order of the limit and integration,

$$\lim_{n \rightarrow \infty} \int_a^x f_n(t) dt = \int_a^x \lim_{n \rightarrow \infty} f_n(t) dt.$$

Proof : Let $[c, d] \subseteq I$ be a segment of I containing a . Since $(f_n)_{n \geq 1}$ converges uniformly on $[c, d]$ to f , it follows from Corollary 8.2.2 that f is also continuous on $[c, d]$. Therefore, the primitives φ and φ_n with $n \geq 1$ are well defined on $[c, d]$. For every $n \geq 1$ and $x \in [c, d]$, we have

$$\begin{aligned} \|\varphi_n(x) - \varphi(x)\| &= \left\| \int_a^x (f_n(t) - f(t)) dt \right\| \\ &\leq |x - a| \|f_n - f\|_{\infty, [c, d]} \leq |d - c| \|f_n - f\|_{\infty, [c, d]} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

The convergence to 0 in the above bound does not depend on $x \in [c, d]$, so we have established the uniform convergence of $(\varphi_n)_{n \geq 1}$ to φ on $[c, d]$. \square

Example 8.2.7 : Let $(f_n)_{n \geq 1}$ be a sequence of real-valued continuous functions on $[0, 1]$ that converges uniformly to f . This means that $(f_n)_{n \geq 1}$ is bounded in $\mathcal{B}([0, 1], \mathbb{R})$, so we may find $M > 0$

第二小節 積分

令 $I \subseteq \mathbb{R}$ 為區間使得 $\dot{I} \neq \emptyset$ 。考慮由 I 映射至 Banach 空間 $(W, \|\cdot\|)$ 的函數序列 $(f_n)_{n \geq 1}$ 。

命題 8.2.5 : 令 $(f_n)_{n \geq 1}$ 為連續函數構成的序列，而且會在每個 I 的線段上均勻收斂到 f 。令 $a \in I$ 並定義下面的原函數：

$$\varphi(x) = \int_a^x f(t) dt \quad \text{以及} \quad \varphi_n(x) = \int_a^x f_n(t) dt, \quad \forall n \geq 1.$$

那麼，序列 $(\varphi_n)_{n \geq 1}$ 會在每個 I 的線段上均勻收斂到 φ 。

註解 8.2.6 : 命題 8.2.5 的結論告訴我們可以交換極限和積分的順序，也就是說：

$$\lim_{n \rightarrow \infty} \int_a^x f_n(t) dt = \int_a^x \lim_{n \rightarrow \infty} f_n(t) dt.$$

證明： 令 $[c, d] \subseteq I$ 為 I 的線段，而且包含 a 。由於 $(f_n)_{n \geq 1}$ 會在 $[c, d]$ 上均勻收斂到 f ，從系理 8.2.2 我們得知 f 也會在 $[c, d]$ 上連續。因此，原函數 φ 還有對於 $n \geq 1$ ，原函數 φ_n 在 $[c, d]$ 上都是定義良好的。對於每個 $n \geq 1$ 還有 $x \in [c, d]$ ，我們有

$$\begin{aligned} \|\varphi_n(x) - \varphi(x)\| &= \left\| \int_a^x (f_n(t) - f(t)) dt \right\| \\ &\leq |x - a| \|f_n - f\|_{\infty, [c, d]} \leq |d - c| \|f_n - f\|_{\infty, [c, d]} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

上面會收斂到 0 的上界不取決於 $x \in [c, d]$ ，所以我們證明了 $(\varphi_n)_{n \geq 1}$ 會在 $[c, d]$ 上均勻收斂到 φ 。 \square

範例 8.2.7 : 令 $(f_n)_{n \geq 1}$ 為定義在 $[0, 1]$ 上的連續實函數所構成的序列，而且會均勻收斂到 f 。這代表著 $(f_n)_{n \geq 1}$ 在 $\mathcal{B}([0, 1], \mathbb{R})$ 中有界，所以我們能找到 $M > 0$ 使得 $\|f_n\|_{\infty} \leq M$ 對於所有

such that $\|f_n\|_\infty \leq M$ for all $n \geq 1$. Then, we have

$$\forall x \in [0, 1], \quad |f_n(x)^2 - f(x)^2| \leq 2M|f_n(x) - f(x)|.$$

This means that $(f_n^2)_{n \geq 1}$ converges uniformly to f^2 , so we have

$$\int_0^1 f_n^2 \xrightarrow{n \rightarrow \infty} \int_0^1 f^2.$$

Example 8.2.8 : Let us consider the sequence of functions $(f_n)_{n \geq 1}$ on $[0, 1]$, defined by

$$\forall x \in [0, 1], \quad f_n(x) = x^n.$$

This sequence of functions converges pointwise to the indicator function $f = \mathbb{1}_1$ (Example 8.1.2) which is not continuous, so this convergence is not uniform (Proposition 8.2.1). However, the sequence of integrals converges,

$$\int_0^1 f_n(x) dx = \frac{1}{n+1} \xrightarrow{n \rightarrow \infty} 0 = \int_0^1 \mathbb{1}_1(x) dx.$$

This shows that the notion of uniform convergence is much stronger than the convergence of integrals. Actually, later in Section 8.5, we will see in a more general context, how to obtain the convergence of integrals without having the uniform convergence.

Corollary 8.2.9 : Let $\sum u_n$ be a series of continuous functions from $[a, b]$ to a Banach space $(W, \|\cdot\|)$. If the series $\sum u_n$ converges normally on $[a, b]$, then, for $x \in [a, b]$, we have

$$\int_a^x \left(\sum_{n \geq 1} u_n(t) \right) dt = \sum_{n \geq 1} \left(\int_a^x u_n(t) dt \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\int_a^x u_k(t) dt \right),$$

where the limit on the right side is uniform on $[a, b]$.

Remark 8.2.10 : Corollary 8.2.9 gives us conditions under which we are allowed to interchange the order of integration and series. In such a circumstance, sometimes we also say that “we may integrate the series term by term”.

$n \geq 1$ 。那麼我們會有

$$\forall x \in [0, 1], \quad |f_n(x)^2 - f(x)^2| \leq 2M|f_n(x) - f(x)|.$$

這代表 $(f_n^2)_{n \geq 1}$ 會均勻收斂到 f^2 ，所以我們有

$$\int_0^1 f_n^2 \xrightarrow{n \rightarrow \infty} \int_0^1 f^2.$$

範例 8.2.8 : 讓我們考慮函數定義在 $[0, 1]$ 上的函數序列 $(f_n)_{n \geq 1}$ 如下：

$$\forall x \in [0, 1], \quad f_n(x) = x^n.$$

這個函數序列會逐點收斂到指標函數 $f = \mathbb{1}_1$ (範例 8.1.2)，這個極限不是連續函數，所以這個收斂不是均勻的 (命題 8.2.1)。然而，積分的序列會收斂：

$$\int_0^1 f_n(x) dx = \frac{1}{n+1} \xrightarrow{n \rightarrow \infty} 0 = \int_0^1 \mathbb{1}_1(x) dx.$$

這證明了均勻收斂的概念是比積分的收斂還要來得強的。事實上，稍後在第 8.5 節中，我們會看到在更一般的框架下，如何在沒有均勻收斂的情況，得到積分的收斂。

系理 8.2.9 : 令 $\sum u_n$ 為由 $[a, b]$ 映射至 Banach 空間 $(W, \|\cdot\|)$ 的連續函數級數。如果級數 $\sum u_n$ 會在 $[a, b]$ 上正規收斂，那麼對於 $x \in [a, b]$ ，我們會有

$$\int_a^x \left(\sum_{n \geq 1} u_n(t) \right) dt = \sum_{n \geq 1} \left(\int_a^x u_n(t) dt \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\int_a^x u_k(t) dt \right),$$

其中右方的極限在 $[a, b]$ 上會是均勻的。

註解 8.2.10 : 系理 8.2.9 告訴我們，在什麼樣的條件之下，我們可以交換積分和級數的順序。這樣的情況下，我們有時候也會說「我們可以對級數一項一項積分」。

We also have a more general statement for the behavior of a uniformly convergent sequence of functions in the context of Riemann–Stieltjes integration. The following theorem states that (1) the Riemann–Stieltjes integrability is preserved by the uniform convergence, and (2) the sequence of primitives also converges uniformly.

Theorem 8.2.11 : Let $\alpha \in \mathcal{BV}([a, b])$. Let $(f_n)_{n \geq 1}$ be a sequence of bounded functions from $[a, b]$ to \mathbb{R} such that $f_n \in R(\alpha; a, b)$ for all $n \geq 1$. Suppose that $(f_n)_{n \geq 1}$ converges uniformly to a function $f : [a, b] \rightarrow \mathbb{R}$, and define

$$g(x) = \int_a^x f(t) d\alpha(t) \quad \text{and} \quad g_n(x) = \int_a^x f_n(t) d\alpha(t), \quad \forall n \geq 1.$$

Then, the following properties hold.

- (1) $f \in R(\alpha; a, b)$.
- (2) The sequence $(g_n)_{n \geq 1}$ converges uniformly to g on $[a, b]$.

Proof : By the decomposition theorem of functions with bounded variation, see Theorem 5.1.17 and Corollary 5.3.16, it is enough to show the statement for a strictly increasing function α . We have seen a similar argument in the proof of Theorem 5.3.21.

- (1) Let us prove that f satisfies Riemann's condition with respect to α on $[a, b]$ (Definition 5.3.8).

Let $\varepsilon > 0$. The uniform convergence of $(f_n)_{n \geq 1}$ to f allows us to find $N \geq 1$ such that

$$\|f(x) - f_n(x)\| \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)}, \quad \forall x \in [a, b], \forall n \geq N.$$

This means that for any partition $P \in \mathcal{P}([a, b])$, we have

$$|U_P(f - f_N, \alpha)| \leq \varepsilon \quad \text{and} \quad |L_P(f - f_N, \alpha)| \leq \varepsilon \quad (8.6)$$

Since $f_N \in R(\alpha; a, b)$, we may find a partition $P_\varepsilon \in \mathcal{P}([a, b])$ such that

$$\forall P \supseteq P_\varepsilon, \quad U_P(f_N, \alpha) - L_P(f_N, \alpha) \leq \varepsilon. \quad (8.7)$$

下面這個更一般的敘述，是針對函數均勻收斂的情況下所對應到的 Riemann–Stieltjes 積分。下面定理告訴我們 (1) Riemann–Stieltjes 積分性會被均勻收斂所保存，以及 (2) 原函數的序列也會均勻收斂。

定理 8.2.11 : 令 $\alpha \in \mathcal{BV}([a, b])$ 。令 $(f_n)_{n \geq 1}$ 為由 $[a, b]$ 映射至 \mathbb{R} 的有界函數所構成的序列，而且對於所有 $n \geq 1$ ，我們有 $f_n \in R(\alpha; a, b)$ 。假設 $(f_n)_{n \geq 1}$ 會均勻收斂到函數 $f : [a, b] \rightarrow \mathbb{R}$ ，並且定義

$$g(x) = \int_a^x f(t) d\alpha(t) \quad \text{以及} \quad g_n(x) = \int_a^x f_n(t) d\alpha(t), \quad \forall n \geq 1.$$

那麼下列性質會成立。

- (1) $f \in R(\alpha; a, b)$ 。
- (2) 序列 $(g_n)_{n \geq 1}$ 會在 $[a, b]$ 上均勻收斂到 g 。

證明：藉由有界變差函數的分解定理，參考定理 5.1.17 和系理 5.3.16，我們只需要對嚴格遞增的函數 α 證明即可。在定理 5.3.21 的證明中，我們也是使用類似的方法。

- (1) 讓我們來證明 f 在 $[a, b]$ 上會滿足對於 α 的黎曼條件（定義 5.3.8）。令 $\varepsilon > 0$ 。由於

$(f_n)_{n \geq 1}$ 會均勻收斂到 f ，我們可以找到 $N \geq 1$ 使得

$$\|f(x) - f_n(x)\| \leq \frac{\varepsilon}{\alpha(b) - \alpha(a)}, \quad \forall x \in [a, b], \forall n \geq N.$$

這代表著對於任意分割 $P \in \mathcal{P}([a, b])$ ，我們有

$$|U_P(f - f_N, \alpha)| \leq \varepsilon \quad \text{以及} \quad |L_P(f - f_N, \alpha)| \leq \varepsilon \quad (8.6)$$

由於 $f_N \in R(\alpha; a, b)$ ，我們能找到分割 $P_\varepsilon \in \mathcal{P}([a, b])$ 使得

$$\forall P \supseteq P_\varepsilon, \quad U_P(f_N, \alpha) - L_P(f_N, \alpha) \leq \varepsilon. \quad (8.7)$$

Therefore, for any $P \supseteq P_\varepsilon$, we have

$$\begin{aligned} U_P(f, \alpha) - L_P(f, \alpha) &\leq U_P(f - f_N, \alpha) - L_P(f - f_N, \alpha) + U_P(f_N, \alpha) - L_P(f_N, \alpha) \\ &\leq |U_P(f - f_N, \alpha)| + |L_P(f - f_N, \alpha)| + [U_P(f_N, \alpha) - L_P(f_N, \alpha)] \\ &\leq 3\varepsilon \end{aligned}$$

from Eq. (8.6) and Eq. (8.7). This shows that $f \in R(\alpha; a, b)$.

(2) For $n \geq N$ and $x \in [a, b]$, we have

$$|g_n(x) - g(x)| \leq \int_a^x |f_n(t) - f(t)| d\alpha(t) \leq \|f_n - f\|_\infty [\alpha(x) - \alpha(a)] \leq \|f_n - f\|_\infty [\alpha(b) - \alpha(a)],$$

where the upper bound does not depend on x , and converges to 0 when $n \rightarrow \infty$. \square

Corollary 8.2.12 : Let $\alpha \in \mathcal{BV}([a, b])$. Let $\sum u_n$ be a series of bounded functions from $[a, b]$ to \mathbb{R} such that $u_n \in R(\alpha; a, b)$ for all $n \geq 1$. Suppose that the series $\sum_n u_n$ converges uniformly on $[a, b]$. Then, the following properties hold.

(1) $\sum_n u_n \in R(\alpha; a, b)$.

(2) For $x \in [a, b]$, we have

$$\int_a^x \left(\sum_{n \geq 1} u_n(t) \right) d\alpha(t) = \sum_{n \geq 1} \left(\int_a^x u_n(t) d\alpha(t) \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\int_a^x u_k(t) d\alpha(t) \right),$$

where the convergence on the right side is uniform in $x \in [a, b]$.

因此，對於任意 $P \supseteq P_\varepsilon$ ，我們有

$$\begin{aligned} U_P(f, \alpha) - L_P(f, \alpha) &\leq U_P(f - f_N, \alpha) - L_P(f - f_N, \alpha) + U_P(f_N, \alpha) - L_P(f_N, \alpha) \\ &\leq |U_P(f - f_N, \alpha)| + |L_P(f - f_N, \alpha)| + [U_P(f_N, \alpha) - L_P(f_N, \alpha)] \\ &\leq 3\varepsilon \end{aligned}$$

這是可以從式 (8.6) 和式 (8.7) 所得到的。這證明了 $f \in R(\alpha; a, b)$ 。

(2) 對於 $n \geq N$ 以及 $x \in [a, b]$ ，我們有

$$|g_n(x) - g(x)| \leq \int_a^x |f_n(t) - f(t)| d\alpha(t) \leq \|f_n - f\|_\infty [\alpha(x) - \alpha(a)] \leq \|f_n - f\|_\infty [\alpha(b) - \alpha(a)],$$

其中上界不取決於 x ，而且當 $n \rightarrow \infty$ 時會收斂到 0。 \square

系理 8.2.12 : 令 $\alpha \in \mathcal{BV}([a, b])$ 。令 $\sum u_n$ 為由 $[a, b]$ 映射至 \mathbb{R} 的有界函數所構成的級數，且對於所有 $n \geq 1$ ，我們有 $u_n \in R(\alpha; a, b)$ 。假設級數 $\sum_n u_n$ 會在 $[a, b]$ 上均勻收斂。那麼下列性質會成立。

(1) $\sum_n u_n \in R(\alpha; a, b)$ 。

(2) 對於 $x \in [a, b]$ ，我們有

$$\int_a^x \left(\sum_{n \geq 1} u_n(t) \right) d\alpha(t) = \sum_{n \geq 1} \left(\int_a^x u_n(t) d\alpha(t) \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\int_a^x u_k(t) d\alpha(t) \right),$$

其中右側的收斂對於 $x \in [a, b]$ 是均勻的。

8.2.3 Derivatives

Let $I \subseteq \mathbb{R}$ be an interval such that $\dot{I} \neq \emptyset$. Consider a sequence $(f_n)_{n \geq 1}$ of functions from I to a Banach space $(W, \|\cdot\|)$.

第三小節 微分

令 $I \subseteq \mathbb{R}$ 為區間，滿足 $\dot{I} \neq \emptyset$ 。考慮由 I 映射至 Banach 空間 $(W, \|\cdot\|)$ 的函數序列 $(f_n)_{n \geq 1}$ 。

Theorem 8.2.13 : Let us make the following assumptions.

- (i) For every $n \geq 1$, the function $f_n : I \rightarrow W$ is of class C^1 .
- (ii) The sequence $(f_n)_{n \geq 1}$ converges pointwise to $f \in \mathcal{F}(I, W)$.
- (iii) The sequence $(f'_n)_{n \geq 1}$ converges uniformly to $g \in \mathcal{F}(I, W)$ on every segment of I .

Then, the following properties hold.

- (1) The function f is of class C^1 and $f' = g$.
- (2) The sequence $(f_n)_{n \geq 1}$ converges uniformly on every segment of I .

Proof : Let $a \in I$. From (ii), we know that $f_n(a) \xrightarrow{n \rightarrow \infty} f(a)$.

- (1) First, we note that since $(f'_n)_{n \geq 1}$ converges uniformly to g on every segment of I , it follows from Corollary 8.2.2 that g is continuous on I . By Proposition 8.2.5, for $x \in I$, we have

$$\int_a^x g(t) dt = \lim_{n \rightarrow \infty} \int_a^x f'_n(t) dt = \lim_{n \rightarrow \infty} (f_n(x) - f_n(a)) = f(x) - f(a).$$

This shows that

$$\forall x \in I, \quad f(x) = f(a) + \int_a^x g(t) dt.$$

Since g is continuous, we deduce that f is of class C^1 and $f' = g$.

- (2) To show the uniform convergence of $(f_n)_{n \geq 1}$ to f , let us proceed as follows. For every $n \geq 1$ and $x \in I$, the fundamental theorem of calculus gives us

$$\|f_n(x) - f(x)\| \leq \left\| \int_a^x (f'_n(t) - f'(t)) dt \right\| + \|f_n(a) - f(a)\|.$$

The first term on the right side converges uniformly to 0 by Proposition 8.2.5, and the second term converges to 0 due to the assumption (ii). Therefore, the above rate of convergence does not depend on $x \in I$, so $(f_n)_{n \geq 1}$ converges uniformly to f . \square

定理 8.2.13 : 讓我們做下列假設。

- (i) 對於每個 $n \geq 1$ ，函數 $f_n : I \rightarrow W$ 是 C^1 類的。
- (ii) 序列 $(f_n)_{n \geq 1}$ 會逐點收斂至 $f \in \mathcal{F}(I, W)$ 。
- (iii) 序列 $(f'_n)_{n \geq 1}$ 會在每個 I 的線段上均勻收斂至 $g \in \mathcal{F}(I, W)$ 。

那麼，下列性質會成立。

- (1) 函數 f 是 C^1 類的，而且 $f' = g$ 。
- (2) 序列 $(f_n)_{n \geq 1}$ 會在每個 I 的線段上均勻收斂。

證明 : 令 $a \in I$ 。從 (ii)，我們知道 $f_n(a) \xrightarrow{n \rightarrow \infty} f(a)$ 。

- (1) 首先，我們注意到由於 $(f'_n)_{n \geq 1}$ 會在每個 I 的線段上均勻收斂至 g ，從系理 8.2.2 我們得知 g 在 I 上是連續的。根據命題 8.2.5，對於 $x \in I$ ，我們有

$$\int_a^x g(t) dt = \lim_{n \rightarrow \infty} \int_a^x f'_n(t) dt = \lim_{n \rightarrow \infty} (f_n(x) - f_n(a)) = f(x) - f(a).$$

這證明了

$$\forall x \in I, \quad f(x) = f(a) + \int_a^x g(t) dt.$$

由於 g 是連續的，我們推得 f 是 C^1 類的，而且 $f' = g$ 。

- (2) 再來我們要證明 $(f_n)_{n \geq 1}$ 會均勻收斂至 f ，方法如下。對於每個 $n \geq 1$ 以及 $x \in I$ ，微積分基本定理給我們

$$\|f_n(x) - f(x)\| \leq \left\| \int_a^x (f'_n(t) - f'(t)) dt \right\| + \|f_n(a) - f(a)\|.$$

根據命題 8.2.5，右手邊的第一項會均勻收斂至 0；根據假設 (ii)，第二項會收斂至 0。因此，上面收斂速度並不取決於 $x \in I$ ，所以 $(f_n)_{n \geq 1}$ 會均勻收斂至 f 。 \square

Remark 8.2.14 : From the above proof, we see that the assumption (ii) can be softened to

(ii') there exists $a \in I$ such that $f_n(a) \xrightarrow{n \rightarrow \infty} f(a)$.

Corollary 8.2.15 : Let $p \geq 1$ be an integer, and $(f_n)_{n \geq 1}$ be a sequence of C^p functions from I to W . Suppose that

- (i) for every $0 \leq k \leq p-1$, the sequence $(f_n^{(k)})_{n \geq 1}$ converges pointwise;
- (ii) the sequence $(f_n^{(p)})_{n \geq 1}$ converges uniformly on every segment of I .

Then, the pointwise limit $f := \lim_{n \rightarrow \infty} f_n$ is of class C^p , and for $0 \leq k \leq p$, we have

$$\forall x \in I, \quad f^{(k)}(x) = \lim_{n \rightarrow \infty} f_n^{(k)}(x).$$

Proof : This can be shown by induction using Theorem 8.2.13. □

Corollary 8.2.16 : Let $(u_n)_{n \geq 1}$ be a sequence of C^1 functions from I to W . Suppose that

- (i) the series $\sum u_n$ converges pointwise;
- (ii) the series $\sum u'_n$ converges uniformly on every segment of I .

Then, the function $\sum_{n \geq 1} u_n$ is of class C^1 and

$$\left(\sum_{n \geq 1} u_n \right)' = \sum_{n \geq 1} u'_n. \quad (8.8)$$

Example 8.2.17 : We claim that the Riemann zeta function $s \mapsto \zeta(s)$ is of class C^1 , and

$$\forall s > 1, \quad \zeta'(s) = - \sum_{n=1}^{\infty} \frac{\ln n}{n^s}. \quad (8.9)$$

註解 8.2.14 : 從上面的證明我們可以看出來，假設 (ii) 可以弱化成

(ii') 存在 $a \in I$ 使得 $f_n(a) \xrightarrow{n \rightarrow \infty} f(a)$ 。

系理 8.2.15 : 令 $p \geq 1$ 為整數，且 $(f_n)_{n \geq 1}$ 為由 I 映射至 W 的 C^p 函數所構成的序列。假設

- (i) 對於每個 $0 \leq k \leq p-1$ ，序列 $(f_n^{(k)})_{n \geq 1}$ 會逐點收斂；
- (ii) 序列 $(f_n^{(p)})_{n \geq 1}$ 會在每個 I 的線段上均勻收斂。

那麼，逐點收斂的極限 $f := \lim_{n \rightarrow \infty} f_n$ 會是個 C^p 的函數，而且對於 $0 \leq k \leq p$ ，我們有

$$\forall x \in I, \quad f^{(k)}(x) = \lim_{n \rightarrow \infty} f_n^{(k)}(x).$$

證明：我們可以使用定理 8.2.13 和數學歸納法來證明。 □

系理 8.2.16 : 令 $(u_n)_{n \geq 1}$ 為由 I 映射至 W 的 C^1 函數所構成的序列。假設

- (i) 級數 $\sum u_n$ 會逐點收斂；
- (ii) 級數 $\sum u'_n$ 會在每個 I 的線段上均勻收斂。

那麼，函數 $\sum_{n \geq 1} u_n$ 會是 C^1 類的，而且

$$\left(\sum_{n \geq 1} u_n \right)' = \sum_{n \geq 1} u'_n. \quad (8.8)$$

範例 8.2.17 : 我們想要證明黎曼 ζ 函數 $s \mapsto \zeta(s)$ 是 C^1 類的，而且

$$\forall s > 1, \quad \zeta'(s) = - \sum_{n=1}^{\infty} \frac{\ln n}{n^s}. \quad (8.9)$$

For every $n \geq 1$, let $u_n : s \mapsto n^{-s}$, which is a C^1 function with derivative given by

$$\forall s > 1, \quad u'_n(s) = -\frac{\ln n}{n^s}.$$

The series of functions $\sum u_n$ converges pointwise to ζ . Fix $b > a > 1$, let us show that $\sum u'_n$ converges normally on $[a, b]$, so also uniformly. Let us choose $c \in (1, a)$. We have

$$\|(u'_n)_{|[a,b]}\|_\infty = \frac{\ln n}{n^a} = \mathcal{O}\left(\frac{1}{n^c}\right).$$

Since $\sum n^{-c}$ converges (Proposition 6.2.6), we deduce that $\sum u_n$ converges normally on $[a, b]$. Therefore, Eq. (8.8) gives us Eq. (8.9).

Corollary 8.2.18 : Let $p \geq 1$ be an integer, and $(u_n)_{n \geq 1}$ be a sequence of C^p functions from I to W . Suppose that

- (i) for every $0 \leq k \leq p-1$, the series $\sum u_n^{(k)}$ converges pointwise;
- (ii) the series $\sum u_n^{(p)}$ converges uniformly on every segment of I .

Then, the function $\sum_{n \geq 1} u_n$ is of class C^p and for $0 \leq k \leq p$, we have

$$\left(\sum_{n \geq 1} u_n\right)^{(k)} = \sum_{n \geq 1} u_n^{(k)}. \quad (8.10)$$

Example 8.2.19 : We follow the same notations as in Example 8.2.17, we find, for every $n, p \geq 1$, that

$$\forall s > 1, \quad u_n^{(p)}(s) = (-1)^p \frac{(\ln n)^p}{n^s}.$$

Let us fix $b > a > 1$. We show in the same way that $\sum u_n^{(p)}$ converges normally on $[a, b]$ for all $p \geq 0$, so also converges uniformly and pointwise. We apply Corollary 8.2.18 to conclude that $s \mapsto \zeta(s)$ is of class C^p for all $p \geq 0$, so it is of class C^∞ . Moreover, Eq. (8.10) gives us

$$\forall s > 1, \forall p \geq 1, \quad \zeta^{(p)}(s) = \sum_{n \geq 1} (-1)^p \frac{(\ln n)^p}{n^s}.$$

對於每個 $n \geq 1$ ，令 $u_n : s \mapsto n^{-s}$ ，這是個 C^1 類的函數，而且他的微分寫做

$$\forall s > 1, \quad u'_n(s) = -\frac{\ln n}{n^s}.$$

函數級數 $\sum u_n$ 會逐點收斂至 ζ 。固定 $b > a > 1$ ，讓我們證明 $\sum u'_n$ 會在 $[a, b]$ 上正規收斂，所以也會均勻收斂。我們選 $c \in (1, a)$ 。我們有

$$\|(u'_n)_{|[a,b]}\|_\infty = \frac{\ln n}{n^a} = \mathcal{O}\left(\frac{1}{n^c}\right).$$

由於 $\sum n^{-c}$ 會收斂（命題 6.2.6），我們推得 $\sum u_n$ 會在 $[a, b]$ 上正規收斂。因此，式 (8.8) 讓我們得到式 (8.9)。

系理 8.2.18 : 令 $p \geq 1$ 為整數，而且 $(u_n)_{n \geq 1}$ 為由 I 映射至 W 的 C^p 函數序列。假設

- (i) 對於每個 $0 \leq k \leq p-1$ ，級數 $\sum u_n^{(k)}$ 會逐點收斂；
- (ii) 級數 $\sum u_n^{(p)}$ 在每個 I 的線段上會均勻收斂。

那麼，函數 $\sum_{n \geq 1} u_n$ 會是 C^p 類的，而且對於 $0 \leq k \leq p$ ，我們有

$$\left(\sum_{n \geq 1} u_n\right)^{(k)} = \sum_{n \geq 1} u_n^{(k)}. \quad (8.10)$$

範例 8.2.19 : 我們沿用範例 8.2.17 中的記號，對於每個 $n, p \geq 1$ ，我們會得到

$$\forall s > 1, \quad u_n^{(p)}(s) = (-1)^p \frac{(\ln n)^p}{n^s}.$$

讓我們固定 $b > a > 1$ 。我們使用相同方法來證明對於所有 $p \geq 0$ ， $\sum u_n^{(p)}$ 會在 $[a, b]$ 上正規收斂，所以也會均勻收斂且逐點收斂。我們使用系理 8.2.18 來總結 $s \mapsto \zeta(s)$ 是 C^p 類的，對於所有 $p \geq 0$ ，所以會是 C^∞ 類的。此外，式 (8.10) 給我們

$$\forall s > 1, \forall p \geq 1, \quad \zeta^{(p)}(s) = \sum_{n \geq 1} (-1)^p \frac{(\ln n)^p}{n^s}.$$

Example 8.2.20 : Let $(W, \|\cdot\|_W)$ be a Banach space. We have seen in Theorem 3.2.18 that $\mathcal{L}_c(W) := \mathcal{L}_c(W, W)$ equipped with the operator norm $\|\cdot\|$ is a Banach space, and is also a normed algebra (Definition 6.6.1), that is the operator norm satisfies the submultiplicative property. Given $u \in \mathcal{L}_c(W)$, we may define the following function

$$\begin{aligned} \mathcal{E}_u : \mathbb{R} &\rightarrow \mathcal{L}_c(W) \\ t &\mapsto \sum_{n \geq 0} \frac{t^n}{n!} u^n. \end{aligned}$$

We may denote $u_n(t) = \frac{t^n}{n!} u^n$ for all $n \geq 0$ and $t \in \mathbb{R}$.

- It is straightforward to check that $\mathcal{E}_u(t)$ is well defined for all $t \in \mathbb{R}$, because

$$\forall t \in \mathbb{R}, \quad \sum_{n \geq 0} \frac{|t|^n}{n!} \|u^n\| \leq \sum_{n \geq 0} \frac{|t|^n}{n!} \|u\|^n = \exp(|t| \|u\|).$$

- A similar argument as in Example 8.2.4 shows that for any $M > 0$, the series of functions $\sum_{n \geq 0} u_n$ converges uniformly on $[-M, M]$ to \mathcal{E}_u .
- We have $u_0(t) = 1$ for all $t \in \mathbb{R}$. For every $n \in \mathbb{N}$, we have

$$\forall t \in \mathbb{R}, \quad u'_n(t) = \frac{t^{n-1}}{(n-1)!} u^n = u \cdot u_{n-1}(t).$$

This shows that the series of functions $\sum_{n \geq 0} u'_n = \sum_{n \geq 1} u'_n = \sum_{n \geq 0} u \cdot u_n$ converges pointwise to $u \cdot \mathcal{E}_u(t)$. This convergence is also uniform on every $[-M, M]$ for $M > 0$.

- Let us fix $M > 0$ and apply the uniform convergence of $\sum_{n \geq 0} u_n$ and $\sum_{n \geq 0} u'_n$ on $[-M, M]$ to conclude that \mathcal{E}_u is of class \mathcal{C}^1 on $[-M, M]$ and $\mathcal{E}'_u(t) = u \cdot \mathcal{E}_u(t)$ for $t \in (-M, M)$. This allows us to conclude that \mathcal{E}_u is of class \mathcal{C}^1 on \mathbb{R} and $\mathcal{E}'_u(t) = u \cdot \mathcal{E}_u(t)$ for all $t \in \mathbb{R}$.
- From the relation $\mathcal{E}'_u = u \cdot \mathcal{E}_u$, we deduce that if \mathcal{E}_u is of class \mathcal{C}^k for some $k \geq 1$, then so is \mathcal{E}'_u , meaning that \mathcal{E}_u needs to be of class \mathcal{C}^{k+1} . As a consequence, \mathcal{E}_u is of class \mathcal{C}^∞ .

範例 8.2.20 : 令 $(W, \|\cdot\|_W)$ 為 Banach 空間。在定理 3.2.18 中，我們看過 $\mathcal{L}_c(W) := \mathcal{L}_c(W, W)$ 賦予算子範數 $\|\cdot\|$ 後會是個 Banach 空間，而且也是個賦範代數（定義 6.6.1），也就是說算子範數滿足劣乘性。給定 $u \in \mathcal{L}_c(W)$ ，我們可以定義下列函數

$$\begin{aligned} \mathcal{E}_u : \mathbb{R} &\rightarrow \mathcal{L}_c(W) \\ t &\mapsto \sum_{n \geq 0} \frac{t^n}{n!} u^n. \end{aligned}$$

對於所有 $n \geq 0$ 和 $t \in \mathbb{R}$ ，我們可以記 $u_n(t) = \frac{t^n}{n!} u^n$ 。

- 我們可以直接檢查對於所有 $t \in \mathbb{R}$ ， $\mathcal{E}_u(t)$ 是定義良好的，因為

$$\forall t \in \mathbb{R}, \quad \sum_{n \geq 0} \frac{|t|^n}{n!} \|u^n\| \leq \sum_{n \geq 0} \frac{|t|^n}{n!} \|u\|^n = \exp(|t| \|u\|).$$

- 使用與在範例 8.2.4 相似的技巧，我們可以證明對於任意 $M > 0$ ，函數級數 $\sum_{n \geq 0} u_n$ 會在 $[-M, M]$ 上均勻收斂至 \mathcal{E}_u 。
- 對於所有 $t \in \mathbb{R}$ ，我們有 $u_0(t) = 1$ 。對於每個 $n \in \mathbb{N}$ ，我們有

$$\forall t \in \mathbb{R}, \quad u'_n(t) = \frac{t^{n-1}}{(n-1)!} u^n = u \cdot u_{n-1}(t).$$

這證明了函數級數 $\sum_{n \geq 0} u'_n = \sum_{n \geq 1} u'_n = \sum_{n \geq 0} u \cdot u_n$ 會逐點收斂到 $u \cdot \mathcal{E}_u(t)$ 。對於每個 $M > 0$ ，這個收斂在 $[-M, M]$ 上會是均勻的。

- 讓我們固定 $M > 0$ 並使用 $\sum_{n \geq 0} u_n$ 和 $\sum_{n \geq 0} u'_n$ 在 $[-M, M]$ 上的均勻收斂性，我們可以得證 \mathcal{E}_u 在 $[-M, M]$ 上是 \mathcal{C}^1 類的，而且 $\mathcal{E}'_u(t) = u \cdot \mathcal{E}_u(t)$ 對於 $t \in (-M, M)$ 。這可以讓我們總結 \mathcal{E}_u 在 \mathbb{R} 上是 \mathcal{C}^1 類的，而且 $\mathcal{E}'_u(t) = u \cdot \mathcal{E}_u(t)$ 對於所有 $t \in \mathbb{R}$ 。
- 從關係式 $\mathcal{E}'_u = u \cdot \mathcal{E}_u$ ，我們推得如果對於某個 $k \geq 1$ ， \mathcal{E}_u 是 \mathcal{C}^k 類的，那麼 \mathcal{E}'_u 也會是，這代表著 \mathcal{E}_u 會是 \mathcal{C}^{k+1} 類的。所以我們就能推得 \mathcal{E}_u 是 \mathcal{C}^∞ 類的。

8.3 Power series

In this section, we are going to study a particular form of series of functions, called *power series*. We restrict ourselves to real-valued and complexed-valued power series, but you need to keep in mind that all the notions are still valid if we replace $(\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$ by a normed algebra.

8.3.1 Definitions and radius of convergence

We define a few topological notions in $(\mathbb{C}, |\cdot|)$. An open ball centered at c with radius $r > 0$ is also called an *open disk* centered at c with the same radius r , denoted $D(c, r) := B(c, r)$. We also define the notion of closed disks in the same way.

Definition 8.3.1 : Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers and $c \in \mathbb{C}$.

- A series of functions of the form $\sum_{n \geq 0} a_n(z - c)^n$ is called a power series (冪級數) centered at c , where $z \in \mathbb{C}$ is the variable of the functions.
- If the sequence $(a_n)_{n \geq 0}$ is real-valued and $c \in \mathbb{R}$, we may use $x \in \mathbb{R}$ as the variable of the power series, and write $\sum_{n \geq 0} a_n(x - c)^n$. Then, this power series takes values in \mathbb{R} .

We are going to develop some theories for power series centered at $c = 0$. For a general power series centered at $c \in \mathbb{C}$, all the corresponding notions and properties can be obtained by a shift $z \mapsto z + c$. The properties and theorems are stated in terms of complex-valued power series, but you should also know that the exact same proofs apply to the real-valued power series.

Proposition 8.3.2 (Abel's lemma) : Let $\sum a_n z^n$ be a power series and $z_0 \in \mathbb{C}$ be such that the sequence $(a_n z_0^n)_{n \geq 0}$ is bounded. Then, the following properties hold.

- (1) For every $z \in \mathbb{C}$ with $|z| < |z_0|$, the series $\sum a_n z^n$ is absolutely convergent.
- (2) For every $r \in (0, |z_0|)$, the series of functions $\sum a_n z^n$ is normally convergent in the closed disk $\overline{D}(0, r) := \overline{B}(0, r)$.

第三節 冪級數

在這個章節中，我們會討論函數級數的特例，稱作冪級數。我們會侷限在取值為實數或複數的冪級數，但要記得的是，很多我們所討論的概念，在我們把 $(\mathbb{R}, |\cdot|)$ 或 $(\mathbb{C}, |\cdot|)$ 換成賦範代數時，也還是會成立的。

第一小節 定義與收斂半徑

我們定義在 $(\mathbb{C}, |\cdot|)$ 中的一些拓撲概念。我們把中心為 c 半徑為 $r > 0$ 的開球稱作中心為 c 半徑為 r 的開圓盤，記作 $D(c, r) := B(c, r)$ 。我們也用相同方式來定義閉圓盤。

定義 8.3.1 : 令 $(a_n)_{n \geq 0}$ 為複數序列以及 $c \in \mathbb{C}$ 。

- 我們把寫成 $\sum_{n \geq 0} a_n(z - c)^n$ 形式的函數級數稱作中心在 c 的冪級數 (power series)，其中 $z \in \mathbb{C}$ 是函數的變數。
- 如果序列 $(a_n)_{n \geq 0}$ 是實數序列，而且 $c \in \mathbb{R}$ ，我們可以用 $x \in \mathbb{R}$ 來代表冪集數的變數，並且記 $\sum_{n \geq 0} a_n(x - c)^n$ 。這個時候，冪級數會取值在 \mathbb{R} 中。

接著我們要討論中心在 $c = 0$ 的冪級數所構成的理論。對一般中心在 $c \in \mathbb{C}$ 的冪級數，所有對應到的概念和性質都可以透過平移 $z \mapsto z + c$ 來得到。下面的性質和定理會對複數冪級數來敘述，但你要知道的是，相同的證明對於實數冪級數也是對的。

命題 8.3.2 【Abel 引理】 : 令 $\sum a_n z^n$ 為冪級數，而且 $z_0 \in \mathbb{C}$ 使得序列 $(a_n z_0^n)_{n \geq 0}$ 有界。那麼下列性質成立。

- (1) 對於每個 $z \in \mathbb{C}$ 滿足 $|z| < |z_0|$ ，級數 $\sum a_n z^n$ 會絕對收斂。
- (2) 對於每個 $r \in (0, |z_0|)$ ，函數級數 $\sum a_n z^n$ 在閉圓盤 $\overline{D}(0, r) := \overline{B}(0, r)$ 中會正規收斂。

Proof : Let $M > 0$ be such that $|a_n||z_0|^n \leq M$ for every $n \geq 0$. For $z \in \mathbb{C}$ such that $|z| < |z_0|$, we have

$$\forall n \geq 0, \quad |a_n z^n| = \left| \frac{z}{z_0} \right|^n |a_n||z_0|^n \leq M \left| \frac{z}{z_0} \right|^n,$$

where the right-hand side is a convergent series (geometric series with ratio strictly smaller than 1). \square

Definition 8.3.3 : Let $\sum a_n z^n$ be a power series. The following quantity

$$R = R(\sum a_n z^n) := \sup\{r \geq 0 : (|a_n| r^n)_{n \geq 0} \text{ is bounded}\} \in [0, +\infty]$$

is called the *radius of convergence* (收斂半徑) of $\sum a_n z^n$.

Remark 8.3.4 : We note that if we add phases to the sequence $(a_n)_{n \geq 0}$ defining the power series $\sum a_n z^n$, its radius of convergence remains unchanged.

Proposition 8.3.5 : Let $\sum a_n z^n$ be a power series and R be its radius of convergence. Then, we have the following properties.

- (1) For $z \in \mathbb{C}$ with $|z| < R$, the series $\sum a_n z^n$ converges absolutely.
- (2) For $z \in \mathbb{C}$ with $|z| > R$, the series $\sum a_n z^n$ diverges.
- (3) For $r \in [0, R)$, the series $\sum a_n z^n$ converges normally on the closed disk $\overline{D}(0, r)$.

And the open disk $D(0, R)$ is called the disk of convergence (收斂圓盤) of the power series $\sum a_n z^n$.

Remark 8.3.6 :

- (1) When $R = +\infty$, the power series $\sum a_n z^n$ converges for every $z \in \mathbb{C}$, so it defines a function from \mathbb{C} to \mathbb{C} . Such a function is called an entire function (整函數).
- (2) When $R < +\infty$, on the boundary of the disk of convergence, that is when $z \in \partial D(0, R)$, the power series may have any possible behavior, see Example 8.3.9.

證明 : 令 $M > 0$ 使得 $|a_n||z_0|^n \leq M$ 對於所有 $n \geq 0$ 。對於 $z \in \mathbb{C}$ 滿足 $|z| < |z_0|$ ，我們有

$$\forall n \geq 0, \quad |a_n z^n| = \left| \frac{z}{z_0} \right|^n |a_n||z_0|^n \leq M \left| \frac{z}{z_0} \right|^n,$$

其中右式是個收斂級數（公比嚴格小於 1 的幾何級數）。 \square

定義 8.3.3 : 令 $\sum a_n z^n$ 為冪級數。我們定義

$$R = R(\sum a_n z^n) := \sup\{r \geq 0 : (|a_n| r^n)_{n \geq 0} \text{ 有界}\} \in [0, +\infty]$$

稱作 $\sum a_n z^n$ 的收斂半徑 (radius of convergence)。

註解 8.3.4 : 我們注意到，如果我們對序列 $(a_n)_{n \geq 0}$ 加上相位，這並不會改變冪級數 $\sum a_n z^n$ 的收斂半徑。

命題 8.3.5 : 令 $\sum a_n z^n$ 為冪級數，且 R 為他的收斂半徑。那麼我們有下列性質。

- (1) 對於 $z \in \mathbb{C}$ 滿足 $|z| < R$ ，級數 $\sum a_n z^n$ 會絕對收斂。
- (2) 對於 $z \in \mathbb{C}$ 滿足 $|z| > R$ ，級數 $\sum a_n z^n$ 會發散。
- (3) 對於 $r \in [0, R)$ ，級數 $\sum a_n z^n$ 會在閉圓盤 $\overline{D}(0, r)$ 上正規收斂。

我們把開圓盤 $D(0, R)$ 稱作是冪級數 $\sum a_n z^n$ 的收斂圓盤 (disk of convergence)。

註解 8.3.6 :

- (1) 當 $R = +\infty$ 時，冪級數 $\sum a_n z^n$ 對每個 $z \in \mathbb{C}$ 都收斂，所以他定義了從 \mathbb{C} 映射到 \mathbb{C} 的函數。這樣的函數稱作整函數 (entire function)。
- (2) 當 $R < +\infty$ ，在收斂圓盤的邊界上，也就是當 $z \in \partial D(0, R)$ 時，冪級數可以有各種可能的行為，見範例 8.3.9。

Proof :

- (1) It is a direct consequence of Proposition 8.3.2 (1).
- (2) For $z \in \mathbb{C} \setminus \overline{D}(0, R)$, since $(|a_n||z|^n)_{n \geq 0}$ is not bounded, we do not have $a_n z^n \xrightarrow{n \rightarrow \infty} 0$, so the series $\sum a_n z^n$ diverges.
- (3) It is a direct consequence of Proposition 8.3.2 (2). \square

Proposition 8.3.7 (D'Alembert's criterion, ratio test) : Let $\sum a_n z^n$ be a power series, and R be its radius of convergence. Suppose that the following limit exists,

$$\ell := \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \in [0, +\infty].$$

Then, $R = \ell^{-1}$.

Proof : It is a direct consequence of Theorem 6.3.1. \square

Proposition 8.3.8 (Cauchy's criterion, root test) : Let $\sum a_n z^n$ be a power series, and R be its radius of convergence. Let

$$\lambda := \limsup_{n \rightarrow \infty} |a_n|^{1/n} \in [0, +\infty].$$

Then, $R = \frac{1}{\lambda}$.

Proof : It is a direct consequence of Corollary 6.3.8. \square

Example 8.3.9 : The following three series have the same radius of convergence 1, that can be obtained by either the ratio test or the root test. However, they have totally different behaviors on the *boundary* of the disk of convergence.

- (1) The series $\sum z^n$ has radius of convergence 1. For $z \in \mathbb{C}$ with $|z| = 1$, the series $\sum z^n$ never converges.

證明 :

- (1) 這是命題 8.3.2 (1) 的直接結果。
- (2) 對於 $z \in \mathbb{C} \setminus \overline{D}(0, R)$ ，由於 $(|a_n||z|^n)_{n \geq 0}$ 不是有界的，我們不會有 $a_n z^n \xrightarrow{n \rightarrow \infty} 0$ ，所以級數 $\sum a_n z^n$ 發散。
- (3) 這是命題 8.3.2 (2) 的直接結果。 \square

命題 8.3.7 【D'Alembert 準則，商檢測法】：令 $\sum a_n z^n$ 為冪級數，且 R 為他的收斂半徑。假設下列極限存在：

$$\ell := \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \in [0, +\infty].$$

那麼 $R = \ell^{-1}$ 。

證明：這是定理 6.3.1 的直接結果。 \square

命題 8.3.8 【柯西準則，根檢測法】：令 $\sum a_n z^n$ 為冪級數，且 R 為他的收斂半徑。令

$$\lambda := \limsup_{n \rightarrow \infty} |a_n|^{1/n} \in [0, +\infty].$$

那麼 $R = \frac{1}{\lambda}$ 。

證明：這是系理 6.3.8 的直接結果。 \square

範例 8.3.9 : 下面的三個級數有相同的收斂半徑 1，這可以透過商檢測法或是根檢測法來得到。然而，他們在收斂圓盤的邊界上，有完全不同的行為。

- (1) 級數 $\sum z^n$ 的收斂半徑為 1。對於 $z \in \mathbb{C}$ 滿足 $|z| = 1$ ，級數 $\sum z^n$ 永遠不會收斂。
- (2) 級數 $\sum \frac{z^n}{n^2}$ 的收斂半徑為 1。對於 $z \in \mathbb{C}$ 滿足 $|z| = 1$ ，級數 $\sum \frac{z^n}{n^2}$ 會正規收斂，所以收

- (2) The series $\sum \frac{z^n}{n^2}$ has radius of convergence 1. For $z \in \mathbb{C}$ with $|z| = 1$, the series $\sum \frac{z^n}{n^2}$ converges normally, so converges.
- (3) The series $\sum \frac{z^n}{n}$ has radius of convergence 1. For $z = 1$, the series $\sum \frac{z^n}{n}$ diverges. For $z \in \mathbb{C}$ such that $|z| = 1$ and $z \neq 1$, the series $\sum \frac{z^n}{n}$ converges by Example 6.4.9.

8.3.2 Operations on power series

Proposition 8.3.10 : Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ be power series with radius of convergence R_f and R_g . Let R be the radius of convergence of $\sum (a_n + b_n) z^n$. Then,

$$R \geq \min(R_f, R_g).$$

Moreover, if $R_f \neq R_g$, we have $R = \min(R_f, R_g)$. For any $z \in \mathbb{C}$ with $|z| < \min(R_f, R_g)$, we also have

$$\sum_{n \geq 0} (a_n + b_n) z^n = \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} b_n z^n. \quad (8.11)$$

Proof : Let $z \in \mathbb{C}$ such that $|z| < \min(R_f, R_g)$. It follows from Proposition 8.3.5 that both $\sum a_n z^n$ and $\sum b_n z^n$ converges absolutely, so the series $\sum (a_n + b_n) z^n$ also converges absolutely. This means that Eq. (8.11) holds. Moreover, this also implies that $R \geq \min(R_f, R_g)$.

Suppose that $R_f \neq R_g$, for example, $R_f < R_g$. Let $z \in \mathbb{C}$ such that $R_f < |z| < R_g$. Since $(b_n z^n)_{n \geq 1}$ is bounded and $(a_n z^n)_{n \geq 1}$ is unbounded, we deduce that $((a_n + b_n) z^n)_{n \geq 1}$ is unbounded, so $|z| \geq R$. By taking infimum over $z \in \mathbb{C}$ satisfying $R_f < |z| < R_g$, we find that $R_f \geq R$. \square

Definition 8.3.11 : Let $\sum a_n z^n$ and $\sum b_n z^n$ be power series. Their *Cauchy product* is the power series $\sum c_n z^n$, where the coefficients $(c_n)_{n \geq 1}$ are given by

$$\forall n \geq 0, \quad c_n = \sum_{k=0}^n a_k b_{n-k}.$$

斂。

- (3) 級數 $\sum \frac{z^n}{n}$ 的收斂半徑為 1。對於 $z = 1$ ，級數 $\sum \frac{z^n}{n}$ 會發散。對於 $z \in \mathbb{C}$ 滿足 $|z| = 1$ 且 $z \neq 1$ ，根據範例 6.4.9，級數 $\sum \frac{z^n}{n}$ 會收斂。

第二小節 冪級數的運算

命題 8.3.10 : 令 $f(z) = \sum a_n z^n$ 和 $g(z) = \sum b_n z^n$ 為收斂半徑為 R_f 和 R_g 的冪級數。令 R 為 $\sum (a_n + b_n) z^n$ 的收斂半徑。那麼

$$R \geq \min(R_f, R_g).$$

此外，如果 $R_f \neq R_g$ ，我們有 $R = \min(R_f, R_g)$ 。對於任意 $z \in \mathbb{C}$ 滿足 $|z| < \min(R_f, R_g)$ ，我們也有

$$\sum_{n \geq 0} (a_n + b_n) z^n = \sum_{n \geq 0} a_n z^n + \sum_{n \geq 0} b_n z^n. \quad (8.11)$$

證明 : 令 $z \in \mathbb{C}$ 滿足 $|z| < \min(R_f, R_g)$ 。從命題 8.3.5，我們知道 $\sum a_n z^n$ 和 $\sum b_n z^n$ 都會絕對收斂，所以級數 $\sum (a_n + b_n) z^n$ 也會絕對收斂。這代表著式 (8.11) 會成立。此外，這也蘊含 $R \geq \min(R_f, R_g)$ 。

假設 $R_f \neq R_g$ ，例如 $R_f < R_g$ 。令 $z \in \mathbb{C}$ 滿足 $R_f < |z| < R_g$ 。由於 $(b_n z^n)_{n \geq 1}$ 有界且 $(a_n z^n)_{n \geq 1}$ 沒有界，我們推得 $((a_n + b_n) z^n)_{n \geq 1}$ 沒有界，所以 $|z| \geq R$ 。藉由對 $z \in \mathbb{C}$ 滿足 $R_f < |z| < R_g$ 取最大下界，我們得到 $R_f \geq R$ 。 \square

定義 8.3.11 : 令 $\sum a_n z^n$ 和 $\sum b_n z^n$ 為冪級數。他們的柯西積是由冪級數 $\sum c_n z^n$ 所給定的，其中係數 $(c_n)_{n \geq 1}$ 定義如下：

$$\forall n \geq 0, \quad c_n = \sum_{k=0}^n a_k b_{n-k}.$$

Proposition 8.3.12 : Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ be power series with radius of convergence R_f and R_g . Let $\sum c_n z^n$ be their Cauchy product. For every $z \in \mathbb{C}$ with $|z| < \min(R_f, R_g)$, we have

$$f(z)g(z) = \left(\sum_{n \geq 0} a_n z^n \right) \left(\sum_{n \geq 0} b_n z^n \right) = \sum_{n \geq 0} \left(\sum_{k=0}^n a_k b_{n-k} \right) z^n = \sum_{n \geq 0} c_n z^n. \quad (8.12)$$

In particular, if R is the radius of convergence of $\sum c_n z^n$, then we have

$$R \geq \min(R_f, R_g).$$

Proof : Let $z \in \mathbb{C}$ such that $|z| < \min(R_f, R_g)$. From Proposition 8.3.5, we know that both $\sum a_n z^n$ and $\sum b_n z^n$ converges absolutely, then by Theorem 6.6.3, we know that their Cauchy product $\sum c_n z^n$ converges absolutely, and satisfies Eq. (8.12). Additionally, this implies that $R \geq \min(R_f, R_g)$. \square

8.3.3 Regularity

Here, let $f := \sum a_n z^n$ be a power series with radius of convergence $R > 0$. We have seen in Proposition 8.3.5 that f is well defined on $D(0, R)$.

Theorem 8.3.13 : The function $f : z \mapsto \sum_{n \geq 0} a_n z^n$ is continuous on the disk of convergence $D(0, R)$.

Proof : Fix $z \in D(0, R)$. Let us consider a closed disk $\overline{D}(z, r)$ centered at z with radius $r < R - |z|$. Then, for any $w \in \overline{D}(z, r)$, we have $|w| \leq |w - z| + |z| \leq |z| + r < R$, which means that $\overline{D}(z, r) \subseteq D(0, R)$. It follows Proposition 8.3.5 (3) that the power series $\sum a_n z^n$ converges normally on $\overline{D}(z, r)$. Since the partial sums defining f are continuous (polynomial functions), we use Proposition 8.2.1 to conclude that the limit f is continuous at z . \square

Theorem 8.3.14 (Abel's theorem) : Let $\sum a_n z^n$ be a power series with radius of convergence $R > 0$. Suppose that the series $\sum a_n R^n$ converges. Then, the function $x \mapsto \sum_{n \geq 0} a_n x^n$ defined on $[0, R]$ is

命題 8.3.12 : 令 $f(z) = \sum a_n z^n$ 和 $g(z) = \sum b_n z^n$ 為收斂半徑為 R_f 和 R_g 的冪級數。令 $\sum c_n z^n$ 為他們的柯西積。對於每個 $z \in \mathbb{C}$ 滿足 $|z| < \min(R_f, R_g)$ ，我們有

$$f(z)g(z) = \left(\sum_{n \geq 0} a_n z^n \right) \left(\sum_{n \geq 0} b_n z^n \right) = \sum_{n \geq 0} \left(\sum_{k=0}^n a_k b_{n-k} \right) z^n = \sum_{n \geq 0} c_n z^n. \quad (8.12)$$

如果我們把 $\sum c_n z^n$ 的收斂半徑記作 R ，那麼我們有

$$R \geq \min(R_f, R_g).$$

證明 : 令 $z \in \mathbb{C}$ 滿足 $|z| < \min(R_f, R_g)$ 。從命題 8.3.5，我們知道 $\sum a_n z^n$ 和 $\sum b_n z^n$ 兩者皆會絕對收斂，接著從定理 6.6.3，我們知道他們的柯西積 $\sum c_n z^n$ 會絕對收斂，而且滿足式 (8.12)。此外，這也蘊含了 $R \geq \min(R_f, R_g)$ 。 \square

第三小節 規律性

這裡，令 $f := \sum a_n z^n$ 為收斂半徑為 $R > 0$ 的冪級數。從命題 8.3.5，我們已知 f 在 $D(0, R)$ 上是定義良好的。

定理 8.3.13 : 函數 $f : z \mapsto \sum_{n \geq 0} a_n z^n$ 在收斂圓盤 $D(0, R)$ 上是連續的。

證明 : 固定 $z \in D(0, R)$ 。讓我們考慮中心在 z ，半徑為 $r < R - |z|$ 的閉圓盤 $\overline{D}(z, r)$ 。那麼對於任意 $w \in \overline{D}(z, r)$ ，我們有 $|w| \leq |w - z| + |z| \leq |z| + r < R$ ，這代表著 $\overline{D}(z, r) \subseteq D(0, R)$ 。從命題 8.3.5 (3) 我們得知冪級數 $\sum a_n z^n$ 會在 $\overline{D}(z, r)$ 上正規收斂。由於定義 f 的部份和都是連續的（多項式函數），我們可以使用命題 8.2.1 來總結極限 f 在 z 會是連續的。 \square

定理 8.3.14 【Abel 定理】 : 令 $\sum a_n z^n$ 為收斂半徑為 $R > 0$ 的冪級數。假設級數 $\sum a_n R^n$ 收

continuous. In other words, we have

$$\sum_{n \geq 0} a_n x^n \xrightarrow{x \rightarrow R^-} \sum_{n \geq 0} a_n R^n.$$

Proof : For every $n \in \mathbb{N}_0$, let $u_n : [0, R] \rightarrow \mathbb{C}$ be defined by

$$\forall x \in [0, R], \quad u_n(x) = a_n x^n, \quad \text{and} \quad R_n = \sum_{k \geq n+1} a_k R^k.$$

By the assumption, the series of functions $\sum u_n$ converges pointwise on $[0, R]$. We want to show that this convergence is uniform, then we can conclude by Proposition 8.2.1. By rewriting each u_n as $u_n = a_n R^n \left(\frac{x}{R}\right)^n$, we may assume that $R = 1$.

Let $\varepsilon > 0$. Since $\sum a_n$ is convergent, we may find $N \geq 1$ such that $|R_n| \leq \varepsilon$ for all $n \geq N$. For $m, n \in \mathbb{N}$ with $m > n \geq N$, and $x \in [0, 1]$, we establish the Abel's transform using the remainders of the convergent series $\sum a_k$,

$$\begin{aligned} \sum_{k=n+1}^m a_k x^k &= \sum_{k=n+1}^m (R_{k-1} - R_k) x^k = \sum_{k=n}^{m-1} R_k x^{k+1} - \sum_{k=n+1}^m R_k x^k \\ &= R_n x^{n+1} - R_m x^m + \sum_{k=n+1}^{m-1} R_k (x^{k+1} - x^k). \end{aligned}$$

Since $R_m \xrightarrow{m \rightarrow \infty} 0$ and $(x_m)_{m \geq 0}$ is bounded, we have $R_m x^m \xrightarrow{m \rightarrow \infty} 0$. Moreover, we have $|R_k(x^{k+1} - x^k)| \leq \varepsilon(x^k - x^{k+1})$, and the series $\sum_k (x^k - x^{k+1})$ converges, so $\sum R_k(x^{k+1} - x^k)$ converges absolutely. Thus, for $n \in \mathbb{N}$ and $x \in [0, 1]$, the remainder of the power series writes

$$r_n(x) = R_n x^{n+1} + \sum_{k \geq n+1} R_k (x^{k+1} - x^k).$$

For $n \geq N$ and $x \in [0, 1]$, we have

$$\begin{aligned} |R_n x^{n+1}| &\leq |R_n| \leq \varepsilon, \\ \sum_{k \geq n+1} |R_k (x^{k+1} - x^k)| &\leq \sum_{k \geq n+1} \varepsilon (x^k - x^{k+1}) = \varepsilon x^{n+1} \leq \varepsilon. \end{aligned}$$

So $|r_n(x)| \leq 2\varepsilon$ for all $n \geq N$ and $x \in [0, 1]$. This means that $r_n \xrightarrow{n \rightarrow \infty} 0$ uniformly. By Proposition 8.1.17, we have shown that $\sum u_n$ converges uniformly on $[0, R]$. \square

斂。那麼，定義在 $[0, R]$ 上的函數 $x \mapsto \sum_{n \geq 0} a_n x^n$ 是連續的。換句話說，我們有

$$\sum_{n \geq 0} a_n x^n \xrightarrow{x \rightarrow R^-} \sum_{n \geq 0} a_n R^n.$$

證明：對於每個 $n \in \mathbb{N}_0$ ，令 $u_n : [0, R] \rightarrow \mathbb{C}$ 定義做

$$\forall x \in [0, R], \quad u_n(x) = a_n x^n, \quad \text{以及} \quad R_n = \sum_{k \geq n+1} a_k R^k.$$

根據假設，函數級數 $\sum u_n$ 會在 $[0, R]$ 上逐點收斂。如果我們可以證明這個收斂是均勻的，那麼我們就可以使用命題 8.2.1 來總結。我們還可以把每一項 u_n 寫成 $u_n = a_n R^n \left(\frac{x}{R}\right)^n$ ，這樣一來，我們可以假設 $R = 1$ 。

令 $\varepsilon > 0$ 。由於 $\sum a_n$ 會收斂，我們能找到 $N \geq 1$ 使得 $|R_n| \leq \varepsilon$ 對於所有 $n \geq N$ 。對於 $m, n \in \mathbb{N}$ 滿足 $m > n \geq N$ 以及 $x \in [0, 1]$ ，我們用收斂級數 $\sum a_k$ 的餘項來寫下 Abel 變換：

$$\begin{aligned} \sum_{k=n+1}^m a_k x^k &= \sum_{k=n+1}^m (R_{k-1} - R_k) x^k = \sum_{k=n}^{m-1} R_k x^{k+1} - \sum_{k=n+1}^m R_k x^k \\ &= R_n x^{n+1} - R_m x^m + \sum_{k=n+1}^{m-1} R_k (x^{k+1} - x^k). \end{aligned}$$

由於 $R_m \xrightarrow{m \rightarrow \infty} 0$ 且 $(x_m)_{m \geq 0}$ 有界，我們有 $R_m x^m \xrightarrow{m \rightarrow \infty} 0$ 。此外，我們有 $|R_k(x^{k+1} - x^k)| \leq \varepsilon(x^k - x^{k+1})$ ，而且級數 $\sum_k (x^k - x^{k+1})$ 會收斂，所以 $\sum R_k(x^{k+1} - x^k)$ 會絕對收斂。因此，對於 $n \in \mathbb{N}$ 以及 $x \in [0, 1]$ ，幕級數的餘項寫做

$$r_n(x) = R_n x^{n+1} + \sum_{k \geq n+1} R_k (x^{k+1} - x^k).$$

對於 $n \geq N$ 以及 $x \in [0, 1]$ ，我們有

$$\begin{aligned} |R_n x^{n+1}| &\leq |R_n| \leq \varepsilon, \\ \sum_{k \geq n+1} |R_k (x^{k+1} - x^k)| &\leq \sum_{k \geq n+1} \varepsilon (x^k - x^{k+1}) = \varepsilon x^{n+1} \leq \varepsilon. \end{aligned}$$

所以 $|r_n(x)| \leq 2\varepsilon$ 對於所有 $n \geq N$ 和 $x \in [0, 1]$ 。這代表著收斂 $r_n \xrightarrow{n \rightarrow \infty} 0$ 是均勻的。根據命

The following Tauber's theorem gives a converse of the above Abel's theorem.

Theorem 8.3.15 (Tauber's theorem): Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence $R > 0$. Suppose that $f(x) \xrightarrow{x \rightarrow R^-} \ell$ and $na_n \xrightarrow{n \rightarrow \infty} 0$. Then, the series $\sum a_n R^n$ converges to ℓ .

Proof: Without loss of generality, we may assume that $R = 1$. Let us denote by $(S_n)_{n \geq 0}$ the partial sums of the series $\sum a_n$. For any $n \in \mathbb{N}_0$ and $x \in (-1, 1)$, we have

$$S_n - f(x) = \sum_{k=1}^n a_k(1 - x^k) - \sum_{k \geq n+1} a_k x^k.$$

For $x \in (0, 1)$, we have

$$1 - x^k = (1 - x)(1 + x + \cdots + x^{k-1}) \leq k(1 - x).$$

Therefore, for any $n \in \mathbb{N}_0$ and $x \in (0, 1)$, we have

$$|S_n - f(x)| \leq (1 - x) \sum_{k=1}^n k|a_k| + \sum_{k \geq n+1} |a_k| x^k.$$

Given $\varepsilon > 0$ and choose $N \geq 1$ such that $n|a_n| \leq \varepsilon$ for all $n \geq N$. For any $n \geq N$, we have

$$\sum_{k \geq n+1} |a_k| x^k \leq \varepsilon \sum_{k \geq n+1} \frac{x^k}{k} \leq \frac{\varepsilon}{n} \sum_{k \geq n+1} x^k \leq \frac{\varepsilon}{n(1-x)}.$$

For $n \geq N$, let us choose $x_n = 1 - \frac{1}{n}$. Then, we find

$$|S_n - f(x_n)| \leq \frac{1}{n} \sum_{k=1}^n k|a_k| + \varepsilon.$$

Since $n|a_n| \xrightarrow{n \rightarrow \infty} 0$, it follows from Exercise 6.1 that the first term¹ on the right side converges to 0. Therefore,

$$\limsup_{n \rightarrow \infty} |S_n - f(x_n)| \leq \varepsilon.$$

題 8.1.17，我們可以總結 $\sum u_n$ 會在 $[0, R]$ 上均勻收斂。□

下面的 Tauber 定理給我們 Abel 定理的一個逆命題。

定理 8.3.15 【Tauber 定理】：令 $f(z) = \sum a_n z^n$ 為收斂半徑為 $R > 0$ 的冪級數。假設 $f(x) \xrightarrow{x \rightarrow R^-} \ell$ 且 $na_n \xrightarrow{n \rightarrow \infty} 0$ 。那麼，級數 $\sum a_n R^n$ 會收斂至 ℓ 。

證明：不失一般性，我們可以假設 $R = 1$ 。讓我們把級數 $\sum a_n$ 的部份和記作 $(S_n)_{n \geq 0}$ 。對於任意 $n \in \mathbb{N}_0$ 以及 $x \in (-1, 1)$ ，我們有

$$S_n - f(x) = \sum_{k=1}^n a_k(1 - x^k) - \sum_{k \geq n+1} a_k x^k.$$

對於 $x \in (0, 1)$ ，我們有

$$1 - x^k = (1 - x)(1 + x + \cdots + x^{k-1}) \leq k(1 - x).$$

因此，對於任意 $n \in \mathbb{N}_0$ 以及 $x \in (0, 1)$ ，我們有

$$|S_n - f(x)| \leq (1 - x) \sum_{k=1}^n k|a_k| + \sum_{k \geq n+1} |a_k| x^k.$$

給定 $\varepsilon > 0$ 並選擇 $N \geq 1$ 使得 $n|a_n| \leq \varepsilon$ 對於所有 $n \geq N$ 。對於任意 $n \geq N$ ，我們有

$$\sum_{k \geq n+1} |a_k| x^k \leq \varepsilon \sum_{k \geq n+1} \frac{x^k}{k} \leq \frac{\varepsilon}{n} \sum_{k \geq n+1} x^k \leq \frac{\varepsilon}{n(1-x)}.$$

對於 $n \geq N$ ，我們取 $x_n = 1 - \frac{1}{n}$ 。這樣一來，我們得到

$$|S_n - f(x_n)| \leq \frac{1}{n} \sum_{k=1}^n k|a_k| + \varepsilon.$$

由於 $n|a_n| \xrightarrow{n \rightarrow \infty} 0$ ，從習題 6.1 我們知道右式中的第一項¹會收斂到 0。因此

$$\limsup_{n \rightarrow \infty} |S_n - f(x_n)| \leq \varepsilon.$$

Since $\varepsilon > 0$ can be made arbitrarily small, we find

$$\lim_{n \rightarrow \infty} |S_n - f(x_n)| = 0.$$

That is, $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} f(x_n) = \lim_{x \rightarrow 1-} f(x) = \ell$. \square

The following is a generalization of Theorem 6.6.3 and Exercise 6.24.

Corollary 8.3.16 : Let $\sum a_n$ and $\sum b_n$ be convergent series. For $n \in \mathbb{N}_0$, let $c_n = \sum_{k=0}^n a_k b_{n-k}$. Suppose that $\sum c_n$ is convergent. Then,

$$\sum_{n \geq 0} c_n = \left(\sum_{n \geq 0} a_n \right) \left(\sum_{n \geq 0} b_n \right).$$

Proof : Let $\sum a_n z^n$, $\sum b_n z^n$, and $\sum c_n z^n$ be power series. Their radii of convergence are at least 1, because both $(a_n |z|^n)_{n \geq 0}$ and $(b_n |z|^n)_{n \geq 0}$ are bounded for $z \in D(0, 1)$. It follows from Proposition 8.3.12 that the radius of convergence of the power series $\sum c_n z^n$ is greater or equal to 1. By Theorem 8.3.14, we know that

$$\sum_{n \geq 0} a_n x^n \xrightarrow{x \rightarrow 1-} \sum_{n \geq 0} a_n, \quad \sum_{n \geq 0} b_n x^n \xrightarrow{x \rightarrow 1-} \sum_{n \geq 0} b_n, \quad \text{and} \quad \sum_{n \geq 0} c_n x^n \xrightarrow{x \rightarrow 1-} \sum_{n \geq 0} c_n.$$

Moreover, Proposition 8.3.12 gives the following identity,

$$\forall x \in (-1, 1), \quad \sum_{n \geq 0} c_n x^n = \left(\sum_{n \geq 0} a_n x^n \right) \left(\sum_{n \geq 0} b_n x^n \right)$$

By taking the limit $x \rightarrow 1-$ in the above identity, we establish the identity we want. \square

Let us also introduce the notion of *differentiability* in a complex variable.

Definition 8.3.17 : Let $A \subseteq \mathbb{C}$ and $f : A \rightarrow \mathbb{C}$. We say that f is \mathbb{C} -differentiable (or simply differen-

由於 $\varepsilon > 0$ 可以任意小，我們得到

$$\lim_{n \rightarrow \infty} |S_n - f(x_n)| = 0.$$

也就是說 $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} f(x_n) = \lim_{x \rightarrow 1-} f(x) = \ell$. \square

下面是定理 6.6.3 和習題 6.24 的推廣。

系理 8.3.16 : 令 $\sum a_n$ 和 $\sum b_n$ 為收斂級數。對於 $n \in \mathbb{N}_0$ ，令 $c_n = \sum_{k=0}^n a_k b_{n-k}$ 。假設 $\sum c_n$ 會收斂。那麼，我們有

$$\sum_{n \geq 0} c_n = \left(\sum_{n \geq 0} a_n \right) \left(\sum_{n \geq 0} b_n \right).$$

證明 : 令 $\sum a_n z^n$ 、 $\sum b_n z^n$ 和 $\sum c_n z^n$ 為冪級數。他們的收斂半徑至少為 1，因為 $(a_n |z|^n)_{n \geq 0}$ 和 $(b_n |z|^n)_{n \geq 0}$ 對於 $z \in D(0, 1)$ 有界。從命題 8.3.12 我們得知，冪級數 $\sum c_n z^n$ 的收斂半徑大於等於 1。根據定理 8.3.14，我們得到

$$\sum_{n \geq 0} a_n x^n \xrightarrow{x \rightarrow 1-} \sum_{n \geq 0} a_n, \quad \sum_{n \geq 0} b_n x^n \xrightarrow{x \rightarrow 1-} \sum_{n \geq 0} b_n, \quad \text{以及} \quad \sum_{n \geq 0} c_n x^n \xrightarrow{x \rightarrow 1-} \sum_{n \geq 0} c_n.$$

此外，命題 8.3.12 會給我們下列關係式：

$$\forall x \in (-1, 1), \quad \sum_{n \geq 0} c_n x^n = \left(\sum_{n \geq 0} a_n x^n \right) \left(\sum_{n \geq 0} b_n x^n \right)$$

藉由對上式中 $x \rightarrow 1-$ 取極限，我們得到所要證明的關係式。 \square

我們再來也引入對複數變數可微分性的概念。

¹The sum $\frac{1}{n} \sum_{k=1}^n k a_k$ is called the Cesàro sum of $(n a_n)_{n \geq 1}$.

¹我們把 $\frac{1}{n} \sum_{k=1}^n k a_k$ 稱作 $(n a_n)_{n \geq 1}$ 的 Cesàro 和。

tiable) at $z_0 \in \mathring{A}$ if the following limit exists,

$$\frac{df}{dz}(z_0) = \frac{d}{dz}f(z_0) = f'(z_0) := \lim_{\substack{z \rightarrow z_0 \\ z \in \mathbb{C}}} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C},$$

which is also called the \mathbb{C} -derivative of f at z_0 .

Remark 8.3.18 : We may identify \mathbb{C} as a two-dimensional real vector space. If we compare the notion of differential from Definition 4.1.1, we may notice that the \mathbb{C} -derivative introduced here is much stronger. In fact, if a function $f : A \rightarrow \mathbb{C}$ is differentiable at z_0 in the sense of Definition 4.1.1, its differential is a continuous linear map. However, if the same function is \mathbb{C} -differentiable at z_0 , its \mathbb{C} -derivative is given by a complex number, which, seen as a differential, is a composition between a rotation and a dilation (in \mathbb{R}^2). It is not hard to see that a composition between a rotation and a dilation is a continuous linear map, but the converse fails to hold in general. In Complex Analysis, you will see that if a function is \mathbb{C} -differentiable in an open subset $A \subseteq \mathbb{C}$, then it can be differentiated as many times as we want in A . Such functions are called *holomorphic functions*.

A power series contains only polynomials functions, and it is not hard to check that the \mathbb{C} -derivative of a polynomial function is the same as its usual \mathbb{R} -derivative. In other words, we have

$$\forall n \in \mathbb{N}_0, \quad \frac{d(z^n)}{dz} = nz^{n-1}.$$

Theorem 8.3.19 : The function $f : D(0, R) \rightarrow \mathbb{C}, z \mapsto \sum_{n \geq 0} a_n z^n$ is of class \mathcal{C}^1 . The power series $\sum_{n \geq 1} n a_n z^{n-1}$ has the same radius of convergence as $\sum_{n \geq 0} a_n z^n$, that is

$$R\left(\sum_{n \geq 1} n a_n z^{n-1}\right) = R\left(\sum_{n \geq 0} a_n z^n\right).$$

We also have

$$\forall z \in D(0, R), \quad f'(z) = \sum_{n \geq 1} n a_n z^{n-1}. \quad (8.13)$$

定義 8.3.17 : 令 $A \subseteq \mathbb{C}$ 以及 $f : A \rightarrow \mathbb{C}$ 。給定 $z_0 \in \mathring{A}$ 。如果下面極限存在：

$$\frac{df}{dz}(z_0) = \frac{d}{dz}f(z_0) = f'(z_0) := \lim_{\substack{z \rightarrow z_0 \\ z \in \mathbb{C}}} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C},$$

則我們稱他為 f 在 z_0 點的複數微分，也說 f 在 \mathbb{C} 中可微分（或簡稱可微）。

註解 8.3.18 : 我們可以把 \mathbb{C} 看作是二維的實數向量空間。如果我們把定義 4.1.1 當中微分的概念拿來比較，我們注意到這裡引進的複數微分概念是比較強的。實際上，如果函數 $f : A \rightarrow \mathbb{C}$ 在定義 4.1.1 的意義下，在 z_0 點可微，那麼他的微分會是個連續線性映射。然而，如果相同的函數在 z_0 點是複數可微的，那麼他的複數微分會被一個複數所給出，也就代表說他的微分（在 \mathbb{R}^2 中），會由旋轉和縮放所合成。我們不難看出來，旋轉和縮放的合成函數是個連續線性映射，但這個逆命題一般來說是不對的。在複分析的課程中，你會看到如果函數在開子集 $A \subseteq \mathbb{C}$ 中是複數可微的，那麼他在 A 中可以被微分任何多次，這樣的函數稱作全純函數。

冪級數當中只有多項式函數，我們不難檢查多項式的複數微分與實數微分是相同的。換句話說，我們會有：

$$\forall n \in \mathbb{N}_0, \quad \frac{d(z^n)}{dz} = nz^{n-1}.$$

定理 8.3.19 : 函數 $f : D(0, R) \rightarrow \mathbb{C}, z \mapsto \sum_{n \geq 0} a_n z^n$ 是 \mathcal{C}^1 類的。冪級數 $\sum_{n \geq 1} n a_n z^{n-1}$ 和 $\sum_{n \geq 0} a_n z^n$ 有相同的收斂半徑，也就是說

$$R\left(\sum_{n \geq 1} n a_n z^{n-1}\right) = R\left(\sum_{n \geq 0} a_n z^n\right).$$

我們也會有

$$\forall z \in D(0, R), \quad f'(z) = \sum_{n \geq 1} n a_n z^{n-1}. \quad (8.13)$$

Remark 8.3.20 : This theorem is of particular interest. It means that we can *always* differentiate term by term a power series, which is not the case of a general series of functions, where additional assumptions are needed (Corollary 8.2.16).

Proof : Let R' be the radius of convergence of $\sum na_n z^{n-1}$. For any $r \in [0, R')$, we know from Definition 8.3.3 that $(na_n r^{n-1})_{n \geq 1}$ is bounded, so $(a_n r^n)_{n \geq 0}$ is also bounded, which implies that $r < R$. By taking the limit $r \rightarrow R'-$, we find $R' \leq R$. For the converse, let $r \in (0, R)$ and $r_0 \in (r, R)$. Again by Definition 8.3.3, we know that $(a_n r_0^n)_{n \geq 0}$ is bounded. We have

$$na_n r^{n-1} = n(a_n r_0^n) \left(\frac{r}{r_0}\right)^{n-1} \xrightarrow{n \rightarrow \infty} 0,$$

so we also know that $(na_n r^{n-1})_{n \geq 1}$ is bounded, that is $r < R'$. When we take $r \rightarrow R-$, we find $R \leq R'$. Now, we can deduce Eq. (8.13) as a direct consequence of Corollary 8.2.16 and Proposition 8.3.5. \square

Corollary 8.3.21 : The power series $f(z) = \sum_{n \geq 0} a_n z^n$ is of class \mathcal{C}^∞ on $D(0, R)$. For every $p \in \mathbb{N}_0$, the p -th derivative of the power series has the same radius of convergence and writes

$$\forall z \in D(0, R), \quad f^{(p)}(z) = \sum_{n \geq p} n(n-1) \cdots (n-p+1) a_n z^{n-p} = \sum_{n \geq p} \binom{n}{p} p! a_n z^{n-p}.$$

In particular, this gives

$$\forall p \in \mathbb{N}_0, \quad a_p = \frac{f^{(p)}(0)}{p!},$$

and

$$\forall z \in D(0, R), \quad f(z) = \sum_{p \geq 0} \frac{f^{(p)}(0)}{p!} z^p.$$

Proof : It is a direct consequence of Theorem 8.3.19 with an induction. \square

註解 8.3.20 : 這個定理非常有趣。這代表著我們永遠可以對冪級數一項一項微分，而這個對一般的函數級數來說，沒有額外假設的話會是錯的（系理 8.2.16）。

證明：令 R' 為 $\sum na_n z^{n-1}$ 的收斂半徑。對於任意 $r \in [0, R')$ ，我們從定義 8.3.3 知道 $(na_n r^{n-1})_{n \geq 1}$ 會是有界的，所以 $(a_n r^n)_{n \geq 0}$ 也會是有界的，這蘊含 $r < R$ 。我們取極限 $r \rightarrow R'-$ ，會得到 $R' \leq R$ 。要證明另一個不等式，我們考慮 $r \in (0, R)$ 以及 $r_0 \in (r, R)$ 。再次使用定義 8.3.3，我們知道 $(a_n r_0^n)_{n \geq 0}$ 是有界的。我們有

$$na_n r^{n-1} = n(a_n r_0^n) \left(\frac{r}{r_0}\right)^{n-1} \xrightarrow{n \rightarrow \infty} 0,$$

所以我們知道 $(na_n r^{n-1})_{n \geq 1}$ 也是有界的，也就是說 $r < R'$ 。當我們取 $r \rightarrow R-$ 時，會得到 $R \leq R'$ 。現在，我們可以把式 (8.13) 看作是系理 8.2.16 和命題 8.3.5 的直接結果。 \square

系理 8.3.21 : 冪級數 $f(z) = \sum_{n \geq 0} a_n z^n$ 在 $D(0, R)$ 上是 \mathcal{C}^∞ 類的。對於每個 $p \in \mathbb{N}_0$ ，冪級數的第 p 階微分會有相同的收斂半徑，而且寫做

$$\forall z \in D(0, R), \quad f^{(p)}(z) = \sum_{n \geq p} n(n-1) \cdots (n-p+1) a_n z^{n-p} = \sum_{n \geq p} \binom{n}{p} p! a_n z^{n-p}.$$

這讓我們得到

$$\forall p \in \mathbb{N}_0, \quad a_p = \frac{f^{(p)}(0)}{p!},$$

和

$$\forall z \in D(0, R), \quad f(z) = \sum_{p \geq 0} \frac{f^{(p)}(0)}{p!} z^p.$$

證明：搭配上數學歸納法後，這是定理 8.3.19 的直接結果。 \square

Example 8.3.22 : We have the following identity,

$$\forall z \in D(0, 1), \quad \frac{1}{1-z} = \sum_{n \geq 0} z^n.$$

Theorem 8.3.19 allows us to differentiate the identity, giving us

$$\forall z \in D(0, 1), \quad \frac{1}{(1-z)^2} = \sum_{n \geq 1} n z^{n-1} = \sum_{n \geq 0} (n+1) z^n. \quad (8.14)$$

By taking higher-order derivatives, for every $p \in \mathbb{N}$, by Corollary 8.3.21, we find

$$\forall z \in D(0, 1), \quad \frac{p!}{(1-z)^{p+1}} = \sum_{n \geq 0} (n+1) \dots (n+p) z^n \quad \text{or} \quad \frac{1}{(1-z)^{p+1}} = \sum_{n \geq 0} \binom{n+p}{p} z^n$$

If we multiply Eq. (8.14) by z then differentiate again, we find

$$\forall z \in D(0, 1), \quad \frac{1+z}{(1-z)^3} = \sum_{n \geq 1} n^2 z^{n-1} = \sum_{n \geq 0} (n+1)^2 z^n.$$

In particular, when $z = \frac{1}{2}$, we find the following identity,

$$\sum_{n \geq 1} \frac{n^2}{2^n} = 6.$$

Corollary 8.3.21 gives us following direct consequences, which are very useful when we deal with power series.

Corollary 8.3.23 : The power series

$$F : D(0, R) \rightarrow \mathbb{C} \\ z \mapsto \sum_{n \geq 1} \frac{a_n}{n+1} z^{n+1}$$

has the same radius of convergence as $\sum a_n z^n$. Moreover, we have $F' = f$ on $D(0, R)$.

8.3.4 Coefficients of power series

範例 8.3.22 : 我們有下列關係式：

$$\forall z \in D(0, 1), \quad \frac{1}{1-z} = \sum_{n \geq 0} z^n.$$

定理 8.3.19 讓我們可以對這個關係式微分，得到

$$\forall z \in D(0, 1), \quad \frac{1}{(1-z)^2} = \sum_{n \geq 1} n z^{n-1} = \sum_{n \geq 0} (n+1) z^n. \quad (8.14)$$

透過系理 8.3.21，我們也可以算更高階的微分。對於每個 $p \in \mathbb{N}$ ，我們有

$$\forall z \in D(0, 1), \quad \frac{p!}{(1-z)^{p+1}} = \sum_{n \geq 0} (n+1) \dots (n+p) z^n \quad \text{或} \quad \frac{1}{(1-z)^{p+1}} = \sum_{n \geq 0} \binom{n+p}{p} z^n$$

如果我們把式 (8.14) 乘上 z 再微分一次，我們得到

$$\forall z \in D(0, 1), \quad \frac{1+z}{(1-z)^3} = \sum_{n \geq 1} n^2 z^{n-1} = \sum_{n \geq 0} (n+1)^2 z^n.$$

當我們取特別的值 $z = \frac{1}{2}$ ，我們得到下面這個關係式：

$$\sum_{n \geq 1} \frac{n^2}{2^n} = 6.$$

系理 8.3.21 給我們下面這個直接結果，這對冪級數來說也是非常有用的。

系理 8.3.23 : 冪級數

$$F : D(0, R) \rightarrow \mathbb{C} \\ z \mapsto \sum_{n \geq 1} \frac{a_n}{n+1} z^{n+1}$$

和 $\sum a_n z^n$ 有相同的收斂半徑。此外，在 $D(0, R)$ 上，我們有 $F' = f$ 。

第四小節 冪級數的係數

Corollary 8.3.24 (Uniqueness of power series) : Let $f(z) = \sum_{n \geq 0} a_n z^n$ and $g(z) = \sum_{n \geq 0} b_n z^n$ be two power series with radius of convergence

$$R_f := R\left(\sum_{n \geq 0} a_n z^n\right) > 0, \quad \text{and} \quad R_g := R\left(\sum_{n \geq 0} b_n z^n\right) > 0.$$

Suppose that there exists $r > 0$ and $r \leq \min(R_f, R_g)$ such that $f \equiv g$ on $(-r, r) \subseteq \mathbb{R}$. Then, we have $a_n = b_n$ for all $n \in \mathbb{N}_0$.

Proof : Let $R = \min(R_f, R_g)$ and consider the following functions defined on $(-R, R)$,

$$\forall z \in (-R, R), \quad f(z) = \sum_{n \geq 0} a_n z^n, \quad \text{and} \quad g(z) = \sum_{n \geq 0} b_n z^n.$$

It follows from Corollary 8.3.21 that both f and g are \mathcal{C}^∞ functions, and their coefficients are given by

$$\forall n \in \mathbb{N}_0, \quad a_n = \frac{f^{(n)}(0)}{n!}, \quad \text{and} \quad b_n = \frac{g^{(n)}(0)}{n!}.$$

By the assumption that $f \equiv g$ on $(-r, r)$ for some $r \in (0, R]$, we deduce that $f^{(n)}(0) = g^{(n)}(0)$ for all $n \geq 0$, so we also have $a_n = b_n$ for all $n \geq 0$. \square

Example 8.3.25 : Let $f : D(0, R) \rightarrow \mathbb{C}, z \mapsto \sum_{n \geq 0} a_n z^n$ be a power series with $R > 0$. Suppose that f is an even function, that is $f(z) = f(-z)$ for $z \in (-R, R)$. In other words,

$$\forall z \in (-R, R), \quad \sum_{n \geq 0} a_n (-z)^n = \sum_{n \geq 0} a_n z^n.$$

This implies that

$$\forall n \in \mathbb{N}_0, \quad (-1)^n a_n = a_n.$$

In other words, $a_n = 0$ if n is an odd integer.

系理 8.3.24 【冪級數的唯一性】 : 令 $f(z) = \sum_{n \geq 0} a_n z^n$ 與 $g(z) = \sum_{n \geq 0} b_n z^n$ 為冪級數，他們的收斂半徑記作：

$$R_f := R\left(\sum_{n \geq 0} a_n z^n\right) > 0, \quad \text{以及} \quad R_g := R\left(\sum_{n \geq 0} b_n z^n\right) > 0.$$

假設存在 $r > 0$ 以及 $r \leq \min(R_f, R_g)$ 使得在 $(-r, r) \subseteq \mathbb{R}$ 上我們有 $f \equiv g$ 。那麼對於所有 $n \in \mathbb{N}_0$ ，我們有 $a_n = b_n$ 。

證明 : 令 $R = \min(R_f, R_g)$ 並考慮下列定義在 $(-R, R)$ 上的函數：

$$\forall z \in (-R, R), \quad f(z) = \sum_{n \geq 0} a_n z^n, \quad \text{以及} \quad g(z) = \sum_{n \geq 0} b_n z^n.$$

從系理 8.3.21 我們知道， f 和 g 都是 \mathcal{C}^∞ 函數，而且他們的係數由下列關係式給定：

$$\forall n \in \mathbb{N}_0, \quad a_n = \frac{f^{(n)}(0)}{n!}, \quad \text{以及} \quad b_n = \frac{g^{(n)}(0)}{n!}.$$

由於存在 $r \in (0, R]$ 使得 $f \equiv g$ 在 $(-r, r)$ 上相等的假設，我們推得 $f^{(n)}(0) = g^{(n)}(0)$ 對於所有 $n \geq 0$ ，所以我們也有 $a_n = b_n$ 對於所有 $n \geq 0$ 。 \square

範例 8.3.25 : 令 $f : D(0, R) \rightarrow \mathbb{C}, z \mapsto \sum_{n \geq 0} a_n z^n$ 為冪級數，且 $R > 0$ 。假設 f 是個偶函數，也就是說 $f(z) = f(-z)$ 對於 $z \in (-R, R)$ 。換句話說，我們有

$$\forall z \in (-R, R), \quad \sum_{n \geq 0} a_n (-z)^n = \sum_{n \geq 0} a_n z^n.$$

這蘊含

$$\forall n \in \mathbb{N}_0, \quad (-1)^n a_n = a_n.$$

換句話說，如果 n 是奇數，則我們有 $a_n = 0$ 。

Theorem 8.3.26 (Cauchy's formula) : Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence $R > 0$. Then, for any $r \in (0, R)$ and $n \in \mathbb{N}_0$, we have

$$r^n a_n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

Proof : Let us fix $r \in (0, R)$ and $n \in \mathbb{N}_0$. We have

$$\int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta = \int_0^{2\pi} \left(\sum_{p \geq 0} a_p r^p e^{i(p-n)\theta} \right) d\theta.$$

Since $\sum |a_p| r^p$ converges, the series of functions $\theta \mapsto \sum a_p r^p e^{i(p-n)\theta}$ converges normally on $[0, 2\pi]$. We deduce from Corollary 8.2.9 that we may interchange the order between integration and summation. As a consequence,

$$\int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta = \sum_{p \geq 0} a_p r^p \int_0^{2\pi} e^{i(p-n)\theta} d\theta = \sum_{p \geq 0} a_p r^p (2\pi) \mathbb{1}_{p=n} = 2\pi r^n a_n. \quad \square$$

Remark 8.3.27 : This provides another proof of Corollary 8.3.24 if, using its notations, $f \equiv g$ on $D(0, r)$ for some $r \in (0, R)$.

8.3.5 Expansion in power series

In the previous subsections, we were given power series and discussed their properties. In this subsection, we are going to see when and which functions can be written (or expanded) as a power series.

Definition 8.3.28 : Let $A \subseteq \mathbb{C}$ be an open set and a function $f : A \rightarrow \mathbb{C}$.

- Let $R > 0$. If $0 \in A$ and there exists a power series $\sum a_n z^n$ such that

$$\forall z \in D(0, R), \quad f(z) = \sum_{n \geq 0} a_n z^n, \quad (8.15)$$

then we say that f can be written (or expanded) as a power series around 0, or on $D(0, R)$. In

定理 8.3.26 【柯西公式】 : 令 $f(z) = \sum a_n z^n$ 為收斂半徑為 $R > 0$ 的冪級數。那麼，對於任意 $r \in (0, R)$ 以及 $n \in \mathbb{N}_0$ ，我們有

$$r^n a_n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

證明 : 讓我們固定 $r \in (0, R)$ 以及 $n \in \mathbb{N}_0$ 。我們有

$$\int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta = \int_0^{2\pi} \left(\sum_{p \geq 0} a_p r^p e^{i(p-n)\theta} \right) d\theta.$$

由於 $\sum |a_p| r^p$ 收斂，函數級數 $\theta \mapsto \sum a_p r^p e^{i(p-n)\theta}$ 在 $[0, 2\pi]$ 上會正規收斂。從系理 8.2.9 我們推得可以交換積分和取和的順序，所以會有

$$\int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta = \sum_{p \geq 0} a_p r^p \int_0^{2\pi} e^{i(p-n)\theta} d\theta = \sum_{p \geq 0} a_p r^p (2\pi) \mathbb{1}_{p=n} = 2\pi r^n a_n. \quad \square$$

註解 8.3.27 : 使用系理 8.3.24 的記號，如果存在 $r \in (0, R)$ 使得 $f \equiv g$ 在 $D(0, r)$ 上，那麼這個定理給我們另一個證明 f 與 g 兩個冪級數中的係數相同的方式。

第五小節 冪級數展開

在前面的小節中，我們給定冪級數，並且討論他們的性質。在這個小節中，我們會看到什麼時候還有哪種函數可以被寫成（或展開）為冪級數。

定義 8.3.28 : 令 $A \subseteq \mathbb{C}$ 為開集，且 $f : A \rightarrow \mathbb{C}$ 為函數。

- 令 $R > 0$ 。如果 $0 \in A$ 而且存在冪級數 $\sum a_n z^n$ 使得

$$\forall z \in D(0, R), \quad f(z) = \sum_{n \geq 0} a_n z^n, \quad (8.15)$$

那麼我們說 f 可以在 0 附近，或在 $D(0, R)$ 上寫成（或展開成）冪級數。特別來說，這

particular, such a function needs to be C^∞ at 0, which is a direct consequence of Corollary 8.3.21.

- Let $z_0 \in A$. We say that f can be written (or expanded) as a power series around z_0 if $z \mapsto f(z + z_0)$ can be written as a power series around 0.

Proposition 8.3.29 : Let $A \subseteq \mathbb{C}$ be an open set containing 0 and a function $f : A \rightarrow \mathbb{C}$. Then, the following properties are equivalent.

- (1) f can be written as a power series around 0.
- (2) There exists $r > 0$ such that the series of remainders $(R_n)_{n \geq 0}$ converges pointwise to 0 on $D(0, r)$, where

$$\forall n \in \mathbb{N}_0, \forall z \in D(0, r), \quad R_n(z) = f(z) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} z^k. \quad (8.16)$$

When (2) holds, it means that the power series $\sum \frac{f^{(n)}(0)}{n!} z^n$ has radius of convergence R satisfying $R \geq r$, and f is equal to the series on $D(0, r)$.

Remark 8.3.30 :

- (1) To check Proposition 8.3.29 (2), we use Taylor–Lagrange or Taylor integral formula (Section 4.3.1) to write the remainder as

$$R_n(z) = \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(\theta z), \quad \theta \in (0, 1), \quad \text{or} \quad R_n(z) = z^{n+1} \int_0^1 \frac{(1-t)^n}{n!} f^{(n+1)}(tz) dt.$$

- (2) We note that to check Proposition 8.3.29 (2), it is not sufficient to check that the radius of convergence of $\sum \frac{f^{(n)}(0)}{n!} z^n$ is strictly positive. Actually, there are functions such that this power series has a strictly positive radius of convergence without Eq. (8.15) holds, see Example 8.3.32 for an example. However, if this radius of convergence is 0, it tells us that f cannot be written as a power series around 0.

Proof : There is nothing to show for $(1) \Rightarrow (2)$. Suppose that (2) holds, let us show (1). Let $r > 0$ satisfying Eq. (8.16). Let $z \in D(0, r)$. The condition $R_n(z) \xrightarrow{n \rightarrow \infty} 0$ implies that $f(z) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n$. Therefore, the sequence $(\frac{f^{(n)}(0)}{n!} z^n)_{n \geq 0}$ tends to 0, so is bounded, so the radius of convergence R of the

樣的函數一定要在 0 點為 C^∞ 的，這會是系理 8.3.21 的直接結果。

- 令 $z_0 \in A$ 。如果 $z \mapsto f(z + z_0)$ 可以在 0 點附近寫成冪級數，則我們說 f 可以在 z_0 附近寫成（或展開成）冪級數。

命題 8.3.29 : 令 $A \subseteq \mathbb{C}$ 為包含 0 點的開集，以及函數 $f : A \rightarrow \mathbb{C}$ 。那麼下列性質是等價的。

- (1) f 可以在 0 點附近寫成冪級數。
- (2) 存在 $r > 0$ 使得餘項級數 $(R_n)_{n \geq 0}$ 會在 $D(0, r)$ 上逐點收斂至 0，其中

$$\forall n \in \mathbb{N}_0, \forall z \in D(0, r), \quad R_n(z) = f(z) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} z^k. \quad (8.16)$$

當 (2) 成立，這代表著冪級數 $\sum \frac{f^{(n)}(0)}{n!} z^n$ 的收斂半徑 R 滿足 $R \geq r$ ，且 f 在 $D(0, r)$ 上與級數相等。

註解 8.3.30 :

- (1) 我們可以使用 Taylor–Lagrange 或 Taylor 積分公式（第 4.3.1 小節）來檢查命題 8.3.29 (2)，這可以讓我們把餘項寫做：

$$R_n(z) = \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(\theta z), \quad \theta \in (0, 1), \quad \text{或} \quad R_n(z) = z^{n+1} \int_0^1 \frac{(1-t)^n}{n!} f^{(n+1)}(tz) dt.$$

- (2) 我們注意到，只檢查 $\sum \frac{f^{(n)}(0)}{n!} z^n$ 的收斂半徑是嚴格為正的，與檢查命題 8.3.29 (2) 是不同的。事實上，存在函數使得他所對應到的冪級數有嚴格為正的收斂半徑，但式 (8.15) 卻不會成立，在範例 8.3.32 中我們可以看到一個這樣的例子。然而，如果這個冪級數收斂半徑為 0，這告訴我們 f 在 0 點附近無法寫成冪級數。

證明 : $(1) \Rightarrow (2)$ 並不需要證明。假設 (2) 成立，讓我們來證明 (1)。令 $r > 0$ 滿足式 (8.16)。令 $z \in D(0, r)$ 。條件 $R_n(z) \xrightarrow{n \rightarrow \infty} 0$ 會蘊含 $f(z) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n$ 。因此，序列 $(\frac{f^{(n)}(0)}{n!} z^n)_{n \geq 0}$ 會

corresponding power series satisfies $R \geq |z|$ (Definition 8.3.3). By taking supremum over $z \in D(0, r)$, we find $R \geq r$. \square

Example 8.3.31 : The following functions can be written as a power series around 0.

(1) The exponential function $z \mapsto \exp(z)$,

$$\forall z \in \mathbb{C}, \quad e^z = \sum_{n \geq 0} \frac{z^n}{n!}.$$

In fact, for any $z \in \mathbb{C}$ and $n \geq 0$, the n -th remainder writes

$$|R_n(z)| = \frac{|z|^{n+1}}{(n+1)!} |f^{(n+1)}(\theta z)| = \frac{|z|^{n+1}}{(n+1)!} e^{\theta \operatorname{Re}(z)} \xrightarrow{n \rightarrow \infty} 0.$$

(2) The function $z \mapsto \frac{1}{1-z}$ is defined on $\mathbb{C} \setminus \{1\}$, and we have

$$\forall z \in D(0, 1), \quad \frac{1}{1-z} = \sum_{n \geq 0} z^n.$$

In fact, for any $z \in D(0, 1)$ and $n \geq 0$, the n -th remainder writes

$$|R_n(z)| = \left| \frac{z^{n+1}}{1-z} \right| \leq \frac{|z|^{n+1}}{|1-z|} \xrightarrow{n \rightarrow \infty} 0.$$

(3) Any polynomial function $P \in \mathbb{C}[X]$ satisfies

$$\forall z \in \mathbb{C}, \quad P(z) = \sum_{n \geq 0} \frac{P^{(n)}(0)}{n!} z^n.$$

Actually, the above power series contains only finitely many terms.

Example 8.3.32 : Let us consider the function f defined as below,

$$f: \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

收斂到 0，所以有界，所以相對應冪級數的收斂半徑 R 會滿足 $R \geq |z|$ (定義 8.3.3)。藉由對 $z \in D(0, r)$ 取最小上界，我們得到 $R \geq r$ 。 \square

範例 8.3.31 : 下面的函數可以在 0 點附近寫成冪級數。

(1) 指數函數 $z \mapsto \exp(z)$:

$$\forall z \in \mathbb{C}, \quad e^z = \sum_{n \geq 0} \frac{z^n}{n!}.$$

我們可以檢查，對於任意 $z \in \mathbb{C}$ 還有 $n \geq 0$ ，第 n 個餘項寫做：

$$|R_n(z)| = \frac{|z|^{n+1}}{(n+1)!} |f^{(n+1)}(\theta z)| = \frac{|z|^{n+1}}{(n+1)!} e^{\theta \operatorname{Re}(z)} \xrightarrow{n \rightarrow \infty} 0.$$

(2) 函數 $z \mapsto \frac{1}{1-z}$ 是定義在 $\mathbb{C} \setminus \{1\}$ 上的，我們有：

$$\forall z \in D(0, 1), \quad \frac{1}{1-z} = \sum_{n \geq 0} z^n.$$

我們可以檢查，對於任意 $z \in D(0, 1)$ 還有 $n \geq 0$ ，第 n 個餘項寫做：

$$|R_n(z)| = \left| \frac{z^{n+1}}{1-z} \right| \leq \frac{|z|^{n+1}}{|1-z|} \xrightarrow{n \rightarrow \infty} 0.$$

(3) 任意多項式函數 $P \in \mathbb{C}[X]$ 滿足：

$$\forall z \in \mathbb{C}, \quad P(z) = \sum_{n \geq 0} \frac{P^{(n)}(0)}{n!} z^n.$$

我們可以檢查，上面的冪級數中只會有有限多個項。

範例 8.3.32 : 讓我們考慮定義如下的函數 f :

$$f: \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto \begin{cases} e^{-1/x} & \text{若 } x > 0, \\ 0 & \text{若 } x \leq 0. \end{cases}$$

For $k \in \mathbb{N}_0$, we may compute the k -th derivative of f on $(0, +\infty)$,

$$\forall x > 0, \quad f^{(k)}(x) = P_k\left(\frac{1}{x}\right)e^{-1/x}, \quad (8.17)$$

where P_k is a polynomial satisfying $\deg(P_k) \leq 2k$. Therefore, for each $k \geq 0$, we may extend $f^{(k)}$ continuously to 0 by the value 0, so f is a C^∞ function on \mathbb{R} . Therefore, the power series $\sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n$ is the zero function. Its radius of convergence is $+\infty$, and is not equal to f on $(0, r)$ for any $r > 0$.

Proposition 8.3.33 : If f can be written as a power series in $D(0, R)$ for some $R > 0$, then for any $z_0 \in D(0, R)$, f can also be written as a power series around z_0 .

Proof : Let f be a function, $R > 0$, and a power series $\sum a_n z^n$ such that

$$\forall z \in D(0, R), \quad f(z) = \sum_{n \geq 0} a_n z^n.$$

Let $z_0 \in D(0, R)$ and $r = R - |z_0|$. It is not hard to see that $D(0, r) \subseteq D(0, R)$. Let $z \in D(z_0, r)$, we write

$$\begin{aligned} \sum_{n \geq 0} a_n z^n &= \sum_{n \geq 0} a_n (z_0 + (z - z_0))^n = \sum_{n \geq 0} a_n \sum_{k=0}^n \binom{n}{k} z_0^{n-k} (z - z_0)^k \\ &= \sum_{n \geq 0} \sum_{k \geq 0} a_n \mathbb{1}_{n \geq k} \binom{n}{k} z_0^{n-k} (z - z_0)^k \end{aligned}$$

We may check that for every $n \geq 0$, the series $\sum_{k \geq 0} a_n \mathbb{1}_{n \geq k} \binom{n}{k} z_0^{n-k} (z - z_0)^k$ converges absolutely (finite series). Additionally, we have

$$\sum_{n \geq 0} \sum_{k \geq 0} |a_n| \mathbb{1}_{n \geq k} \binom{n}{k} |z_0|^{n-k} |z - z_0|^k = \sum_{n \geq 0} |a_n| (|z_0| + |z - z_0|)^n$$

which converges because $|z_0| + |z - z_0| < |z_0| + r = R$. Therefore, Theorem 6.7.4 allows us to interchange the order of summations. We find,

$$\sum_{n \geq 0} a_n z^n = \sum_{k \geq 0} \sum_{n \geq k} a_n \mathbb{1}_{n \geq k} \binom{n}{k} z_0^{n-k} (z - z_0)^k = \sum_{k \geq 0} \left(\sum_{n \geq k} a_n \binom{n}{k} z_0^{n-k} \right) (z - z_0)^k,$$

對於 $k \in \mathbb{N}_0$ ，我們可以計算 f 在 $(0, +\infty)$ 上的第 k 階微分：

$$\forall x > 0, \quad f^{(k)}(x) = P_k\left(\frac{1}{x}\right)e^{-1/x}, \quad (8.17)$$

其中 P_k 是個多項式，滿足 $\deg(P_k) \leq 2k$ 。因此，對於每個 $k \geq 0$ ，我們可以把 $f^{(k)}$ 以 0 的值連續延拓到 0，所以 f 在 \mathbb{R} 上是個 C^∞ 函數。所以，冪級數 $\sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n$ 是個零函數。他的收斂半徑是 $+\infty$ ，但對於任意 $r > 0$ ，在 $(0, r)$ 上不會與 f 相等。

命題 8.3.33 : 如果對於某個 $R > 0$ ，函數 f 在 $D(0, R)$ 上可以寫成冪級數，則對於任意 $z_0 \in D(0, R)$ ，函數 f 在 z_0 附近也可以寫成冪級數。

證明：令 f 為函數， $R > 0$ 以及冪級數 $\sum a_n z^n$ 滿足

$$\forall z \in D(0, R), \quad f(z) = \sum_{n \geq 0} a_n z^n.$$

令 $z_0 \in D(0, R)$ 以及 $r = R - |z_0|$ 。我們不難看出來 $D(0, r) \subseteq D(0, R)$ 。令 $z \in D(z_0, r)$ ，我們寫

$$\begin{aligned} \sum_{n \geq 0} a_n z^n &= \sum_{n \geq 0} a_n (z_0 + (z - z_0))^n = \sum_{n \geq 0} a_n \sum_{k=0}^n \binom{n}{k} z_0^{n-k} (z - z_0)^k \\ &= \sum_{n \geq 0} \sum_{k \geq 0} a_n \mathbb{1}_{n \geq k} \binom{n}{k} z_0^{n-k} (z - z_0)^k \end{aligned}$$

我們可以檢查對於每個 $n \geq 0$ ，級數 $\sum_{k \geq 0} a_n \mathbb{1}_{n \geq k} \binom{n}{k} z_0^{n-k} (z - z_0)^k$ 會絕對收斂（有限級數）。

此外，我們有

$$\sum_{n \geq 0} \sum_{k \geq 0} |a_n| \mathbb{1}_{n \geq k} \binom{n}{k} |z_0|^{n-k} |z - z_0|^k = \sum_{n \geq 0} |a_n| (|z_0| + |z - z_0|)^n$$

這會收斂，因為 $|z_0| + |z - z_0| < |z_0| + r = R$ 。因此，定理 6.7.4 允許我們交換取和的順序。我

which is a power series centered at z_0 . □

8.3.6 Applications to ODEs

Power series can be used to solve linear ordinary differential equations with polynomial coefficients. We have two cases.

- We know that the solution can be written as a power series, and we look for recurrence relations between coefficients of the power series. Then, the uniqueness of the coefficients (Corollary 8.3.24) allows us to find this unique solution. See Example 8.3.34.
- We do not know whether the solution can be written as a power series and want to show that there exists such a solution. We apply the same method as in the previous point, and show that the corresponding power series has a strictly positive radius of convergence. This gives us the unique solution that can be written as a power series, see Example 8.3.35. Note that this does not prove any result about the uniqueness of the solution.

Example 8.3.34 : We want to look for a power series expansion of the following function around 0,

$$\begin{aligned} f: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto e^{x^2} \int_0^x e^{-t^2} dt. \end{aligned}$$

The function f can be written as a power series centered at 0 with radius of convergence equal to $+\infty$, because it consists of multiplication and integration of such functions. Additionally, by the fundamental theorem of calculus, we have

$$\forall x \in \mathbb{R}, \quad f'(x) = 2xf(x) + 1, \quad \text{and} \quad f(0) = 0.$$

Suppose that $f(x) = \sum_{n \geq 0} a_n x^n$. Then, we have

$$\forall x \in \mathbb{R}, \quad f'(x) = \sum_{n \geq 1} n a_n x^{n-1}, \quad \text{and} \quad x f(x) = \sum_{n \geq 0} a_n x^{n+1} = \sum_{n \geq 2} a_{n-2} x^{n-1}.$$

們會得到

$$\sum_{n \geq 0} a_n z^n = \sum_{k \geq 0} \sum_{n \geq k} a_n \mathbb{1}_{n \geq k} \binom{n}{k} z_0^{n-k} (z - z_0)^k = \sum_{k \geq 0} \left(\sum_{n \geq k} a_n \binom{n}{k} z_0^{n-k} \right) (z - z_0)^k,$$

這就會是中心在 z_0 的冪級數。 □

第六小節 在 ODE 上的應用

我們可以拿冪級數來解係數為多項式的線性常微分方程。我們有兩種情況：

- 我們知道解可以寫成冪級數，因此我們可以尋找冪級數係數之間的遞迴關係。接著，係數的唯一性（系理 8.3.24）讓我們可以找到這個唯一解。見範例 8.3.34。
- 我們並不知道解是否可以寫成冪級數，不過想要證明存在一個這樣的解。如同前項，我們使用相同的方式，並證明所對應到的冪級數的收斂半徑嚴格為正。這給我們能夠寫成冪級數時的唯一解，見範例 8.3.35。注意到，這並沒有給我們任何關於解唯一性的結果。

範例 8.3.34 : 我們想要對下列函數，在 0 附近做冪級數的展開：

$$\begin{aligned} f: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto e^{x^2} \int_0^x e^{-t^2} dt. \end{aligned}$$

函數 f 可以寫成中心在 0 的冪級數，收斂半徑為 $+\infty$ ，因為他是這種類型函數的乘積還有積分。此外，根據微積分基本定理，我們有：

$$\forall x \in \mathbb{R}, \quad f'(x) = 2xf(x) + 1, \quad \text{以及} \quad f(0) = 0.$$

假設 $f(x) = \sum_{n \geq 0} a_n x^n$ 。那麼我們有

$$\forall x \in \mathbb{R}, \quad f'(x) = \sum_{n \geq 1} n a_n x^{n-1}, \quad \text{以及} \quad x f(x) = \sum_{n \geq 0} a_n x^{n+1} = \sum_{n \geq 2} a_{n-2} x^{n-1}.$$

Therefore,

$$\forall x \in \mathbb{R}, \quad f'(x) - 2xf(x) = a_1 + \sum_{n \geq 2} (na_n - 2a_{n-2})x^{n-1}.$$

The initial condition $f(0) = 0$ gives $a_0 = 0$. By Corollary 8.3.24, we know that

$$a_1 = 1, \quad \text{and} \quad \forall n \geq 2, \quad a_n = \frac{2}{n}a_{n-2}.$$

Thus, by induction, we find that

$$\forall n \geq 0, \quad a_{2n} = 0, \quad \text{and} \quad a_{2n+1} = \frac{4^n n!}{(2n+1)!}.$$

We check again (even though not necessary in this example) that the power series define by this sequence of $(a_n)_{n \geq 0}$ indeed has radius of convergence equal to $+\infty$, so

$$\forall x \in \mathbb{R}, \quad f(x) = \sum_{n \geq 0} \frac{4^n n!}{(2n+1)!} x^{2n+1}.$$

Note that this solution can also be expanded around every $a \in \mathbb{R}$ as a power series.

因此

$$\forall x \in \mathbb{R}, \quad f'(x) - 2xf(x) = a_1 + \sum_{n \geq 2} (na_n - 2a_{n-2})x^{n-1}.$$

初始條件 $f(0) = 0$ 給我們 $a_0 = 0$ 。根據系理 8.3.24，我們知道

$$a_1 = 1, \quad \text{以及} \quad \forall n \geq 2, \quad a_n = \frac{2}{n}a_{n-2}.$$

因此，透過歸納法，我們得到

$$\forall n \geq 0, \quad a_{2n} = 0, \quad \text{以及} \quad a_{2n+1} = \frac{4^n n!}{(2n+1)!}.$$

我們可以再次檢查（即使在這個範例中，這不是必須的），透過序列 $(a_n)_{n \geq 0}$ 定義出來的冪級數的收斂半徑會等於 $+\infty$ ，因此

$$\forall x \in \mathbb{R}, \quad f(x) = \sum_{n \geq 0} \frac{4^n n!}{(2n+1)!} x^{2n+1}.$$

注意到這個解也可以在任何 $a \in \mathbb{R}$ 附近展開成冪級數。

Example 8.3.35 : Let $\alpha \in \mathbb{C}$. We want to look for a power series expansion of the following function around 0,

$$\begin{aligned} f : (-1, 1) &\rightarrow \mathbb{C} \\ x &\mapsto (1+x)^\alpha. \end{aligned}$$

This function f satisfies the following first-order linear ordinary differential equation,

$$\forall x \in (-1, 1), \quad (1+x)f'(x) = \alpha f(x), \quad \text{and} \quad f(0) = 1.$$

Such a differential equation has a unique solution (Theorem 8.4.17). Suppose that $f(x) = \sum_{n \geq 0} a_n x^n$ with radius of convergence $R > 0$. Then, we have

$$\forall x \in (-R, R), \quad f'(x) = \sum_{n \geq 1} na_n x^{n-1} = \sum_{n \geq 0} (n+1)a_{n+1}x^n, \quad \text{and} \quad xf'(x) = \sum_{n \geq 1} na_n x^n.$$

範例 8.3.35 : 令 $\alpha \in \mathbb{C}$ 。我們想要找出下列函數在 0 附近的冪級數展開：

$$\begin{aligned} f : (-1, 1) &\rightarrow \mathbb{C} \\ x &\mapsto (1+x)^\alpha. \end{aligned}$$

這個函數 f 滿足下面這個一階線性長微分方程：

$$\forall x \in (-1, 1), \quad (1+x)f'(x) = \alpha f(x), \quad \text{以及} \quad f(0) = 1.$$

這樣的微分方程有唯一的解（定理 8.4.17）。假設 $f(x) = \sum_{n \geq 0} a_n x^n$ 的收斂半徑為 $R > 0$ 。那麼我們有

$$\forall x \in (-R, R), \quad f'(x) = \sum_{n \geq 1} na_n x^{n-1} = \sum_{n \geq 0} (n+1)a_{n+1}x^n, \quad \text{以及} \quad xf'(x) = \sum_{n \geq 1} na_n x^n.$$

Therefore,

$$\forall x \in (-R, R), \quad (1+x)f'(x) - \alpha f(x) = \sum_{n \geq 0} ((n+1)a_{n+1} + na_n - \alpha a_n)x^n.$$

From the initial condition $f(0) = 1$, we have $a_0 = 1$. By the uniqueness of the coefficients (Corollary 8.3.24), we find

$$\forall n \in \mathbb{N}_0, \quad a_{n+1} = \frac{\alpha - n}{n+1} a_n.$$

By induction, we deduce that

$$\forall n \in \mathbb{N}_0, \quad a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} = \binom{\alpha}{n}. \quad (8.18)$$

By d'Alembert's criterion, we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\alpha - n}{n+1} \right| \xrightarrow{n \rightarrow \infty} 1$$

Therefore, the power series $\sum a_n x^n$ defined by the coefficients in Eq. (8.18) has radius of convergence equal to 1, and we conclude that

$$\forall x \in (-1, 1), \quad (1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n = \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n.$$

This generalizes the binomial expansion to the case with a complex-valued exponent.

因此，

$$\forall x \in (-R, R), \quad (1+x)f'(x) - \alpha f(x) = \sum_{n \geq 0} ((n+1)a_{n+1} + na_n - \alpha a_n)x^n.$$

根據初始條件 $f(0) = 1$ ，我們得到 $a_0 = 1$ 。根據係數的唯一性（系理 8.3.24），我們得到

$$\forall n \in \mathbb{N}_0, \quad a_{n+1} = \frac{\alpha - n}{n+1} a_n.$$

使用數學歸納法，我們推得

$$\forall n \in \mathbb{N}_0, \quad a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} = \binom{\alpha}{n}. \quad (8.18)$$

根據 d'Alembert 檢測法，我們得到

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\alpha - n}{n+1} \right| \xrightarrow{n \rightarrow \infty} 1$$

因此，透過式 (8.18) 當中係數所定義的冪級數 $\sum a_n x^n$ 收斂半徑為 1，所以我們得到結論：

$$\forall x \in (-1, 1), \quad (1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n = \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n.$$

這個式子把二項式展開推廣到指數為複數的情形。

8.4 Advanced theorems on uniform convergence

8.4.1 Arzelà-Ascoli theorem

Arzelà-Ascoli theorem is an important theorem in functional analysis, and it allows us to characterize when a subset of continuous functions is compact. In particular, it turns out to be useful to show the existence of solution for some differential equations, see Theorem 8.4.14. First, let us introduce the notion of *equicontinuity*.

第四節 均勻收斂的進階定理

第一小節 Arzelà-Ascoli 定理

Arzelà-Ascoli 定理是個泛函分析中的重要定理，他讓我們可以刻劃什麼時候連續函數的子集合會是緊緻的。這個是可以拿來證明某些微分方程的解有存在性，見定理 8.4.14。首先，讓我們引入等度連續的概念。

Definition 8.4.1 : Let (K, d) be a metric space. In addition, if K is a compact space, the space of continuous functions $\mathcal{C}(K, \mathbb{R})$ is a subset of $\mathcal{B}(K, \mathbb{R})$. We have equipped $\mathcal{B}(K, \mathbb{R})$ with the supremum norm in Definition 8.1.9, which we may induce on the subspace $\mathcal{C}(K, \mathbb{R})$. A subset $\mathcal{F} \subseteq \mathcal{C}(K, \mathbb{R})$ is said to be *equicontinuous* (等度連續) if

$$\forall \varepsilon > 0, \forall x \in M, \exists \delta > 0, \forall f \in \mathcal{F}, \quad y \in B(x, \delta) \Rightarrow |f(x) - f(y)| < \varepsilon. \quad (8.19)$$

Remark 8.4.2 : We note that the definition in Eq. (8.19) is much stronger than just requiring that all the functions $f \in \mathcal{F}$ are continuous. Once $\varepsilon > 0$ and $x \in M$ are fixed, this condition needs the choice of $\delta > 0$ to be *uniform* in $f \in \mathcal{F}$.

Example 8.4.3 :

- (1) A subset of finitely many continuous functions is equicontinuous.
- (2) For every $L > 0$, the set of all the L -Lipschitz continuous functions is equicontinuous.

Theorem 8.4.4 (Arzelà–Ascoli theorem) : Let (K, d) be a compact metric space and $\mathcal{F} \subseteq \mathcal{C}(K, \mathbb{R})$ be a subset. Then, we have the following properties.

- (1) \mathcal{F} is compact if and only if \mathcal{F} is bounded, closed, and equicontinuous.
- (2) \mathcal{F} is precompact if and only if \mathcal{F} is bounded and equicontinuous.

Remark 8.4.5 :

- (1) We recall that a compact space is necessarily bounded and closed (Proposition 3.1.6), and a bounded and closed set may not be compact (Remark 3.1.34), except that we are in a finite-dimensional normed vector space (Corollary 3.2.24). If the compact metric space K is consisted of a finite number of points, it is clear that $\mathcal{C}(K, \mathbb{R})$ is isomorphic to \mathbb{R}^n for $n = \text{Card}(K)$, which is a finite-dimensional normed vector space, and the theorem becomes trivial. However, for a generic compact metric space K , the

定義 8.4.1 : 令 (K, d) 為賦距空間。此外，如果 K 是個緊緻空間，連續函數所構成的空間 $\mathcal{C}(K, \mathbb{R})$ 是個 $\mathcal{B}(K, \mathbb{R})$ 的子集合。在定義 8.1.9 中，我們賦予 $\mathcal{B}(K, \mathbb{R})$ 最小上界範數，這也可以被引導在子空間 $\mathcal{C}(K, \mathbb{R})$ 上。給定子集合 $\mathcal{F} \subseteq \mathcal{C}(K, \mathbb{R})$ 。如果下列成立：

$$\forall \varepsilon > 0, \forall x \in M, \exists \delta > 0, \forall f \in \mathcal{F}, \quad y \in B(x, \delta) \Rightarrow |f(x) - f(y)| < \varepsilon, \quad (8.19)$$

則我們說 \mathcal{F} 是等度連續 (equicontinuous) 的。

註解 8.4.2 : 我們注意到，式 (8.19) 中的定義比要求所有 $f \in \mathcal{F}$ 都要連續來得強。只要固定了 $\varepsilon > 0$ 和 $x \in M$ ，這個條件要求的是選擇 $\delta > 0$ 使得他對 $f \in \mathcal{F}$ 來說是均勻的。

範例 8.4.3 :

- (1) 由有限多個連續函數構成的子集合是等度連續的。
- (2) 對於每個 $L > 0$ ，由所有 L -Lipschitz 連續函數構成的集合是等度連續的。

定理 8.4.4 【Arzelà–Ascoli 定理】：令 (K, d) 為緊緻賦距空間，以及 $\mathcal{F} \subseteq \mathcal{C}(K, \mathbb{R})$ 為子集合。那麼我們有下列性質。

- (1) 若且唯若 \mathcal{F} 是有界、閉集，且等度連續的，則 \mathcal{F} 是緊緻的。
- (2) 若且唯若 \mathcal{F} 是有界且等度連續的，則 \mathcal{F} 是預緊緻的。

註解 8.4.5 :

- (1) 我們不要忘記，緊緻空間一定會是個有界閉集（命題 3.1.6），但給定一個有界閉集，他不一定是緊緻的（註解 3.1.34），除非說我們在一個有限維度的賦範向量空間中（系理 3.2.24）。如果緊緻賦距空間 K 中只有有限多個點，我們記 $n = \text{Card}(K)$ ，那麼 $\mathcal{C}(K, \mathbb{R})$ 顯然會和 \mathbb{R}^n 同構，所以是個有限維度賦範向量空間，這個定理就變得顯然。然而，對於一般的賦距空間 K 來

space of continuous functions $\mathcal{C}(K, \mathbb{R})$ is not of finite-dimensional.

- (2) From Exercise 3.21, we know that a metric space is compact if and only if it is precompact and complete. Moreover, in Exercise 8.30, we can check that if \mathcal{F} is equicontinuous, then so is $\overline{\mathcal{F}}$. Moreover, since $\mathcal{C}(K, \mathbb{R})$ is a Banach space, we see that (2) is a direct consequence of (1).
- (3) We also note that \mathbb{R} can be replaced by any Banach space, and the following proof can be adapted accordingly.

Proof :

- Suppose that \mathcal{F} is compact. We already know that it is bounded and closed, so we only need to show that it is equicontinuous. A compact set is also relatively compact (or precompact), see Lemma 3.1.22. Let $\varepsilon > 0$. We may find $N \geq 1$ and $f_1, \dots, f_N \in \mathcal{F}$ such that $\mathcal{F} \subseteq \bigcup_{i=1}^N B(f_i, \varepsilon)$. Additionally, the finite set of functions $\{f_1, \dots, f_N\}$ is equicontinuous.

Let $x \in M$. We may find $\delta > 0$ such that

$$\forall i = 1, \dots, N, \quad y \in B(x, \delta) \quad \Rightarrow \quad |f_i(x) - f_i(y)| \leq \varepsilon.$$

For any given $f \in \mathcal{F}$, we may find $1 \leq i \leq N$ such that $f \in B(f_i, \varepsilon)$. Then, for any $y \in B(x, \delta)$, we have

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \leq 3\varepsilon.$$

This allows us to conclude that \mathcal{F} is equicontinuous.

- Suppose that \mathcal{F} is bounded, closed, and equicontinuous. In order to show that \mathcal{F} is compact, it is sufficient to show that it satisfies the Bolzano–Weierstraß property (Definition 3.1.19), see Theorem 3.1.20.

Let $(f_n)_{n \geq 1}$ be a sequence in \mathcal{F} . Since K is compact, we may find a dense sequence in K , that we denote by $(x_n)_{n \geq 1}$ ². We are going to use a diagonal argument to extract a subsequence of $(f_n)_{n \geq 1}$ which converges at every x_k for $k \geq 1$.

- The sequence $(f_n(x_1))_{n \geq 1}$ is bounded in \mathbb{R} , so by the Bolzano–Weierstraß theorem (Theorem 2.2.5), we may find a convergent subsequence, that we denote by $(f_{\varphi_1(n)}(x_1))_{n \geq 1}$,

說，連續函數構成的空間 $\mathcal{C}(K, \mathbb{R})$ 不是有限維度的。

- (2) 從習題 3.21，我們知道給定一個賦距空間，若且唯若他是預緊緻且是完備的，則他是緊緻的。此外，在習題 8.30 中，我們可以檢查如果 \mathcal{F} 是等度連續的，那麼 $\overline{\mathcal{F}}$ 也是。還有，由於 $\mathcal{C}(K, \mathbb{R})$ 是個 Banach 空間，我們可以看出來 (2) 是可以從 (1) 所得到的直接結果。
- (3) 我們也注意到，可以把 \mathbb{R} 替換成任意的 Banach 空間，而證明也可以在這個情況下被推廣。

證明：

- 假設 \mathcal{F} 是緊緻的。由於我們已經知道他是個有界閉集，我們只需要證明他是等度連續的。緊緻集合也是相對緊緻（預緊緻）的，見引理 3.1.22。令 $\varepsilon > 0$ 。我們能找到 $N \geq 1$ 以及 $f_1, \dots, f_N \in \mathcal{F}$ 使得 $\mathcal{F} \subseteq \bigcup_{i=1}^N B(f_i, \varepsilon)$ 。此外，由有限多個函數構成的集合 $\{f_1, \dots, f_N\}$ 是等度連續的。

令 $x \in M$ 。我們能找到 $\delta > 0$ 使得

$$\forall i = 1, \dots, N, \quad y \in B(x, \delta) \quad \Rightarrow \quad |f_i(x) - f_i(y)| \leq \varepsilon.$$

對於任意給定的 $f \in \mathcal{F}$ ，我們能找到 $1 \leq i \leq N$ 滿足 $f \in B(f_i, \varepsilon)$ 。那麼，對於任意 $y \in B(x, \delta)$ ，我們有

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \leq 3\varepsilon.$$

這讓我們可以總結 \mathcal{F} 是等度連續的。

- 假設 \mathcal{F} 是有界閉集，且是等度連續的。如果要證明 \mathcal{F} 是緊緻的，我們只需要證明他會滿足 Bolzano–Weierstraß 性質（定義 3.1.19）即可，見定理 3.1.20。

令 $(f_n)_{n \geq 1}$ 為在 \mathcal{F} 裡面的序列。由於 K 是緊緻的，我們能找到在 K 中稠密的序列，我們把他記作 $(x_n)_{n \geq 1}$ ²。我們會使用對角論證法來萃取 $(f_n)_{n \geq 1}$ 的子序列，使得他對於每個 $k \geq 1$ 都會在 x_k 收斂。

- 序列 $(f_n(x_1))_{n \geq 1}$ 在 \mathbb{R} 中有界，所以根據 Bolzano–Weierstraß 定理（定理 2.2.5），我們能夠找到收斂子序列，記作 $(f_{\varphi_1(n)}(x_1))_{n \geq 1}$ ，其中 $\varphi_1 : \mathbb{N} \rightarrow \mathbb{N}$ 是個萃取函數。

where $\varphi_1 : \mathbb{N} \rightarrow \mathbb{N}$ is an extraction.

- Let $m \geq 1$. Suppose that we have already constructed extractions $\varphi_1, \dots, \varphi_m$ such that $(f_{\psi_m(n)}(x_k))_{n \geq 1}$ converges for all $1 \leq k \leq m$, where $\psi_m := \varphi_1 \circ \dots \circ \varphi_m$. Then, the sequence $(f_{\psi_m(n)}(x_{m+1}))_{n \geq 1}$ is bounded, so we may find an extraction $\varphi_{m+1} : \mathbb{N} \rightarrow \mathbb{N}$ such that $(f_{\psi_m \circ \varphi_{m+1}(n)}(x_{m+1}))_{n \geq 1}$ converges. It is clear that for $1 \leq k \leq m$, the sequence $(f_{\psi_m \circ \varphi_{m+1}(n)}(x_k))_{n \geq 1}$ still converges, being a subsequence of a convergent sequence.
- For $n \geq 1$, let $\psi(n) := \varphi_1 \circ \dots \circ \varphi_n(n)$ and $g_n = f_{\psi(n)}$. Then, $(g_n)_{n \geq 1}$ is a subsequence of $(f_n)_{n \geq 1}$. From above, for every $k \geq 1$, the sequence $(g_n(x_k) = f_{\psi(n)}(x_k))_{n \geq k}$ is a subsequence of the convergent sequence $(f_{\psi_k(n)}(x_k))_{n \geq 1}$, so the sequence $(g_n(x_k))_{n \geq 1}$ converges. We may denote by $f(x_k)$ for the above limit for every $k \geq 1$.

Now, we need to show that this convergence can be extended to every $x \in K$, and that this convergence is uniform, so the limit is still in $\mathcal{C}(K, \mathbb{R})$.

Let us fix $\varepsilon > 0$.

- For every $k \geq 1$, from the convergence of the sequence $(g_n(x_k))_{n \geq 1}$, we may find $N(\varepsilon, x_k) \geq 1$ such that

$$\forall m, n \geq N(\varepsilon, x_k), \quad |g_m(x_k) - g_n(x_k)| \leq \varepsilon. \quad (8.20)$$

- By the equicontinuity of \mathcal{F} , for every $z \in K$, we may find $\delta_z > 0$ such that for every $n \geq 1$, we have

$$y \in B(z, \delta_z) \Rightarrow |g_n(z) - g_n(y)| \leq \varepsilon. \quad (8.21)$$

The open balls $B(z, \delta_z)$ form an open covering of K , and by the compactness of K , we may find $L \geq 1$ and $z_1, \dots, z_L \in K$ such that

$$K = \bigcup_{i=1}^L B(z_i, \delta_{z_i}).$$

For every $1 \leq i \leq L$, we may also find $n_i \geq 1$ such that $x_{n_i} \in B(z_i, \delta_{z_i})$.

- We may take $N := \max\{N(\varepsilon, x_{n_1}), \dots, N(\varepsilon, x_{n_L})\}$. This implies that we have a uniform

- 令 $m \geq 1$ 。假設我們已經構造好了萃取函數 $\varphi_1, \dots, \varphi_m$ 使得對於每個 $1 \leq k \leq m$ ，以及 $\psi_m := \varphi_1 \circ \dots \circ \varphi_m$ ，序列 $(f_{\psi_m(n)}(x_k))_{n \geq 1}$ 皆會收斂。那麼，序列 $(f_{\psi_m(n)}(x_{m+1}))_{n \geq 1}$ 是有界的，所以我們能找到萃取函數 $\varphi_{m+1} : \mathbb{N} \rightarrow \mathbb{N}$ 使得 $(f_{\psi_m \circ \varphi_{m+1}(n)}(x_{m+1}))_{n \geq 1}$ 會收斂。我們可以顯然得到，對於 $1 \leq k \leq m$ ，序列 $(f_{\psi_m \circ \varphi_{m+1}(n)}(x_k))_{n \geq 1}$ 還是會收斂，因為他是收斂序列的一個子序列。
- 對於 $n \geq 1$ ，令 $\psi(n) := \varphi_1 \circ \dots \circ \varphi_n(n)$ 以及 $g_n = f_{\psi(n)}$ 。那麼， $(g_n)_{n \geq 1}$ 是個 $(f_n)_{n \geq 1}$ 的子序列。從上面我們得到，對於每個 $k \geq 1$ ，序列 $(g_n(x_k) = f_{\psi(n)}(x_k))_{n \geq k}$ 是收斂序列 $(f_{\psi_k(n)}(x_k))_{n \geq 1}$ 的子序列，所以序列 $(g_n(x_k))_{n \geq 1}$ 會收斂。對於每個 $k \geq 1$ ，我們可以把上面極限記作 $f(x_k)$ 。

現在，我們需要證明這個收斂可以推廣到所有的 $x \in K$ ，而且這個收斂會是均勻的，所以極限還會是在 $\mathcal{C}(K, \mathbb{R})$ 當中。

讓我們固定 $\varepsilon > 0$ 。

- 對於每個 $k \geq 1$ ，從序列 $(g_n(x_k))_{n \geq 1}$ 的收斂，我們能找到 $N(\varepsilon, x_k) \geq 1$ 使得

$$\forall m, n \geq N(\varepsilon, x_k), \quad |g_m(x_k) - g_n(x_k)| \leq \varepsilon. \quad (8.20)$$

- 使用 \mathcal{F} 的等度連續性，對於每個 $z \in K$ ，我們能找到 $\delta_z > 0$ 使得對於每個 $n \geq 1$ ，我們有

$$y \in B(z, \delta_z) \Rightarrow |g_n(z) - g_n(y)| \leq \varepsilon. \quad (8.21)$$

開球 $B(z, \delta_z)$ 構成 K 的開覆蓋，再使用 K 的緊緻性，我們能找到 $L \geq 1$ 還有 $z_1, \dots, z_L \in K$ 滿足

$$K = \bigcup_{i=1}^L B(z_i, \delta_{z_i}).$$

對於每個 $1 \leq i \leq L$ ，我們也能找到 $n_i \geq 1$ 使得 $x_{n_i} \in B(z_i, \delta_{z_i})$ 。

- 我們取 $N := \max\{N(\varepsilon, x_{n_1}), \dots, N(\varepsilon, x_{n_L})\}$ 。這告訴我們均勻柯西條件（命

²We use the precompactness of K . For every $n \geq 1$, we may find finitely many balls with radius $\frac{1}{n}$ that cover K . The union of the centers of these balls over all the integers $n \geq 1$ is a countable dense set in K .

²我們使用 K 的預緊緻性。對於每個 $n \geq 1$ ，我們能找到有限多顆半徑為 $\frac{1}{n}$ 的球把 K 覆蓋住。我們把這些球的球心構成的集合，對於所有整數 $n \geq 1$ 聯集起來，我們得到的是 K 中可數稠密的集合。

Cauchy condition (Proposition 8.1.8) on x_{n_1}, \dots, x_{n_L} ,

$$\forall i = 1, \dots, L, \forall m, n \geq N, \quad |g_m(x_{n_i}) - g_n(x_{n_i})| \leq \varepsilon.$$

– Let $x \in K$ and $1 \leq i \leq L$ such that $x \in B(z_i, \delta_{z_i})$. For $m, n \geq N$, we have

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq |g_m(x) - g_m(z_i)| + |g_m(z_i) - g_m(x_{n_i})| + |g_m(x_{n_i}) - g_n(x_{n_i})| \\ &\quad + |g_n(x_{n_i}) - g_n(z_i)| + |g_n(z_i) - g_n(x)| \\ &\leq 5\varepsilon, \end{aligned}$$

where for the middle (third) term, we use Eq. (8.20); and for the other terms, we use Eq. (8.21) and the fact that $x, x_{n_i} \in B(z_i, \delta_{z_i})$.

Therefore, for every $x \in K$, the sequence $(g_n(x))_{n \geq 1}$ is Cauchy, and we saw from above that the choice of N is independent from the choice of $x \in K$. From this we can deduce that $(g_n(x))_{n \geq 1}$ converges for every $x \in K$, and this convergence is uniform, so the limit function is still an element of $\mathcal{C}(K, \mathbb{R})$. \square

8.4.2 Stone–Weierstraß theorem

The following Stone–Weierstraß theorem allows us to find sets of functions that can approximate continuous functions uniformly on compact spaces.

Theorem 8.4.6 (Stone–Weierstraß theorem): *Let X be a compact metric space and $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . The space of continuous functions $\mathcal{C}(X, \mathbb{K})$ equipped with the supremum norm $\|\cdot\|_\infty$ is a normed vector space and a normed algebra. Let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{K})$ be a subalgebra of $\mathcal{C}(X, \mathbb{K})$. Suppose that*

- $1 \in \mathcal{A}$;
- \mathcal{A} separates points, that is for any $x \neq y \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$;
- (in the case $\mathbb{K} = \mathbb{C}$) $f \in \mathcal{A}$ if and only if $\bar{f} \in \mathcal{A}$.

Then, \mathcal{A} is dense in $\mathcal{C}(X, \mathbb{K})$.

題 8.1.8) 對於 x_{n_1}, \dots, x_{n_L} 會成立：

$$\forall i = 1, \dots, L, \forall m, n \geq N, \quad |g_m(x_{n_i}) - g_n(x_{n_i})| \leq \varepsilon.$$

– 令 $x \in K$ 以及 $1 \leq i \leq L$ 使得 $x \in B(z_i, \delta_{z_i})$ 。對於 $m, n \geq N$ ，我們有

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq |g_m(x) - g_m(z_i)| + |g_m(z_i) - g_m(x_{n_i})| + |g_m(x_{n_i}) - g_n(x_{n_i})| \\ &\quad + |g_n(x_{n_i}) - g_n(z_i)| + |g_n(z_i) - g_n(x)| \\ &\leq 5\varepsilon, \end{aligned}$$

其中我們對中間的項（第三項）使用了式 (8.20)；對於其他的項，我們使用了式 (8.21)

還有 $x, x_{n_i} \in B(z_i, \delta_{z_i})$ 。

因此，對於每個 $x \in K$ ，序列 $(g_n(x))_{n \geq 1}$ 是柯西的，且我們從前面看到， N 的選擇與 $x \in K$ 的選擇無關。所以我們可以總結，對於每個 $x \in K$ ， $(g_n(x))_{n \geq 1}$ 會收斂，而且這個收斂是均勻的，所以極限函數還是會在 $\mathcal{C}(K, \mathbb{R})$ 當中。 \square

第二小節 Stone–Weierstraß 定理

接下來的 Stone–Weierstraß 定理讓我們知道什麼樣的函數集合，能夠近似定義在緊緻集合上面的連續函數。

定理 8.4.6 【Stone–Weierstraß 定理】：令 X 為緊緻賦距空間以及 $\mathbb{K} = \mathbb{R}$ 或 \mathbb{C} 。連續函數所構成的空間 $\mathcal{C}(X, \mathbb{K})$ 賦予最小上界範數 $\|\cdot\|_\infty$ 後，會是個賦範向量空間，也會是個賦範代數。令 $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{K})$ 為 $\mathcal{C}(X, \mathbb{K})$ 的子代數。假設

- $1 \in \mathcal{A}$ ；
- \mathcal{A} 會分離點，也就是說對於任意 $x \neq y \in X$ ，存在 $f \in \mathcal{A}$ 滿足 $f(x) \neq f(y)$ ；
- 【當 $\mathbb{K} = \mathbb{C}$ 時】 $f \in \mathcal{A}$ 若且唯若 $\bar{f} \in \mathcal{A}$ 。

Example 8.4.7 : Below are some examples for which the Stone–Weierstraß theorem applies.

- (1) Let $I = [a, b]$ be a segment with $\mathbb{K} = \mathbb{R}$. The set of polynomials $\mathbb{K}[X]$ viewed as functions defined on I is dense in $\mathcal{C}(I, \mathbb{R})$.
- (2) Let $I = [a, b]$ be a segment with $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . The set of all the Lipschitz continuous functions is dense in $\mathcal{C}(I, \mathbb{K})$.
- (3) Let $\mathcal{C}_{\text{per}}(\mathbb{R}, \mathbb{C})$ be the set of 2π -periodic continuous functions on \mathbb{R} . The set of trigonometric functions, which is spanned by the set $\{x \mapsto e^{inx} : n \in \mathbb{Z}\}$, is dense in $\mathcal{C}_{\text{per}}(\mathbb{R}, \mathbb{C})$.

The proof of the Stone–Weierstraß theorem is quite involved. We are going to state a particular example of this theorem, called *Weierstraß approximation theorem*, and prove it using a more elementary approach. After this, we need a few lemmas (Lemma 8.4.11 and Lemma 8.4.12) that allow us to prove the Stone–Weierstraß theorem.

Theorem 8.4.8 (Weierstraß approximation theorem) : Let $I = [a, b]$ be a segment and $\mathcal{C}(I, \mathbb{R})$ be equipped with the supremum norm $\|\cdot\|_{\infty}$. Let \mathcal{P} be the set of all polynomial functions. Then, \mathcal{P} is dense in $\mathcal{C}(I, \mathbb{R})$. In other words, for any $f \in \mathcal{C}(I, \mathbb{R})$, we may find a sequence of polynomials $(P_n)_{n \geq 1}$ such that

$$\|P_n - f\|_{\infty} \xrightarrow{n \rightarrow \infty} 0.$$

Remark 8.4.9 :

- (1) It is not hard to check that the set of all polynomials \mathcal{P} is a subalgebra of $\mathcal{C}(I, \mathbb{R})$ and it satisfies the conditions in Theorem 8.4.6. Thus, the Weierstraß approximation theorem can be seen as a special case of the Stone–Weierstraß theorem.
- (2) It is important to take $I = [a, b]$ to be a segment. For example, in Exercise 8.6 we have seen that this theorem does not hold if $I = \mathbb{R}$.

那麼， \mathcal{A} 在 $\mathcal{C}(X, \mathbb{K})$ 裡面會是稠密的。

範例 8.4.7 : 下面我們給一些可以使用 Stone–Weierstraß 定理的例子。

- (1) 令 $I = [a, b]$ 為線段且 $\mathbb{K} = \mathbb{R}$ 。當我們把多項式集合 $\mathbb{K}[X]$ 看作定義在 I 上的函數時，他在 $\mathcal{C}(I, \mathbb{R})$ 中會是稠密的。
- (2) 令 $I = [a, b]$ 為線段且 $\mathbb{K} = \mathbb{R}$ 或 \mathbb{C} 。由所有 Lipschitz 連續函數所構成的集合在 $\mathcal{C}(I, \mathbb{K})$ 中會是稠密的。
- (3) 令 $\mathcal{C}_{\text{per}}(\mathbb{R}, \mathbb{C})$ 為由 \mathbb{R} 上週期為 2π 的連續函數所構成的空間。由 $\{x \mapsto e^{inx} : n \in \mathbb{Z}\}$ 線性展開所得到的集合，或是稱作三角函數集合，在 $\mathcal{C}_{\text{per}}(\mathbb{R}, \mathbb{C})$ 中會是稠密的。

Stone–Weierstraß 定理的證明不簡單。我們會先敘述這個定理的特例，稱作 *Weierstraß 近似定理*，並用比較基礎的方法來證明他。在這個之後，我們會需要幾個引理（引理 8.4.11 和引理 8.4.12），然後我們就可以證明 Stone–Weierstraß 定理了。

定理 8.4.8 【Weierstraß 近似定理】 : 令 $I = [a, b]$ 為線段，以及 $\mathcal{C}(I, \mathbb{R})$ 賦予最小上界範數 $\|\cdot\|_{\infty}$ 。令 \mathcal{P} 為由所有多項式函數所構成的集合。那麼， \mathcal{P} 在 $\mathcal{C}(I, \mathbb{R})$ 裡面是稠密的。換句話說，對於任意 $f \in \mathcal{C}(I, \mathbb{R})$ ，我們可以找到多項式序列 $(P_n)_{n \geq 1}$ 使得

$$\|P_n - f\|_{\infty} \xrightarrow{n \rightarrow \infty} 0.$$

註解 8.4.9 :

- (1) 我們不難看出來，由所有多項式所構成的集合 \mathcal{P} 是個 $\mathcal{C}(I, \mathbb{R})$ 的子代數，而且他滿足定理 8.4.6 中的條件。所以，Weierstraß 近似定理可以視為是 Stone–Weierstraß 定理的特例。
- (2) 取 $I = [a, b]$ 為線段是很重要的。例如，在習題 8.6 中，我們有看到這個定理對 $I = \mathbb{R}$ 不會成立。

The original proof from Weierstraß uses *convolution*, that we do not discuss in this class. The proof we give below is from Bernstein, which can be reformulated using a probabilistic language, in terms of the law of large numbers for Bernoulli random variables.

Proof : Without loss of generality, we may assume that $I = [0, 1]$. For every integer $0 \leq k \leq n$, let us define

$$\begin{aligned} b_{n,k} : I &\rightarrow \mathbb{R} \\ x &\mapsto \binom{n}{k} x^k (1-x)^{n-k}, \end{aligned}$$

and for $n \in \mathbb{N}_0$, define

$$\begin{aligned} B_n : \mathcal{C}(I, \mathbb{R}) &\rightarrow \mathbb{R}[x] \\ f &\mapsto \sum_{k=0}^n f\left(\frac{k}{n}\right) b_{n,k}(x). \end{aligned}$$

We are going to show that $B_n(f)$ converges to f uniformly.

Given $\varepsilon > 0$. Since f is continuous on the segment I , it is bounded. Let us take $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$. By the Heine–Cantor theorem (Theorem 3.1.17), we may find $\eta > 0$ such that

$$\forall x, y \in I, \quad |x - y| < \eta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Then, for any $n \in \mathbb{N}_0$ and $x \in I$, we have

$$\begin{aligned} |B_n(f)(x) - f(x)| &= |B_n(f)(x) - f(x)B_n(1)| \leq \sum_{k=0}^n \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) \\ &\leq \sum_{k \in K_1} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) + \sum_{k \in K_2} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x), \end{aligned}$$

where

$$K_1 = \left\{ 0 \leq k \leq n : \left| \frac{k}{n} - x \right| \geq \eta \right\}, \quad \text{and} \quad K_2 = \left\{ 0 \leq k \leq n : \left| \frac{k}{n} - x \right| < \eta \right\}.$$

Using the uniform continuity, the second sum involving indices in K_2 can be bounded from above,

$$\sum_{k \in K_2} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) \leq \sum_{k \in K_2} \varepsilon b_{n,k}(x) \leq \sum_{k=0}^n \varepsilon b_{n,k}(x) = \varepsilon.$$

Weierstraß 原始的證明會使用捲積，但我們這堂課不會討論這樣的概念。下面我們要給的定理證明來自 Bernstein，這也可以用機率的語言來描述，並看作是對應到 Bernoulli 隨機變數的大數法則。

證明：不失一般性，我們可以假設 $I = [0, 1]$ 。對於每個整數 $0 \leq k \leq n$ ，讓我們定義

$$\begin{aligned} b_{n,k} : I &\rightarrow \mathbb{R} \\ x &\mapsto \binom{n}{k} x^k (1-x)^{n-k}, \end{aligned}$$

還有對於 $n \in \mathbb{N}_0$ ，定義

$$\begin{aligned} B_n : \mathcal{C}(I, \mathbb{R}) &\rightarrow \mathbb{R}[x] \\ f &\mapsto \sum_{k=0}^n f\left(\frac{k}{n}\right) b_{n,k}(x). \end{aligned}$$

我們再來要證明 $B_n(f)$ 會均勻收斂到 f 。

給定 $\varepsilon > 0$ 。由於 f 在線段 I 上連續，他會有界。讓我們取 $M > 0$ 滿足 $|f(x)| \leq M$ 對於所有 $x \in I$ 。根據 Heine–Cantor 定理（定理 3.1.17），我們能找到 $\eta > 0$ 使得

$$\forall x, y \in I, \quad |x - y| < \eta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$

那麼，對於任意 $n \in \mathbb{N}_0$ 還有 $x \in I$ ，我們有

$$\begin{aligned} |B_n(f)(x) - f(x)| &= |B_n(f)(x) - f(x)B_n(1)| \leq \sum_{k=0}^n \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) \\ &\leq \sum_{k \in K_1} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) + \sum_{k \in K_2} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x), \end{aligned}$$

其中

$$K_1 = \left\{ 0 \leq k \leq n : \left| \frac{k}{n} - x \right| \geq \eta \right\}, \quad \text{以及} \quad K_2 = \left\{ 0 \leq k \leq n : \left| \frac{k}{n} - x \right| < \eta \right\}.$$

使用均勻連續性，第二個和當中的項是下標在 K_2 中的情況，我們可以得到上界：

$$\sum_{k \in K_2} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) \leq \sum_{k \in K_2} \varepsilon b_{n,k}(x) \leq \sum_{k=0}^n \varepsilon b_{n,k}(x) = \varepsilon.$$

For the sum involving indices in K_1 , we are going to use the following square trick,

$$\begin{aligned} \sum_{k \in K_1} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) &\leq 2M \sum_{k \in K_1} b_{n,k}(x) \leq \frac{2M}{\eta^2} \sum_{k \in K_1} \left(\frac{k}{n} - x\right)^2 b_{n,k}(x) \\ &\leq \frac{2M}{\eta^2} \sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 b_{n,k}(x) \\ &= \frac{2M}{\eta^2} [B_n(x^2) - 2xB_n(x) + x^2B_n(1)]. \end{aligned}$$

Consider the following identity,

$$F(a, b) = [a + (1 - b)]^n = \sum_{k=0}^n \binom{n}{k} a^k (1 - b)^{n-k}.$$

Then, we may compute $B_n(1)$, $B_n(x)$, and $B_n(x^2)$ as follow,

$$\begin{aligned} B_n(1) &= \sum_{k=0}^n b_{n,k}(x) = F(x, x) = 1, \\ B_n(x) &= \sum_{k=0}^n \frac{k}{n} b_{n,k}(x) = \frac{x}{n} \sum_{k=1}^n k \binom{n}{k} x^{k-1} (1-x)^{n-k} \\ &= \frac{x}{n} \frac{\partial}{\partial a} F(x, x) = \frac{x}{n} n [x + (1-x)]^{n-1} = x, \\ B_n(x^2) &= \sum_{k=0}^n \left(\frac{k}{n}\right)^2 b_{n,k}(x) = \sum_{k=0}^n \left(\frac{k(k-1)}{n^2} + \frac{k}{n^2}\right) b_{n,k}(x) \\ &= \frac{x^2}{n^2} \frac{\partial^2}{\partial a^2} F(x, x) + \frac{x}{n^2} \frac{\partial}{\partial a} F(x, x) \\ &= \frac{x^2}{n^2} [n(n-1)(x + (1-x))^{n-2}] + \frac{x}{n^2} n [x + (1-x)]^{n-1} \\ &= x^2 + \frac{x(1-x)}{n}. \end{aligned}$$

Therefore, we find

$$\sum_{k \in K_1} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) \leq \frac{2M}{\eta^2} \frac{x(1-x)}{n} \leq \frac{M}{2n\eta^2}.$$

Putting all the inequalities together, we obtain

$$|B_n(f)(x) - f(x)| \leq \varepsilon + \frac{M}{2n\eta^2}.$$

By taking the supremum norm then \limsup over n , we find

$$\limsup_{n \rightarrow \infty} \|B_n(f) - f\|_{\infty} \leq \varepsilon.$$

對於下標在 K_1 中的項，我們使用下面的平方技巧：

$$\begin{aligned} \sum_{k \in K_1} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) &\leq 2M \sum_{k \in K_1} b_{n,k}(x) \leq \frac{2M}{\eta^2} \sum_{k \in K_1} \left(\frac{k}{n} - x\right)^2 b_{n,k}(x) \\ &\leq \frac{2M}{\eta^2} \sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 b_{n,k}(x) \\ &= \frac{2M}{\eta^2} [B_n(x^2) - 2xB_n(x) + x^2B_n(1)]. \end{aligned}$$

考慮下面這個關係式：

$$F(a, b) = [a + (1 - b)]^n = \sum_{k=0}^n \binom{n}{k} a^k (1 - b)^{n-k}.$$

我們可以用下列方法來計算 $B_n(1)$ 、 $B_n(x)$ 和 $B_n(x^2)$ ：

$$\begin{aligned} B_n(1) &= \sum_{k=0}^n b_{n,k}(x) = F(x, x) = 1, \\ B_n(x) &= \sum_{k=0}^n \frac{k}{n} b_{n,k}(x) = \frac{x}{n} \sum_{k=1}^n k \binom{n}{k} x^{k-1} (1-x)^{n-k} \\ &= \frac{x}{n} \frac{\partial}{\partial a} F(x, x) = \frac{x}{n} n [x + (1-x)]^{n-1} = x, \\ B_n(x^2) &= \sum_{k=0}^n \left(\frac{k}{n}\right)^2 b_{n,k}(x) = \sum_{k=0}^n \left(\frac{k(k-1)}{n^2} + \frac{k}{n^2}\right) b_{n,k}(x) \\ &= \frac{x^2}{n^2} \frac{\partial^2}{\partial a^2} F(x, x) + \frac{x}{n^2} \frac{\partial}{\partial a} F(x, x) \\ &= \frac{x^2}{n^2} [n(n-1)(x + (1-x))^{n-2}] + \frac{x}{n^2} n [x + (1-x)]^{n-1} \\ &= x^2 + \frac{x(1-x)}{n}. \end{aligned}$$

因此，我們得到

$$\sum_{k \in K_1} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) \leq \frac{2M}{\eta^2} \frac{x(1-x)}{n} \leq \frac{M}{2n\eta^2}.$$

把所有不等式放在一起，我們得到

$$|B_n(f)(x) - f(x)| \leq \varepsilon + \frac{M}{2n\eta^2}.$$

Since the above holds for any arbitrary $\varepsilon > 0$, we deduce that $\limsup_{n \rightarrow \infty} \|B_n(f) - f\|_\infty = 0$. \square

We need to introduce the notion of *lattice*, and state the lattice version of the Stone–Weierstraß theorem. This will allow us to recover the original version in Theorem 8.4.6.

Definition 8.4.10 : Let X be a compact metric space and $\mathcal{L} \subseteq \mathcal{C}(X, \mathbb{R})$ be a subset. We say that \mathcal{L} is a *lattice* if

$$\forall f, g \in \mathcal{L}, \quad \max\{f, g\}, \min\{f, g\} \in \mathcal{L}.$$

Lemma 8.4.11 : For any $a > 0$, there exists a sequence of polynomials that converges uniformly on $[-a, a]$ to the function $x \mapsto |x|$.

Proof : There are two ways to prove this lemma. It can either be seen as a direct consequence of the Weierstraß approximation theorem (Theorem 8.4.8), or be proven by construction.

By scaling, we may assume that $a = 1$. We note that for $x \in [-1, 1]$, and $u = 1 - x^2 \in [0, 1]$, we have

$$|x| = \sqrt{x^2} = \sqrt{1 - (1 - x^2)} = \sqrt{1 - u}$$

If $|u| < 1$, we have

$$\sqrt{1 - u} = \sum_{n \geq 0} a_n (-u)^n, \quad \text{where } a_n = \binom{1/2}{n}, \quad (8.22)$$

where the power series comes from Example 8.3.35, and it has radius of convergence equal to 1. We want to show that this power series converges uniformly for $u \in [0, 1]$. We may check that it converges normally, then the uniform convergence follows, see Proposition 8.1.22. For this, it suffices to check

我們先取最小上界範數，再對 n 取 \limsup ，可以得到

$$\limsup_{n \rightarrow \infty} \|B_n(f) - f\|_\infty \leq \varepsilon.$$

由於上面對於任意 $\varepsilon > 0$ 皆成立，我們推得 $\limsup_{n \rightarrow \infty} \|B_n(f) - f\|_\infty = 0$ 。 \square

再來我們要引入網格的概念，並且給出網格版本的 Stone–Weierstraß 定理。這可以讓我們推得定理 8.4.6 中原始版本的定理。

定義 8.4.10 : 令 X 為緊緻賦距空間以及 $\mathcal{L} \subseteq \mathcal{C}(X, \mathbb{R})$ 為子集合。如果

$$\forall f, g \in \mathcal{L}, \quad \max\{f, g\}, \min\{f, g\} \in \mathcal{L},$$

則我們說 \mathcal{L} 是個網格。

引理 8.4.11 : 對於任意 $a > 0$ ，存在定義在 $[-a, a]$ 上的多項式序列，使得他會均勻收斂到函數 $x \mapsto |x|$ 。

證明 : 我們有兩種方式來證明這個引理。我們可以把牠看作是 Weierstraß 近似定理（定理 8.4.8）的直接結果，或是使用構造法來證明。

我們可以把問題做縮放，因此可以假設 $a = 1$ 。我們注意到，對於 $x \in [-1, 1]$ 以及 $u = 1 - x^2 \in [0, 1]$ ，我們有

$$|x| = \sqrt{x^2} = \sqrt{1 - (1 - x^2)} = \sqrt{1 - u}$$

如果 $|u| < 1$ ，我們會有

$$\sqrt{1 - u} = \sum_{n \geq 0} a_n (-u)^n, \quad \text{其中 } a_n = \binom{1/2}{n}, \quad (8.22)$$

其中幕級數可以從範例 8.3.35 得到，而且他的收斂半徑等於 1。我們想要證明這個幕級數對

that $\sum a_n$ converges absolutely. For $n \in \mathbb{N}_0$, we have

$$\begin{aligned} a_n &= \frac{\frac{1}{2}(-\frac{1}{2}) \cdots (\frac{1}{2} - n + 1)}{n!} = \frac{(-1)^{n-1} (2n-3)!!}{2^n n!} \\ &= \frac{(-1)^{n-1} (2n-3)!!(2n-2)!!}{2^n n!(2n-2)!!} = \frac{(-1)^{n-1} (2n-2)!}{2^{2n-1} n!(n-1)!}, \end{aligned}$$

and the Stirling's formula gives us $|a_n| \sim \text{cst} \cdot n^{-3/2}$. This means that $\sum a_n$ converges absolutely. \square

Lemma 8.4.12 : Any closed subalgebra $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$ is a lattice.

Proof : Let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$ be a subalgebra. Given $f, g \in \mathcal{A}$, we have

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}, \quad \text{and} \quad \min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

Therefore, it is sufficient to show that for $h \in \mathcal{A}$, we also have $|h| \in \mathcal{A}$ to conclude. Let $h \in \mathcal{A}$. Due to the continuity of h and the compactness of X , we can define $a := \max_{x \in X} |h(x)| < \infty$, see Proposition 3.1.12. By Lemma 8.4.11, we may find a sequence of polynomials $(P_n)_{n \geq 1}$ that converges uniformly to the absolute value function on $[-a, a]$. For every $n \geq 1$, define $h_n = P_n(h) \in \mathcal{A}$. Therefore, $(h_n)_{n \geq 1}$ is a sequence of functions that converges uniformly to $|h|$ on X . Since \mathcal{A} is closed, we conclude that $|h| \in \mathcal{A}$. \square

Theorem 8.4.13 : Let X be a compact metric space with at least two points and $\mathcal{L} \subseteq \mathcal{C}(X, \mathbb{R})$ be a lattice. Suppose that for any $x \neq y \in X$ and $a, b \in \mathbb{R}$, there exists $f \in \mathcal{L}$ with $f(x) = a$ and $f(y) = b$. Then, \mathcal{L} is dense in $\mathcal{C}(X, \mathbb{R})$.

Proof : Let $\mathcal{L} \subseteq \mathcal{C}(X, \mathbb{R})$ be a lattice. Let $g \in \mathcal{C}(X, \mathbb{R})$ and $\varepsilon > 0$. We want to construct a function $f \in \mathcal{L}$ such that $\|f - g\|_\infty \leq \varepsilon$.

For any $a, b \in X$, we may find $f_{a,b} \in \mathcal{L}$ such that $f_{a,b}(a) = g(a)$ and $f_{a,b}(b) = g(b)$. By the continuity of $f_{a,b}$ and g , we know that there exists an open set $U_{a,b}$ containing b such that $f_{a,b} \geq g - \varepsilon$ on $U_{a,b}$. Since $(U_{a,b})_{b \in X}$ is an open covering of the compact space X , by the Borel-Lebesgue

於 $u \in [0, 1]$ 會均勻收斂。我們可以檢查他會正規收斂，那麼就能直接得到均勻收斂，見命題 8.1.22。因此，我們只需要檢查 $\sum a_n$ 會絕對收斂。對於 $n \in \mathbb{N}_0$ ，我們有

$$\begin{aligned} a_n &= \frac{\frac{1}{2}(-\frac{1}{2}) \cdots (\frac{1}{2} - n + 1)}{n!} = \frac{(-1)^{n-1} (2n-3)!!}{2^n n!} \\ &= \frac{(-1)^{n-1} (2n-3)!!(2n-2)!!}{2^n n!(2n-2)!!} = \frac{(-1)^{n-1} (2n-2)!}{2^{2n-1} n!(n-1)!}, \end{aligned}$$

再使用 Stirling 公式，我們會得到 $|a_n| \sim \text{cst} \cdot n^{-3/2}$ 。這代表著 $\sum a_n$ 會絕對收斂。 \square

引理 8.4.12 : 任何閉子代數 $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$ 都是個網格。

證明 : 令 $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$ 為子代數。給定 $f, g \in \mathcal{A}$ ，我們有

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}, \quad \text{以及} \quad \min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

因此，我們只需要證明對於 $h \in \mathcal{A}$ ，我們也會有 $|h| \in \mathcal{A}$ ，就可以總結了。令 $h \in \mathcal{A}$ 。使用 h 的連續性還有 X 的緊緻性，我們可以定義 $a := \max_{x \in X} |h(x)| < \infty$ ，見命題 3.1.12。使用引理 8.4.11，我們可以找到多項式序列 $(P_n)_{n \geq 1}$ ，使得他在 $[-a, a]$ 上會均勻收斂到絕對值函數。對於每個 $n \geq 1$ ，定義 $h_n = P_n(h) \in \mathcal{A}$ 。因此， $(h_n)_{n \geq 1}$ 是個函數序列，而且會在 X 上均勻收斂到 $|h|$ 。由於 \mathcal{A} 是個閉集，我們總結 $|h| \in \mathcal{A}$ 。 \square

定理 8.4.13 : 令 X 為包含至少兩個點的緊緻賦距空間，且 $\mathcal{L} \subseteq \mathcal{C}(X, \mathbb{R})$ 為網格。假設對於任意 $x \neq y \in X$ 以及 $a, b \in \mathbb{R}$ ，存在 $f \in \mathcal{L}$ 滿足 $f(x) = a$ 和 $f(y) = b$ 。那麼， \mathcal{L} 在 $\mathcal{C}(X, \mathbb{R})$ 中是稠密的。

證明 : 令 $\mathcal{L} \subseteq \mathcal{C}(X, \mathbb{R})$ 為網格。令 $g \in \mathcal{C}(X, \mathbb{R})$ 還有 $\varepsilon > 0$ 。我們想要構造函數 $f \in \mathcal{L}$ 使得 $\|f - g\|_\infty \leq \varepsilon$ 。

對於任意 $a, b \in X$ ，我們能找到 $f_{a,b} \in \mathcal{L}$ 使得 $f_{a,b}(a) = g(a)$ 還有 $f_{a,b}(b) = g(b)$ 。使用 $f_{a,b}$ 和 g 的連續性，我們知道存在包含 b 的開集 $U_{a,b}$ 使得 $f_{a,b} \geq g - \varepsilon$ 在 $U_{a,b}$ 上。由於 $(U_{a,b})_{b \in X}$ 是個

property (Definition 3.1.3), we may find $b_1, \dots, b_m \in X$ such that $(U_{a,b_i})_{1 \leq i \leq m}$ covers X . Let $f_a := \sup_{1 \leq i \leq m} f_{a,b_i} \in \mathcal{L}$. Then, we have $f_a(a) = a$ and $f_a \geq g - \varepsilon$ on X . Similarly, by the continuity of f_a and g , there exists an open set V_a containing a such that $f_a \leq g + \varepsilon$ on V_a . Since $(V_a)_{a \in X}$ is an open covering of the compact space X , again by the Borel–Lebesgue property (Definition 3.1.3), we may find $a_1, \dots, a_n \in X$ such that $(V_{a_j})_{1 \leq j \leq n}$ covers X . Let $f := \inf_{1 \leq j \leq n} f_{a_j}$. Then, we may easily check that $g - \varepsilon \leq f \leq g + \varepsilon$ on X , so $\|f - g\|_\infty \leq \varepsilon$. This concludes that \mathcal{L} is dense in $\mathcal{C}(X, \mathbb{R})$. \square

Proof of Proof of Theorem 8.4.6: Let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$ be a subalgebra satisfying the assumptions in Theorem 8.4.6. We write $\overline{\mathcal{A}} = \overline{\mathcal{A}}$, which is still a subalgebra, because addition, multiplication, and scalar multiplication are continuous. It follows from Lemma 8.4.12 that $\overline{\mathcal{A}}$ is a lattice. Now, let us check that the assumptions in Theorem 8.4.13 are satisfied.

Let $x \neq y \in X$ and $a, b \in \mathbb{R}$. By the assumptions in Theorem 8.4.6, we may find $p \in \mathcal{A}$ such that $p(x) \neq p(y)$. Since $1 \in \mathcal{A}$, we may also add $c \times 1 \in \mathcal{A}$ to p , to make $p(x) + c \neq 0$ and $p(y) + c \neq 0$. Without loss of generality, let us assume that $p(x) \neq p(y)$, $p(x) \neq 0$, and $p(y) \neq 0$ for some $p \in \mathcal{A}$. Then, we may look for $f \in \mathcal{A}$ in the form $f = \alpha p + \beta p^2$, where $\alpha, \beta \in \mathbb{R}$ can be chosen properly so that $f(x) = a$ and $f(y) = b$. Therefore, Theorem 8.4.13 tells us that $\overline{\mathcal{A}} = \mathcal{C}(X, \mathbb{R})$, that is $\overline{\mathcal{A}} = \mathcal{C}(X, \mathbb{R})$.

For the complex version of the theorem, we proceed as follows. Let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{C})$ be a subalgebra satisfying the assumptions in Theorem 8.4.6. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the set of real-valued functions in \mathcal{A} , which is a \mathbb{R} -subalgebra of $\mathcal{C}(X, \mathbb{R})$. We want to check that $\overline{\mathcal{A}_0} = \mathcal{C}(X, \mathbb{R})$. First, it is not hard to check that $1 \in \mathcal{A}_0$. Then, for any $f \in \mathcal{A}$, since $\overline{f} \in \mathcal{A}$, we deduce that $\operatorname{Re}(f), \operatorname{Im}(f) \in \mathcal{A}_0$. For any $x \neq y \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$, so we need to have $\operatorname{Re}(f)(x) \neq \operatorname{Re}(f)(y)$ or $\operatorname{Im}(f)(x) \neq \operatorname{Im}(f)(y)$. This means that \mathcal{A}_0 separates points. By the real version of the theorem, we conclude that $\overline{\mathcal{A}_0} = \mathcal{C}(X, \mathbb{R})$. For any function $f \in \mathcal{C}(X, \mathbb{C})$ and $\varepsilon > 0$, we may find $g_1, g_2 \in \mathcal{A}_0$ such that

$$\|\operatorname{Re}(f) - g_1\|_\infty \leq \varepsilon, \quad \text{and} \quad \|\operatorname{Im}(f) - g_2\|_\infty \leq \varepsilon.$$

Since \mathcal{A} is a \mathbb{C} -algebra, we know that $g_1 + i g_2 \in \mathcal{A}$. Moreover,

$$\|f - (g_1 + i g_2)\|_\infty \leq \|\operatorname{Re}(f) - g_1\|_\infty + \|\operatorname{Im}(f) - g_2\|_\infty \leq 2\varepsilon.$$

This shows that \mathcal{A} is dense in $\mathcal{C}(X, \mathbb{C})$. \square

緊緻空間 X 的開覆蓋，根據 Borel–Lebesgue 性質（定義 3.1.3），我們能找到 $b_1, \dots, b_m \in X$ 使得 $(U_{a,b_i})_{1 \leq i \leq m}$ 覆蓋 X 。令 $f_a := \sup_{1 \leq i \leq m} f_{a,b_i} \in \mathcal{L}$ 。那麼，我們有 $f_a(a) = a$ 還有 $f_a \geq g - \varepsilon$ 在 X 上。同理，使用 f_a 和 g 的連續性，存在包含 a 的開集 V_a 使得 $f_a \leq g + \varepsilon$ 在 V_a 上。由於 $(V_a)_{a \in X}$ 是個緊緻空間 X 的開覆蓋，再次使用 Borel–Lebesgue 性質（定義 3.1.3），我們能找到 $a_1, \dots, a_n \in X$ 使得 $(V_{a_j})_{1 \leq j \leq n}$ 覆蓋 X 。令 $f := \inf_{1 \leq j \leq n} f_{a_j}$ 。那麼，我們可以輕易檢查在 X 上，我們有 $g - \varepsilon \leq f \leq g + \varepsilon$ ，所以 $\|f - g\|_\infty \leq \varepsilon$ 。這讓我們可以總結 \mathcal{L} 在 $\mathcal{C}(X, \mathbb{R})$ 中是稠密的。 \square

定理 8.4.6 的證明的證明：令 $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$ 為滿足定理 8.4.6 當中假設的子代數。我們記 $\overline{\mathcal{A}} = \overline{\mathcal{A}}$ ，這也還會是個子代數，因為加法、乘法和向量乘法都是連續的。從引理 8.4.12 我們得知 $\overline{\mathcal{A}}$ 是個網格。現在，讓我們來檢查定理 8.4.13 中的假設會成立。

令 $x \neq y \in X$ 還有 $a, b \in \mathbb{R}$ 。根據定理 8.4.6 中的假設，我們能找到 $p \in \mathcal{A}$ 使得 $p(x) \neq p(y)$ 。由於 $1 \in \mathcal{A}$ ，我們可以把 $c \times 1 \in \mathcal{A}$ 加到 p ，讓我們能得到 $p(x) + c \neq 0$ 還有 $p(y) + c \neq 0$ 。不失一般性，讓我們假設有 $p \in \mathcal{A}$ ，滿足 $p(x) \neq p(y)$ 、 $p(x) \neq 0$ 還有 $p(y) \neq 0$ 。再來，我們找可以寫成 $f = \alpha p + \beta p^2$ 形式的 $f \in \mathcal{A}$ ，然後選擇 $\alpha, \beta \in \mathbb{R}$ 使得 $f(x) = a$ 還有 $f(y) = b$ 。因此，定理 8.4.13 告訴我們 $\overline{\mathcal{A}} = \mathcal{C}(X, \mathbb{R})$ ，也就是 $\overline{\mathcal{A}} = \mathcal{C}(X, \mathbb{R})$ 。

再來我們討論怎麼證明複數版本的定理。令 $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{C})$ 為滿足定理 8.4.6 當中假設的子代數。令 $\mathcal{A}_0 \subseteq \mathcal{A}$ 為 \mathcal{A} 中的實函數所構成的集合，這會是個 $\mathcal{C}(X, \mathbb{R})$ 的 \mathbb{R} 子代數。我們想要檢查 $\overline{\mathcal{A}_0} = \mathcal{C}(X, \mathbb{R})$ 。首先，我們不難檢查 $1 \in \mathcal{A}_0$ 。再來，對於任意 $f \in \mathcal{A}$ ，由於 $\overline{f} \in \mathcal{A}$ ，我們推得 $\operatorname{Re}(f), \operatorname{Im}(f) \in \mathcal{A}_0$ 。對於任意 $x \neq y \in X$ ，存在 $f \in \mathcal{A}$ 滿足 $f(x) \neq f(y)$ ，所以我們一定會有 $\operatorname{Re}(f)(x) \neq \operatorname{Re}(f)(y)$ 或 $\operatorname{Im}(f)(x) \neq \operatorname{Im}(f)(y)$ 。所以，我們知道 \mathcal{A}_0 會分離點。藉由實數版本的定理，我們總結 $\overline{\mathcal{A}_0} = \mathcal{C}(X, \mathbb{R})$ 。對於任意函數 $f \in \mathcal{C}(X, \mathbb{C})$ 還有 $\varepsilon > 0$ ，我們可以找到 $g_1, g_2 \in \mathcal{A}_0$ 使得

$$\|\operatorname{Re}(f) - g_1\|_\infty \leq \varepsilon, \quad \text{以及} \quad \|\operatorname{Im}(f) - g_2\|_\infty \leq \varepsilon.$$

由於 \mathcal{A} 是個 \mathbb{C} 代數，我們有 $g_1 + i g_2 \in \mathcal{A}$ 。此外，我們還有

$$\|f - (g_1 + i g_2)\|_\infty \leq \|\operatorname{Re}(f) - g_1\|_\infty + \|\operatorname{Im}(f) - g_2\|_\infty \leq 2\varepsilon.$$

8.4.3 Peano existence theorem

As an application of the Arzelà–Ascoli theorem and the Stone–Weierstraß theorem, we have the following Peano existence theorem, which gives us the existence of solution for differential equations.

Theorem 8.4.14 (Peano existence theorem) : Fix an integer $n \geq 1$. Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$ be a non-empty open subset, and $F : \Omega \rightarrow \mathbb{R}^n$ be a continuous function. Let $t_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^n$ such that $(t_0, y_0) \in \Omega$. Let $a, b > 0$ such that

$$\mathcal{R} := \{(t, y) : |t - t_0| \leq a, \|y - y_0\| \leq b\} \subseteq \Omega.$$

Let $M > 0$ and suppose that $\|F(t, y)\| \leq M$ for $(t, y) \in \mathcal{R}$. Then, the following differential equation

$$\begin{cases} y'(t) = F(t, y(t)), & \forall t \in \overset{\circ}{I}, \\ y(t_0) = y_0, \end{cases}$$

has a solution $t \mapsto y(t)$ defined on $I := [t_0 - a', t_0 + a']$ with $a' = \min\{a, \frac{b}{M}\}$.

Remark 8.4.15 : It is important to note that the Peano existence theorem does not guarantee uniqueness, see Example 8.4.16. In order to have a unique solution, the function F needs to satisfy stronger properties, as stated in the Picard–Lindelöf theorem, also known as the Cauchy–Lipschitz theorem, see Theorem 8.4.17.

Proof : The proof consists of three parts: (1) We reformulate the solution to the differential equation as a fixed-point problem; (2) we show the existence of the solution in the case that F is a Lipschitz continuous function; (3) we show the existence in the general setting.

Without loss of generality, we may assume that $t = 0$ and $y_0 = 0 \in \mathbb{R}^n$ by a translation in time and in space.

(1) First, let us reformulate this as a solution to some fixed-point problem. Let us write $\mathcal{X} =$

這證明了 \mathcal{A} 在 $\mathcal{C}(X, \mathbb{C})$ 當中會是稠密的。 \square

第三小節 Peano 存在性定理

再來我們會討論下面的 Peano 存在性定理，這是 Arzelà–Ascoli 定理和 Stone–Weierstraß 定理的應用，他會給我們微分方程式解的存在性。

定理 8.4.14 【Peano 存在性定理】：固定整數 $n \geq 1$ 。令 $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$ 為非空開集，以及 $F : \Omega \rightarrow \mathbb{R}^n$ 為連續函數。令 $t_0 \in \mathbb{R}$ 以及 $y_0 \in \mathbb{R}^n$ 使得 $(t_0, y_0) \in \Omega$ 。令 $a, b > 0$ 滿足

$$\mathcal{R} := \{(t, y) : |t - t_0| \leq a, \|y - y_0\| \leq b\} \subseteq \Omega.$$

令 $M > 0$ 並假設對於 $(t, y) \in \mathcal{R}$ ，我們有 $\|F(t, y)\| \leq M$ 。那麼，下面這個微分方程式

$$\begin{cases} y'(t) = F(t, y(t)), & \forall t \in \overset{\circ}{I}, \\ y(t_0) = y_0, \end{cases}$$

會有個定義在 $I := [t_0 - a', t_0 + a']$ 上的解 $t \mapsto y(t)$ ，其中 $a' = \min\{a, \frac{b}{M}\}$ 。

註解 8.4.15：很重要的是，我們要注意到 Peano 存在性定理並沒有保證解的唯一性，見範例 8.4.16。如果要有解的唯一性，函數 F 需要滿足更強的性質，見定理 8.4.17 當中的 Picard–Lindelöf 定理，也稱作 Cauchy–Lipschitz 定理。

證明：我們把證明分成三部份：(1) 我們把微分方程的解重新描述為固定點問題；(2) 我們在 F 是個 Lipschitz 連續函數的情況下，證明解的存在性；(3) 在一般的設定下，我們證明解的存在性。

不失一般性，我們可以對時間和空間做平移，並假設 $t = 0$ 還有 $y_0 = 0 \in \mathbb{R}^n$ 。

(1) 首先，讓我們把所要證明的重新敘述為固定點問題。我們記 $\mathcal{X} = \mathcal{C}(I, \overline{B}(0, b))$ 。考慮下面

$\mathcal{C}(I, \overline{B}(0, b))$. Consider the following operator,

$$\begin{aligned} T : \mathcal{X} &\rightarrow \mathcal{X} \\ f &\mapsto \int_0^t F(s, f(s)) \, ds. \end{aligned}$$

Let us check that for $f \in \mathcal{X}$, the image $T(f)$ is well defined. We first note that $(s, f(s)) \in \mathcal{R}$ for any $s \in I$, so for any $t \in I$, we have

$$\|T(f)(t)\| = \left\| \int_0^t F(s, f(s)) \, ds \right\| \leq |t|M \leq b.$$

In other words, $T(f)$ is a function from I to $\overline{B}(0, b)$. Moreover, it follows from the fundamental theorem of calculus that $T(f)$ is of class \mathcal{C}^1 , so we do have $T(f) \in \mathcal{X}$. As a consequence, if y is a fixed point of T , that is $T(y) = y$, we deduce that y is of class \mathcal{C}^∞ . Moreover, if y is a fixed point, by taking the derivative at $t \in I$, we find

$$y'(t) = (T(y))'(t) = F(t, y(t)).$$

We may also check easily that $y(0) = T(y)(0) = 0$. Therefore, the conclusion of Theorem 8.4.14 is equivalent to showing that T has at least one fixed point.

- (2) Let us assume that F is an L -Lipschitz continuous function on \mathcal{R} . In this case, we can easily check that T is an (La') -Lipschitz continuous function, so it is continuous.

We are going to define a sequence of functions $(y_n)_{n \geq 1}$ which are elements of \mathcal{X} . First, let y_1 be the constant zero function, which is indeed in \mathcal{X} . For $n \geq 1$, we define $y_{n+1} = T(y_n)$, which is in \mathcal{X} from (1). By induction, we establish a sequence $(y_n)_{n \geq 1}$ in \mathcal{X} . Moreover, for any $t, t' \in I$ and $n \geq 1$, we have

$$\|y_n(t) - y_n(t')\| = \left\| \int_{t'}^t F(s, y_{n-1}(s)) \, ds \right\| \leq M|t - t'|. \quad (8.23)$$

This means that $(y_n)_{n \geq 1}$ is a sequence of equicontinuous functions. The Arzelà–Ascoli theorem³ allows us to find a convergent subsequence $(y_{\varphi(n)})_{n \geq 1}$ with limit $y \in \mathcal{X}$. We want to check that $T(y) = y$.

Let us denote $I_+ = I \cap \mathbb{R}_+ = [0, a']$. For every $n \geq 1$ and $t \in I_+$, let us define

$$M_n(t) := \sup_{0 \leq s \leq t} \|T(y_n)(s) - y_n(s)\| = \sup_{0 \leq s \leq t} \|y_{n+1}(s) - y_n(s)\|.$$

這個運算子：

$$\begin{aligned} T : \mathcal{X} &\rightarrow \mathcal{X} \\ f &\mapsto \int_0^t F(s, f(s)) \, ds. \end{aligned}$$

讓我們來檢查，對於 $f \in \mathcal{X}$ ， $T(f)$ 的像是定義良好的。我們先注意到，對於任意 $s \in I$ 來說，我們有 $(s, f(s)) \in \mathcal{R}$ ，所以對於任意 $t \in I$ ，我們有

$$\|T(f)(t)\| = \left\| \int_0^t F(s, f(s)) \, ds \right\| \leq |t|M \leq b.$$

換句話說， $T(f)$ 是個由 I 映射至 $\overline{B}(0, b)$ 的函數。此外，從微積分基本定理，我們知道 $T(f)$ 是 \mathcal{C}^1 類的，所以我們有 $T(f) \in \mathcal{X}$ 。因此，如果 y 是個 T 的固定點，也就是說會滿足 $T(y) = y$ ，我們推得 y 會是 \mathcal{C}^∞ 類的。此外，如果 y 是個固定點，藉由對 $t \in I$ 微分，我們有

$$y'(t) = (T(y))'(t) = F(t, y(t)).$$

我們也可以檢查 $y(0) = T(y)(0) = 0$ 。因此，定理 8.4.14 的結論與證明 T 至少有一個固定點是等價的。

- (2) 讓我們假設 F 是個在 \mathcal{R} 上的 L -Lipschitz 連續函數。在這個情況下，我們可以輕易檢查 T 是個 (La') -Lipschitz 連續函數，所以也會連續。

我們再來要定義由 \mathcal{X} 中的元素所構成的函數序列 $(y_n)_{n \geq 1}$ 。手線，令 y_1 為常數零函數，這的確是在 \mathcal{X} 中的。對於 $n \geq 1$ ，我們定義 $y_{n+1} = T(y_n)$ ，從 (1) 的討論，我們知道他會在 \mathcal{X} 當中。藉由數學歸納法，我們可以定義在 \mathcal{X} 中的序列 $(y_n)_{n \geq 1}$ 。此外，對於任意 $t, t' \in I$ 還有 $n \geq 1$ ，我們有

$$\|y_n(t) - y_n(t')\| = \left\| \int_{t'}^t F(s, y_{n-1}(s)) \, ds \right\| \leq M|t - t'|. \quad (8.23)$$

這代表著 $(y_n)_{n \geq 1}$ 是個等度連續函數所構成的序列。Arzelà–Ascoli 定理³讓我們可以找到收斂子序列 $(y_{\varphi(n)})_{n \geq 1}$ ，極限記作 $y \in \mathcal{X}$ 。我們想要檢查 $T(y) = y$ 。

For $n \geq 2$ and $s \in I_+$, we have

$$\begin{aligned} \|T(y_n)(s) - y_n(s)\| &= \|T(y_n)(s) - T(y_{n-1})(s)\| \\ &= \left\| \int_0^s (F(u, y_n(u)) - F(u, y_{n-1}(u))) \, du \right\| \\ &\leq \int_0^s LM_{n-1}(u) \, du, \end{aligned}$$

which implies that

$$\forall t \in I_+, \quad M_n(t) \leq L \int_0^t M_{n-1}(u) \, du. \quad (8.24)$$

We may compute M_1 as below,

$$\forall t \in I_+, \quad M_1(t) = \sup_{0 \leq s \leq t} \|y_2(s)\| = \sup_{0 \leq s \leq t} \left\| \int_0^s F(u, 0) \, du \right\| \leq tM.$$

Then, for M_2 , we apply Eq. (8.24) and find

$$\forall t \in I_+, \quad M_2(t) \leq L \int_0^t M_1(u) \, du = \frac{t^2}{2} LM.$$

By induction, we find, for every $n \geq 1$,

$$\forall t \in I_+, \quad M_n(t) \leq \frac{t^n}{n!} L^{n-1} M \leq \frac{(a')^n}{n!} L^{n-1} M \xrightarrow{n \rightarrow \infty} 0.$$

Therefore, this allows us to conclude that $(T(y_{\varphi(n)}) - y_{\varphi(n)})_{n \geq 1}$ uniformly converges to 0 on I_+ .

Then, a similar argument allows us to get the uniform convergence to 0 on $I_- := I \cap \mathbb{R}_-$, so this convergence is uniform on I . Since $y_{\varphi(n)}$ uniformly converges to y and T is continuous, we deduce that $T(y_{\varphi(n)})$ uniformly converges to $T(y)$, giving us $T(y) = y$.

讓我們記 $I_+ = I \cap \mathbb{R}_+ = [0, a']$ 。對於每個 $n \geq 1$ 還有 $t \in I_+$ ，讓我們定義

$$M_n(t) := \sup_{0 \leq s \leq t} \|T(y_n)(s) - y_n(s)\| = \sup_{0 \leq s \leq t} \|y_{n+1}(s) - y_n(s)\|.$$

對於 $n \geq 2$ 還有 $s \in I_+$ ，我們有

$$\begin{aligned} \|T(y_n)(s) - y_n(s)\| &= \|T(y_n)(s) - T(y_{n-1})(s)\| \\ &= \left\| \int_0^s (F(u, y_n(u)) - F(u, y_{n-1}(u))) \, du \right\| \\ &\leq \int_0^s LM_{n-1}(u) \, du, \end{aligned}$$

這蘊含

$$\forall t \in I_+, \quad M_n(t) \leq L \int_0^t M_{n-1}(u) \, du. \quad (8.24)$$

我們可以依下列方式來計算 M_1 ：

$$\forall t \in I_+, \quad M_1(t) = \sup_{0 \leq s \leq t} \|y_2(s)\| = \sup_{0 \leq s \leq t} \left\| \int_0^s F(u, 0) \, du \right\| \leq tM.$$

再來，對於 M_2 ，我們使用式 (8.24)，然後得到：

$$\forall t \in I_+, \quad M_2(t) \leq L \int_0^t M_1(u) \, du = \frac{t^2}{2} LM.$$

透過歸納法，對於 $n \geq 1$ ，我們會有

$$\forall t \in I_+, \quad M_n(t) \leq \frac{t^n}{n!} L^{n-1} M \leq \frac{(a')^n}{n!} L^{n-1} M \xrightarrow{n \rightarrow \infty} 0.$$

因此，我們得知 $(T(y_{\varphi(n)}) - y_{\varphi(n)})_{n \geq 1}$ 會在 I_+ 上均勻收斂至 0。接著，使用相似的方法，

我們可以得到這個收斂到 0 在 $I_- := I \cap \mathbb{R}_-$ 上也是均勻的，所以在整個 I 上會是均勻的。

由於 $y_{\varphi(n)}$ 會均勻收斂到 y ，而且 T 是連續的，我們推得 $T(y_{\varphi(n)})$ 會均勻收斂到 $T(y)$ ，這給我們 $T(y) = y$ 。

³Theorem 8.4.4 (2) tells us that the set $\{y_n : n \geq 1\}$ is a precompact subset. It can be shown that there exists a subsequence of $(y_n)_{n \geq 1}$ which is a Cauchy sequence, see Exercise 8.31. Then, this subsequence converges by the completeness of \mathcal{X} .

³定理 8.4.4 (2) 告訴我們集合 $\{y_n : n \geq 1\}$ 是個預緊緻的子集合。我們可以證明序列 $(y_n)_{n \geq 1}$ 會有個柯西子序列，見習題 8.31。那麼，透過 \mathcal{X} 的完備性，這個子序列就會收斂。

- (3) If F is only continuous, by the Stone–Weierstraß theorem (Theorem 8.4.6), we may find a sequence of Lipschitz continuous functions $(F_n)_{n \geq 1}$ that converges uniformly to F on \mathcal{R} . For every $n \geq 1$, let y_n be the corresponding solution to the differential equation with F replaced by F_n . Then, $(y_n)_{n \geq 1}$ is a sequence in \mathcal{X} . Since $(F_n)_{n \geq 1}$ converges to F uniformly on \mathcal{R} , we know that $(F_n)_{n \geq 1}$ can be uniformly bounded by a constant $M' > 0$. This implies that the sequence of functions $(y_n)_{n \geq 1}$ is equicontinuous due to the same Eq. (8.23), with M replaced by M' . Therefore, the Arzelà–Ascoli theorem gives us a subsequence $(y_{\varphi(n)})_{n \geq 1}$ that converges uniformly to $y \in \mathcal{X}$, and we need to check that $T(y) = y$. To achieve this, we start by checking that the functions in the sequence $(s \mapsto F_{\varphi(n)}(s, y_{\varphi(n)}(s)))_{n \geq 1}$ are equicontinuous.

Let $\varepsilon > 0$. For $n \geq 1$ and $s, t \in I$, we write

$$\begin{aligned} & \|F_n(s, y_n(s)) - F_n(t, y_n(t))\| \\ & \leq \|F_n(s, y_n(s)) - F(s, y_n(s))\| + \|F(s, y_n(s)) - F(s, y(s))\| + \|F(s, y(s)) - F(t, y(t))\| \\ & \quad + \|F(t, y(t)) - F(t, y_n(t))\| + \|F(t, y_n(t)) - F_n(t, y_n(t))\| \end{aligned}$$

Since $s \mapsto F(s, y(s))$ is continuous on the segment I , it is uniformly continuous. Similarly, the map $(t, y) \mapsto F(t, y)$ is also uniformly continuous on \mathcal{R} . We may take $\eta > 0$ such that

$$\begin{aligned} |t - s| \leq \eta & \Rightarrow \|F(s, y(s)) - F(t, y(t))\| \leq \varepsilon, \\ \|(t, y) - (s, x)\| \leq \eta & \Rightarrow \|F(t, y) - F(s, x)\| \leq \varepsilon. \end{aligned}$$

Since $y_{\varphi(n)} \xrightarrow{n \rightarrow \infty} y$ uniformly and $F_{\varphi(n)} \xrightarrow{n \rightarrow \infty} F$ uniformly, there exists $N \geq 1$ such that

$$\forall n \geq N, \quad \|y_{\varphi(n)} - y\|_{\infty} \leq \eta, \quad \text{and} \quad \|F_{\varphi(n)} - F\|_{\infty} \leq \varepsilon.$$

Therefore, for $n \geq N$, and $s, t \in I$ such that $|s - t| \leq \eta$, we find

$$\|F_{\varphi(n)}(s, y_{\varphi(n)}(s)) - F_{\varphi(n)}(t, y_{\varphi(n)}(t))\| \leq 5\varepsilon$$

This means that $(s \mapsto F_{\varphi(n)}(s, y_{\varphi(n)}(s)))_{n \geq 1}$ is equicontinuous, so has a convergent subsequence, and we denote the corresponding extraction by ψ . Therefore, for $t \in I$, we have

$$T(y_{\varphi \circ \psi(n)})(t) = \int_0^t F_{\varphi \circ \psi(n)}(s, y_{\varphi \circ \psi(n)}(s)) \, ds \xrightarrow{n \rightarrow \infty} \int_0^t F(s, y(s)) \, ds = T(y)(t),$$

which is uniform in $t \in I$ by Proposition 8.2.5. We conclude that $T(y) = y$. \square

- (3) 如果 F 只是連續的，根據 Stone–Weierstraß 定理（定理 8.4.6），我們能找到由 Lipschitz 連續函數所構成的序列 $(F_n)_{n \geq 1}$ 會在 \mathcal{R} 上均勻收斂至 F 。對於每個 $n \geq 1$ ，令 y_n 為在微分方程中，把 F 換成 F_n 所得到對應的解。這樣一來， $(y_n)_{n \geq 1}$ 會是個在 \mathcal{X} 當中的序列。由於 $(F_n)_{n \geq 1}$ 會在 \mathcal{R} 上均勻收斂到 F ，我們知道 $(F_n)_{n \geq 1}$ 可以被一個常數 $M' > 0$ 均勻控制住。透過把式 (8.23) 中的 M 換成 M' ，我們知道相同的論述可以告訴我們，函數序列 $(y_n)_{n \geq 1}$ 會是等度連續的。因此，Arzelà–Ascoli 定理給我們子序列 $(y_{\varphi(n)})_{n \geq 1}$ ，會均勻收斂到 $y \in \mathcal{X}$ ，再來我們要檢查 $T(y) = y$ 。首先我們先檢查函數序列 $(s \mapsto F_{\varphi(n)}(s, y_{\varphi(n)}(s)))_{n \geq 1}$ 是等度連續的。

固定 $\varepsilon > 0$ 。對於 $n \geq 1$ 還有 $s, t \in I$ ，我們記

$$\begin{aligned} & \|F_n(s, y_n(s)) - F_n(t, y_n(t))\| \\ & \leq \|F_n(s, y_n(s)) - F(s, y_n(s))\| + \|F(s, y_n(s)) - F(s, y(s))\| + \|F(s, y(s)) - F(t, y(t))\| \\ & \quad + \|F(t, y(t)) - F(t, y_n(t))\| + \|F(t, y_n(t)) - F_n(t, y_n(t))\| \end{aligned}$$

由於 $s \mapsto F(s, y(s))$ 在線段 I 上是連續的，所以也會均勻連續。同理，映射 $(t, y) \mapsto F(t, y)$ 在 \mathcal{R} 上也是均勻連續的。我們可以取 $\eta > 0$ 使得

$$\begin{aligned} |t - s| \leq \eta & \Rightarrow \|F(s, y(s)) - F(t, y(t))\| \leq \varepsilon, \\ \|(t, y) - (s, x)\| \leq \eta & \Rightarrow \|F(t, y) - F(s, x)\| \leq \varepsilon. \end{aligned}$$

由於 $y_{\varphi(n)} \xrightarrow{n \rightarrow \infty} y$ 是均勻的，且 $F_{\varphi(n)} \xrightarrow{n \rightarrow \infty} F$ 是均勻的，存在 $N \geq 1$ 使得

$$\forall n \geq N, \quad \|y_{\varphi(n)} - y\|_{\infty} \leq \eta, \quad \text{以及} \quad \|F_{\varphi(n)} - F\|_{\infty} \leq \varepsilon.$$

因此，對於 $n \geq N$ 還有 $s, t \in I$ 滿足 $|s - t| \leq \eta$ ，我們得到

$$\|F_{\varphi(n)}(s, y_{\varphi(n)}(s)) - F_{\varphi(n)}(t, y_{\varphi(n)}(t))\| \leq 5\varepsilon$$

這代表著 $(s \mapsto F_{\varphi(n)}(s, y_{\varphi(n)}(s)))_{n \geq 1}$ 是等度連續的，所以會有收斂子序列，我們把他所對

Example 8.4.16 : Let us take $n = 1$, and $F(t, y) = \sqrt{|y|}$ with initial condition $(t_0, y_0) = (0, 0)$. In other words, the differential equation we are looking at is

$$y'(t) = \sqrt{|y(t)|} \quad \text{and} \quad y(0) = 0. \quad (8.25)$$

We have many different solutions to Eq. (8.25),

- $y(t) = 0$ for $t \in \mathbb{R}$;
- $y(t) = \frac{t|t|}{4}$ for $t \in \mathbb{R}$;
- for any $a > 0$, $y(t) = \frac{(t-a)^2}{4}$ for $t \geq a$ and $y(t) = 0$ for $t \leq a$.

Indeed, the function $x \mapsto \sqrt{|x|}$ is not locally Lipschitz continuous at 0, so does not satisfy the assumptions of the Picard–Lindelöf theorem (Theorem 8.4.17).

The following Picard–Lindelöf theorem, also known as Cauchy–Lipschitz theorem, gives sufficient conditions for the solution to an ordinary differential equation to be unique.

Theorem 8.4.17 (Picard–Lindelöf theorem or Cauchy–Lipschitz theorem) : *Let us fix the same notations as in the statement of Theorem 8.4.14. In addition, suppose that F is L -Lipschitz continuous in the second variable in \mathcal{R} . Then, apart from the existence provided in Theorem 8.4.14, we also have uniqueness of the solution, in the sense that if J is an interval containing t_0 and $\varphi : J \rightarrow \mathbb{R}^n$ is a solution, then y and φ coincide on $I \cap J$.*

應到的萃取函數記作 ψ 。因此，對於 $t \in I$ ，我們有

$$T(y_{\varphi \circ \psi(n)})(t) = \int_0^t F_{\varphi \circ \psi(n)}(s, y_{\varphi \circ \psi(n)}(s)) \, ds \xrightarrow{n \rightarrow \infty} \int_0^t F(s, y(s)) \, ds = T(y)(t),$$

這個收斂根據命題 8.2.5，對於 $t \in I$ 中是均勻的。我們總結 $T(y) = y$ 。□

範例 8.4.16 : 我們取 $n = 1$ 以及 $F(t, y) = \sqrt{|y|}$ 和初始條件 $(t_0, y_0) = (0, 0)$ 。換句話說，我們想要考慮的微分方程如下：

$$y'(t) = \sqrt{|y(t)|} \quad \text{以及} \quad y(0) = 0. \quad (8.25)$$

式 (8.25) 有很多不同的解：

- $y(t) = 0$ 對於 $t \in \mathbb{R}$ ；
- $y(t) = \frac{t|t|}{4}$ 對於 $t \in \mathbb{R}$ ；
- 對於任意 $a > 0$ ， $y(t) = \frac{(t-a)^2}{4}$ 對於 $t \geq a$ ，然後 $y(t) = 0$ 對於 $t \leq a$ 。

事實上，函數 $x \mapsto \sqrt{|x|}$ 在 0 點附近不是局部 Lipschitz 連續的，所以不會滿足 Picard–Lindelöf 定理（定理 8.4.17）當中的假設。

下面這個 Picard–Lindelöf 定理，也稱作 Cauchy–Lipschitz 定理，給了充分條件使得常微分方程式的解會是唯一的。

定理 8.4.17 【Picard–Lindelöf 定理，Cauchy–Lipschitz 定理】：讓我們使用與在定理 8.4.14 的敘述中相同的記號。此外，假設 F 在 \mathcal{R} 中，對於第二個變數是 L -Lipschitz 連續的。那麼，除了定理 8.4.14 所給我們的存在性結果之外，我們還有唯一性的結果。這個唯一性的意義如下：如果 J 是個包含 t_0 的區間，且 $\varphi : J \rightarrow \mathbb{R}^n$ 是個解，那麼 y 和 φ 在 $I \cap J$ 上會是完全相等的。

Proof : We keep the notations from the proof of Theorem 8.4.14. In particular, we want to show that the map T defined therein has a unique fixed point. More precisely, we want to show that there exists an integer $m \in \mathbb{N}$ such that T^m is a contraction, then we may conclude by Exercise 3.24.

Let $f, g \in \mathcal{X}$. We proceed in a similar way as in (2) in the proof of Theorem 8.4.14. For $n \geq 1$ and $t \in I_+$, let us define

$$K_n(t) := \sup_{0 \leq s \leq t} \|(T^n f)(s) - (T^n g)(s)\|.$$

For $n \geq 2$ and $s \in I_+$, we have

$$\begin{aligned} \|T^n(f)(s) - T^n(g)(s)\| &= \left\| \int_0^s F(u, T^{n-1}(f)(u)) - F(u, T^{n-1}(g)(u)) \, du \right\| \\ &\leq \int_0^s L \|T^{n-1}(f)(u) - T^{n-1}(g)(u)\| \, du \\ &\leq \int_0^s L K_{n-1}(u) \, du, \end{aligned}$$

which implies that

$$\forall t \in I_+, \quad K_n(t) \leq L \int_0^t K_{n-1}(u) \, du.$$

We may compute K_1 as below,

$$\forall t \in I_+, \quad K_1(t) = \sup_{0 \leq s \leq t} \left\| \int_0^s F(u, f(u)) - F(u, g(u)) \, du \right\| \leq Lt \|f - g\|_\infty.$$

By induction, we find, for every $n \geq 1$,

$$\forall t \in I, \quad K_n(t) \leq \frac{t^n}{n!} L^n \|f - g\|_\infty \leq \frac{(a')^n}{n!} L^n \|f - g\|_\infty \xrightarrow{n \rightarrow \infty} 0,$$

which tells us that T^n is a contraction map for large enough n . \square

證明：我們使用與定理 8.4.14 證明當中相同的記號。我們想要證明的是，那裡所定義的映射 T ，會有唯一的固定點。更確切來說，我們想要證明存在整數 $m \in \mathbb{N}$ 使得 T^m 會是個壓縮映射，然後使用習題 3.24 來總結。

令 $f, g \in \mathcal{X}$ 。我們使用與定理 8.4.14 (2) 當中類似的方法來證明。對於 $n \geq 1$ 以及 $t \in I_+$ ，讓我們定義

$$K_n(t) := \sup_{0 \leq s \leq t} \|(T^n f)(s) - (T^n g)(s)\|.$$

對於 $n \geq 2$ 還有 $s \in I_+$ ，我們有

$$\begin{aligned} \|T^n(f)(s) - T^n(g)(s)\| &= \left\| \int_0^s F(u, T^{n-1}(f)(u)) - F(u, T^{n-1}(g)(u)) \, du \right\| \\ &\leq \int_0^s L \|T^{n-1}(f)(u) - T^{n-1}(g)(u)\| \, du \\ &\leq \int_0^s L K_{n-1}(u) \, du, \end{aligned}$$

這讓我們得到

$$\forall t \in I_+, \quad K_n(t) \leq L \int_0^t K_{n-1}(u) \, du.$$

我們可以計算 K_1 如下：

$$\forall t \in I_+, \quad K_1(t) = \sup_{0 \leq s \leq t} \left\| \int_0^s F(u, f(u)) - F(u, g(u)) \, du \right\| \leq Lt \|f - g\|_\infty.$$

透過數學歸納法，我們得到對於每個 $n \geq 1$ ，我們有

$$\forall t \in I, \quad K_n(t) \leq \frac{t^n}{n!} L^n \|f - g\|_\infty \leq \frac{(a')^n}{n!} L^n \|f - g\|_\infty \xrightarrow{n \rightarrow \infty} 0,$$

這告訴我們對於夠大的 n 來說， T^n 會是個壓縮映射。 \square

8.5 Theorems on convergence of integrals

第五節 積分收斂的定理

In Proposition 8.2.5, we saw that the uniform convergence of a sequence of functions implies the uniform convergence of their primitives. As a consequence, the sequence of integrals also converges. In practice, however, we are more interested in the convergence of integrals. We have already seen in Example 8.2.8 that a sequence of integrals may converge without the sequence of integrands converges uniformly. Below we are going to prove the monotone convergence theorem (Theorem 8.5.3) and the dominated convergence theorem (Theorem 8.5.5), which are consequences of Eq. (8.26).

8.5.1 Monotone convergence theorem

We start with the following key lemma.

Lemma 8.5.1 : Let $I \subseteq \mathbb{R}$ be an interval. Let $(u_n)_{n \geq 1}$ be a sequence of piecewise continuous functions from I to a Banach space $(W, \|\cdot\|)$. Suppose that

- (i) for each $n \geq 1$, u_n is integrable on I ;
- (ii) the series of functions $\sum u_n$ converges pointwise to a piecewise continuous function $f : I \rightarrow W$;
- (iii) the series $\sum_n \int_I \|u_n\|$ converges.

Then, f is integrable on I and

$$\int_I \|f\| \leq \sum_{n \geq 1} \int_I \|u_n\|, \quad \text{and} \quad \int_I f = \sum_{n \geq 1} \int_I u_n. \quad (8.26)$$

Proof : We are going to prove this in three steps: (1) I is a segment and all the functions are continuous; (2) I is a segment and all the functions are piecewise continuous; (3) I is an interval and all the functions are piecewise continuous.

- (1) If $I = [a, b]$ is a segment, and all the u_n 's and f are continuous functions, the proof is similar to the Dini's theorem (Theorem 8.1.14).

Let $\varepsilon > 0$ and define

$$\forall n \geq 1, \quad E_n = \{x \in [a, b] : \|f(x)\| - \sum_{k=1}^n \|u_k(x)\| < \varepsilon\}. \quad (8.27)$$

在命題 8.2.5 中，我們看到函數序列的均勻收斂，會蘊含他們原函數的均勻收斂。這樣的結果會告訴我們，他所對應到的積分也會收斂。然而，在實際應用上，我們會對積分的收斂比較有興趣。在範例 8.2.8 中，我們有看過一個例子，積分序列會收斂，但是被積分函數卻不會均勻收斂。接下來，我們會證明單調收斂定理（定理 8.5.3）還有控制收斂定理（定理 8.5.5），他們可以看作是式 (8.26) 的結果。

第一小節 單調收斂定理

我們先從下面這個關鍵引理開始。

引理 8.5.1 : 令 $I \subseteq \mathbb{R}$ 為區間。令 $(u_n)_{n \geq 1}$ 為由 I 映射到 Banach 空間 $(W, \|\cdot\|)$ 的片段連續函數序列。假設

- (i) 對於每個 $n \geq 1$ ，函數 u_n 在 I 上是可積的；
- (ii) 函數級數 $\sum u_n$ 會逐點收斂到片段連續函數 $f : I \rightarrow W$ ；
- (iii) 級數 $\sum_n \int_I \|u_n\|$ 會收斂。

那麼， f 在 I 上可積，而且我們有

$$\int_I \|f\| \leq \sum_{n \geq 1} \int_I \|u_n\|, \quad \text{以及} \quad \int_I f = \sum_{n \geq 1} \int_I u_n. \quad (8.26)$$

證明：我們會分三個步驟來證明這個引理：(1) I 是個線段，而且所有函數都是連續的情況；(2) I 是個線段，而且所有函數都是片段連續的情況；(3) I 是個區間，而且所有函數都是片段連續的情況。

- (1) 如果 $I = [a, b]$ 是個線段，而且所有 u_n 和 f 都是連續的，證明與 Dini 定理（定理 8.1.14）是非常類似的。

The continuity implies that E_n is open for every $n \geq 1$. The pointwise convergence of $\sum u_n$ to f implies that $\bigcup_{n \geq 1} E_n = [a, b]$. Since $[a, b]$ is a compact set, by the Borel–Lebesgue property, we may find $N \geq 1$ such that $\bigcup_{n=1}^N E_n = [a, b]$. Therefore, we have

$$\int_{[a,b]} \|f\| \leq \int_{[a,b]} \left(\sum_{k=1}^N \|u_k\| + \varepsilon \right) = \sum_{k=1}^N \int_{[a,b]} \|u_k\| + \varepsilon(b-a) \leq \sum_{n \geq 1} \int_{[a,b]} \|u_n\| + \varepsilon(b-a).$$

The above inequality holds for any arbitrary $\varepsilon > 0$, so we deduce that

$$\int_{[a,b]} \|f\| \leq \sum_{n \geq 1} \int_{[a,b]} \|u_n\|.$$

(2) Next, we suppose that $I = [a, b]$ is a segment, and all the u_n 's and f are piecewise continuous.

Let $\varepsilon > 0$. From Lemma 8.5.2, we may find continuous functions g and $(v_n)_{n \geq 1}$ such that

$$\begin{aligned} g &\leq \|f\| \quad \text{such that} \quad \int_I \|f\| \leq \varepsilon + \int_I g, \\ \forall n \geq 1, \|u_n\| &\leq v_n \quad \text{such that} \quad \int_I v_n \leq \frac{\varepsilon}{2^n} + \int_I \|u_n\|. \end{aligned}$$

Define the following subsets as in Eq. (8.27), but for the continuous functions g and $(v_n)_{n \geq 1}$,

$$\forall n \geq 1, \quad G_n = \left\{ x \in [a, b] : g(x) - \sum_{k=1}^n v_k(x) < \varepsilon \right\}.$$

Similarly, we know that there exists $N \geq 1$ such that $\bigcup_{n=1}^N G_n = [a, b]$. Therefore, we find

$$\begin{aligned} \int_I \|f\| &\leq \varepsilon + \int_I g \leq \varepsilon + \int_I \left(\sum_{k=1}^N v_k + \varepsilon \right) = (b-a+1)\varepsilon + \sum_{k=1}^N \int_I v_k \\ &\leq (b-a+1)\varepsilon + \sum_{k=1}^N \left(\frac{\varepsilon}{2^k} + \int_I \|u_k\| \right) \leq (b-a+2)\varepsilon + \sum_{k=1}^N \int_I \|u_k\| \\ &\leq (b-a+2)\varepsilon + \sum_{n \geq 1} \int_I \|u_n\|. \end{aligned}$$

Then we conclude as in the previous point.

令 $\varepsilon > 0$ 並定義

$$\forall n \geq 1, \quad E_n = \left\{ x \in [a, b] : \|f(x)\| - \sum_{k=1}^n \|u_k(x)\| < \varepsilon \right\}. \quad (8.27)$$

函數的連續性告訴我們對於每個 $n \geq 1$, E_n 是個開集。 $\sum u_n$ 會逐點收斂到 f , 所以 $\bigcup_{n \geq 1} E_n = [a, b]$ 。由於 $[a, b]$ 是個緊緻集合, 根據 Borel–Lebesgue 性質, 我們能找到 $N \geq 1$ 使得 $\bigcup_{n=1}^N E_n = [a, b]$ 。因此, 我們有

$$\int_{[a,b]} \|f\| \leq \int_{[a,b]} \left(\sum_{k=1}^N \|u_k\| + \varepsilon \right) = \sum_{k=1}^N \int_{[a,b]} \|u_k\| + \varepsilon(b-a) \leq \sum_{n \geq 1} \int_{[a,b]} \|u_n\| + \varepsilon(b-a).$$

由於上面這個不等式對於任意 $\varepsilon > 0$ 皆會是對的, 我們推得

$$\int_{[a,b]} \|f\| \leq \sum_{n \geq 1} \int_{[a,b]} \|u_n\|.$$

(2) 再來, 我們假設 $I = [a, b]$ 是個線段, 然後所有的 u_n 和 f 都是片段連續的。令 $\varepsilon > 0$ 。從

引理 8.5.2, 我們能找到連續函數 g 和 $(v_n)_{n \geq 1}$ 滿足

$$\begin{aligned} g &\leq \|f\| \quad \text{使得} \quad \int_I \|f\| \leq \varepsilon + \int_I g, \\ \forall n \geq 1, \|u_n\| &\leq v_n \quad \text{使得} \quad \int_I v_n \leq \frac{\varepsilon}{2^n} + \int_I \|u_n\|. \end{aligned}$$

如同在式 (8.27) 中, 我們對連續函數 g 和 $(v_n)_{n \geq 1}$ 定義下面這些子集合:

$$\forall n \geq 1, \quad G_n = \left\{ x \in [a, b] : g(x) - \sum_{k=1}^n v_k(x) < \varepsilon \right\}.$$

同理, 我們知道存在 $N \geq 1$ 使得 $\bigcup_{n=1}^N G_n = [a, b]$ 。因此, 我們有

$$\begin{aligned} \int_I \|f\| &\leq \varepsilon + \int_I g \leq \varepsilon + \int_I \left(\sum_{k=1}^N v_k + \varepsilon \right) = (b-a+1)\varepsilon + \sum_{k=1}^N \int_I v_k \\ &\leq (b-a+1)\varepsilon + \sum_{k=1}^N \left(\frac{\varepsilon}{2^k} + \int_I \|u_k\| \right) \leq (b-a+2)\varepsilon + \sum_{k=1}^N \int_I \|u_k\| \\ &\leq (b-a+2)\varepsilon + \sum_{n \geq 1} \int_I \|u_n\|. \end{aligned}$$

最後我們可以使用和上面相同的方法來總結。

(3) For any subsegment $J \subseteq I$, from above, we have

$$\int_J \|f\| \leq \sum_{n \geq 1} \int_J \|u_n\| \leq \sum_{n \geq 1} \int_I \|u_n\| < \infty.$$

Therefore, f is integrable on I and satisfies

$$\int_I \|f\| \leq \sum_{n \geq 1} \int_I \|u_n\| < \infty,$$

which is the first part of Eq. (8.26).

For the second part of Eq. (8.26), let us apply the first part to the remainder $\sum_{k \geq n+1} u_k = f - \sum_{k=1}^n u_k$, and we find

$$\int_I \left\| f - \sum_{k=1}^n u_k \right\| \leq \sum_{k \geq n+1} \int_I \|u_k\| \xrightarrow{n \rightarrow \infty} 0,$$

since the right side in the above relation is the remainder of the convergent series $\sum \int_I \|u_n\|$.

Then, it follows that

$$\left\| \int_I f - \sum_{k=1}^n \int_I u_k \right\| = \left\| \int_I \left(f - \sum_{k=1}^n u_k \right) \right\| \leq \int_I \left\| f - \sum_{k=1}^n u_k \right\| \xrightarrow{n \rightarrow \infty} 0,$$

which gives us the relation

$$\int_I f = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_I u_k = \sum_{n \geq 1} \int_I u_n.$$

□

Lemma 8.5.2 : Let $J = [a, b]$ be a segment of \mathbb{R} and $f \in \mathcal{PC}(J, \mathbb{R})$. For every $\varepsilon > 0$, there exists continuous functions f_- and f_+ on J such that

$$f_- \leq f \leq f_+ \quad \text{and} \quad \left(\int_J f_+ \right) - \varepsilon \leq \int_J f \leq \left(\int_J f_- \right) + \varepsilon.$$

Proof : If f is continuous, then there is nothing to prove. Suppose that f has discontinuities. Let $P = (x_k)_{0 \leq k \leq n}$ be a partition of $[a, b]$ such that f restricted on (x_{k-1}, x_k) can be extended to a continuous function on $[x_{k-1}, x_k]$ for every $1 \leq k \leq n$. From Proposition 7.1.3, we know that f is bounded on

(3) 對於任意子線段 $J \subseteq I$ ，從上面的討論，我們得到

$$\int_J \|f\| \leq \sum_{n \geq 1} \int_J \|u_n\| \leq \sum_{n \geq 1} \int_I \|u_n\| < \infty.$$

因此， f 在 I 上是可積的，而且滿足

$$\int_I \|f\| \leq \sum_{n \geq 1} \int_I \|u_n\| < \infty,$$

這會是式 (8.26) 的第一部份。

再來，我們要證明式 (8.26) 的第二部份。我們把第一部份的結果用在餘項 $\sum_{k \geq n+1} u_k = f - \sum_{k=1}^n u_k$ 上，進而得到

$$\int_I \left\| f - \sum_{k=1}^n u_k \right\| \leq \sum_{k \geq n+1} \int_I \|u_k\| \xrightarrow{n \rightarrow \infty} 0,$$

因為上式右方會是收斂級數 $\sum \int_I \|u_n\|$ 的餘項。那麼，我們會有

$$\left\| \int_I f - \sum_{k=1}^n \int_I u_k \right\| = \left\| \int_I \left(f - \sum_{k=1}^n u_k \right) \right\| \leq \int_I \left\| f - \sum_{k=1}^n u_k \right\| \xrightarrow{n \rightarrow \infty} 0,$$

這給我們關係式：

$$\int_I f = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_I u_k = \sum_{n \geq 1} \int_I u_n.$$

□

引理 8.5.2 : 令 $J = [a, b]$ 為 \mathbb{R} 的線段，以及 $f \in \mathcal{PC}(J, \mathbb{R})$ 。對於每個 $\varepsilon > 0$ ，存在 J 上的連續函數 f_- 和 f_+ 滿足

$$f_- \leq f \leq f_+ \quad \text{以及} \quad \left(\int_J f_+ \right) - \varepsilon \leq \int_J f \leq \left(\int_J f_- \right) + \varepsilon.$$

證明：如果 f 是連續的，則沒有什麼需要證明的。假設 f 有不連續點。令 $P = (x_k)_{0 \leq k \leq n}$ 為 $[a, b]$ 的分割，使得對於每個 $1 \leq k \leq n$ 來說，當 f 限制在 (x_{k-1}, x_k) 上時，可以被拓延為在

$[a, b]$, so we may take

$$M > \sup_{x \in J} f(x) \quad \text{and} \quad m < \inf_{x \in J} f(x).$$

Let $\delta > 0$ with $\delta < \frac{1}{2} \|P\|$, so that we may define disjoint intervals $J_i := B(x_i, \delta) \cap J$ for $0 \leq i \leq n$.

We define a continuous function φ_- on J as below,

$$\varphi_-(x) = \begin{cases} m + (M - m) \frac{|x - x_i|}{\delta} & \text{if } x \in J_i, \\ M & \text{otherwise.} \end{cases}$$

Then, the function $f_- := \min(f, \varphi_-)$ satisfies $f_- \leq f$ on J is continuous. In fact, we can see that

- if $x \neq x_i$ for all i , then f is continuous at x , and $f_- = \frac{1}{2}(f + \varphi_- - |f - \varphi_-|)$ is also continuous at x ;
- if $x = x_i$ for some i , then $\varphi_-(x) = m < \inf_{x \in J} f(x)$, so we may find $\varepsilon > 0$ such that φ_- stays strictly below f on $B(x, \varepsilon)$. This means that $f_- = \varphi_-$ on $B(x, \varepsilon)$, so we get the continuity of f_- at x .

Then, let us compute the following integral,

$$\int_J (f - f_-) = \sum_{i=1}^n \int_{J_i} (f - f_-) \leq \sum_{i=1}^n \int_{J_i} (M - m) \leq 2\delta n(M - m),$$

where the equality is obtained from the fact that when $x \notin J_i$ for all i , $\varphi_-(x) = M > f(x)$, so $f_-(x) = f(x)$. To conclude, for $\varepsilon > 0$, we may choose $\delta \leq \min\{\frac{\varepsilon}{2(M-m)n}, \frac{1}{4} \|P\|\}$, which will give us

$$\int_J (f - f_-) \leq \varepsilon \quad \Leftrightarrow \quad \int_J f \leq \left(\int_J f_- \right) + \varepsilon.$$

For the construction of f_+ , we proceed in a similar way. We consider the following continuous function φ_+ on J ,

$$\varphi_+(x) = \begin{cases} M - (M - m) \frac{|x - x_i|}{\delta} & \text{if } x \in J_i, \\ m & \text{otherwise.} \end{cases}$$

Then, we define $f_+ := \max(f, \varphi_+)$. □

$[x_{k-1}, x_k]$ 上的連續函數。從命題 7.1.3 我們得知 f 在 $[a, b]$ 上是有界的，所以我們可以取

$$M > \sup_{x \in J} f(x) \quad \text{以及} \quad m < \inf_{x \in J} f(x).$$

令 $\delta > 0$ 滿足 $\delta < \frac{1}{2} \|P\|$ ，所以我們可以定義互斥區間 $J_i := B(x_i, \delta) \cap J$ ，其中 $0 \leq i \leq n$ 。

我們定義在 J 上的連續函數 φ_- 如下：

$$\varphi_-(x) = \begin{cases} m + (M - m) \frac{|x - x_i|}{\delta} & \text{若 } x \in J_i, \\ M & \text{其他情況.} \end{cases}$$

那麼，函數 $f_- := \min(f, \varphi_-)$ 在 J 上滿足 $f_- \leq f$ ，也會是連續的。實際上，我們可以看到

- 如果 $x \neq x_i$ 對於所有 i ，那麼 f 在 x 連續，而且 $f_- = \frac{1}{2}(f + \varphi_- - |f - \varphi_-|)$ 也會在 x 點連續；
- 如果 $x = x_i$ 對於某個 i ，那麼 $\varphi_-(x) = m < \inf_{x \in J} f(x)$ ，所以我們能找到 $\varepsilon > 0$ 使得 φ_- 在 $B(x, \varepsilon)$ 上會嚴格保持在 f 之下。這代表著在 $B(x, \varepsilon)$ 上我們有 $f_- = \varphi_-$ ，所以我們得到 f_- 在 x 點的連續性。

再來，讓我們計算下列積分：

$$\int_J (f - f_-) = \sum_{i=1}^n \int_{J_i} (f - f_-) \leq \sum_{i=1}^n \int_{J_i} (M - m) \leq 2\delta n(M - m),$$

其中的等式是因為當 $x \notin J_i$ 對於所有 i ，我們會有 $\varphi_-(x) = M > f(x)$ ，所以 $f_-(x) = f(x)$ 。接著我們來總結：對於 $\varepsilon > 0$ ，我們可以選擇 $\delta \leq \min\{\frac{\varepsilon}{2(M-m)n}, \frac{1}{4} \|P\|\}$ ，這會給我們

$$\int_J (f - f_-) \leq \varepsilon \quad \Leftrightarrow \quad \int_J f \leq \left(\int_J f_- \right) + \varepsilon.$$

再來我們可以使用類似的方法來構造 f_+ 。我們考慮在 J 上的連續函數 φ_+ ：

$$\varphi_+(x) = \begin{cases} M - (M - m) \frac{|x - x_i|}{\delta} & \text{若 } x \in J_i, \\ m & \text{其他情況.} \end{cases}$$

然後定義 $f_+ := \max(f, \varphi_+)$ 。 □

Theorem 8.5.3 (Monotone convergence theorem): Let $I \subseteq \mathbb{R}$ be an interval. Let $(f_n)_{n \geq 1}$ be a sequence of non-negative, piecewise continuous, and integrable functions from I to \mathbb{R}_+ . Suppose that

- (i) for every $x \in I$ and $n \geq 1$, we have $f_n(x) \leq f_{n+1}(x)$;
- (ii) $(f_n)_{n \geq 1}$ converges pointwise to a piecewise continuous function f ;
- (iii) $\int_I f_n$ converges when $n \rightarrow \infty$.

Then,

$$\int_I |f_n - f| \xrightarrow{n \rightarrow \infty} 0, \quad \text{and} \quad \int_I f_n \xrightarrow{n \rightarrow \infty} \int_I f.$$

Remark 8.5.4: We note that this theorem is very similar to Dini's theorem (Theorem 8.1.14), with the following differences.

- (1) We make a weaker assumption in Theorem 8.5.3, which is piecewise continuity.
- (2) We do not get the uniform convergence of the sequence of functions $(f_n)_{n \geq 1}$ to deduce the convergence of the integrals. Actually, we *do not* have the uniform convergence here in general, whereas the convergence of integrals still holds.

Proof: It is a special case of Eq. (8.26). For every $n \geq 1$, let $u_n = f_{n+1} - f_n \geq 0$. We may check the following properties.

- (i) For every $n \geq 1$, u_n is integrable because both f_{n+1} and f_n are integrable.
- (ii) $\sum u_n = \sum (f_{n+1} - f_n)$ converges pointwise to a piecewise continuous function because $(f_n)_{n \geq 1}$ converges pointwise to a piecewise continuous function.
- (iii) We have

$$\sum_{n=1}^N \int_I |u_n| = \sum_{n=1}^N \int_I (f_{n+1} - f_n) = \int_I f_{N+1} - \int_I f_1,$$

where the right side can be uniformly bounded from above due to the convergence of $\int_I f_n$. This shows that $\sum \int_I |u_n|$ converges.

定理 8.5.3 【單調收斂定理】：令 $I \subseteq \mathbb{R}$ 為區間。令 $(f_n)_{n \geq 1}$ 為由 I 映射至 \mathbb{R}_+ 的非負、片段連續且可積函數所構成的序列。假設

- (i) 對於每個 $x \in I$ 還有 $n \geq 1$ ，我們有 $f_n(x) \leq f_{n+1}(x)$ ；
- (ii) $(f_n)_{n \geq 1}$ 會逐點收斂到片段連續函數 f ；
- (iii) 當 $n \rightarrow \infty$ 時， $\int_I f_n$ 會收斂。

那麼，我們有

$$\int_I |f_n - f| \xrightarrow{n \rightarrow \infty} 0, \quad \text{以及} \quad \int_I f_n \xrightarrow{n \rightarrow \infty} \int_I f.$$

註解 8.5.4：我們注意到，這個定理與 Dini 定理（定理 8.1.14）非常類似，差別如下。

- (1) 定理 8.5.3 中的假設比較弱，只需要片段連續性。
- (2) 我們不會使用函數序列 $(f_n)_{n \geq 1}$ 的均勻收斂性來推得積分的收斂。事實上，一般來講，均勻收斂不會成立，但積分還是會收斂。

證明：這是個式 (8.26) 的特例。對於每個 $n \geq 1$ ，令 $u_n = f_{n+1} - f_n \geq 0$ 。我們可以檢查下列性質。

- (i) 對於每個 $n \geq 1$ ，函數 u_n 是可積的，因為 f_{n+1} 和 f_n 皆是可積的。
- (ii) $\sum u_n = \sum (f_{n+1} - f_n)$ 會逐點收斂到片段連續函數，因為 $(f_n)_{n \geq 1}$ 會逐點收斂到片段連續函數。
- (iii) 我們有

$$\sum_{n=1}^N \int_I |u_n| = \sum_{n=1}^N \int_I (f_{n+1} - f_n) = \int_I f_{N+1} - \int_I f_1,$$

其中因為 $\int_I f_n$ 會收斂，右式可以被均勻的上界控制住。這證明了 $\sum \int_I |u_n|$ 會收斂。

Therefore, we may apply Eq. (8.26) to conclude that f is integrable on I and

$$\int_I |f_n - f| = \int_I \left| \sum_{k \geq n} u_k \right| \leq \sum_{k \geq n} \int_I |u_k|.$$

The right side in the above inequality is the remainder of a convergent series, so goes to 0 when n goes to ∞ . \square

8.5.2 Dominated convergence theorem

Theorem 8.5.5 (Dominated convergence theorem) : Let $I \subseteq \mathbb{R}$ be an interval and W be a Banach space. Let $(f_n)_{n \geq 1}$ be a sequence of piecewise continuous functions from I to W . Suppose that

- (i) There exists a piecewise continuous non-negative integrable function $\varphi : I \rightarrow \mathbb{R}_+$ such that $\|f_n\| \leq \varphi$ for every $n \geq 1$.
- (ii) The sequence $(f_n)_{n \geq 1}$ converges pointwise to a piecewise continuous function $f : I \rightarrow W$.

Then, each f_n and f are integrable on I and we have

$$\lim_{n \rightarrow \infty} \int_I \|f_n - f\| \xrightarrow{n \rightarrow \infty} 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_I f_n = \int_I f.$$

Proof : Suppose that the theorem holds when $(W, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $(f_n)_{n \geq 1}$ are non-negative functions, and $f \equiv 0$ is the zero function. For all $n \geq 1$, let $h_n = \|f_n - f\|$, which is still a piecewise continuous function on I . Then, $h_n \leq 2\varphi$ and $(h_n)_{n \geq 1}$ converges pointwise to the zero function. So we find

$$\left\| \int_I f_n - \int_I f \right\| \leq \int_I \|f_n - f\| = \int_I h_n \xrightarrow{n \rightarrow \infty} \int_I 0 = 0.$$

Now, let us prove the theorem with the assumption that $(W, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $(f_n)_{n \geq 1}$ are non-negative functions, and f is the zero function. For every $n \geq 1$ and $p \geq n$, let

$$f_{n,p} := \max\{f_n, f_{n+1}, \dots, f_p\},$$

which is still a piecewise continuous function and satisfies $f_{n,p} \leq \varphi$.

- Fix $n \geq 1$. Since $(f_{n,p})_{p \geq n}$ is an increasing sequence, the sequence $(I_{n,p})_{p \geq n}$ defined by $I_{n,p} =$

因此，我們可以使用式 (8.26) 來總結 f 在 I 上是可積的，以及

$$\int_I |f_n - f| = \int_I \left| \sum_{k \geq n} u_k \right| \leq \sum_{k \geq n} \int_I |u_k|.$$

上面不等式的右式，會是個收斂級數的餘項，所以當 n 趨近於 ∞ 時，會收斂到 0。 \square

第二小節 控制收斂定理

定理 8.5.5 【控制收斂定理】 : 令 $I \subseteq \mathbb{R}$ 為區間，且 W 為 Banach 空間。令 $(f_n)_{n \geq 1}$ 為由 I 映射至 W 的片段連續函數序列。假設

- (i) 存在非負片段連續可積函數 $\varphi : I \rightarrow \mathbb{R}_+$ 使得對於所有 $n \geq 1$ ，我們有 $\|f_n\| \leq \varphi$ 。
- (ii) 序列 $(f_n)_{n \geq 1}$ 會逐點收斂到片段連續函數 $f : I \rightarrow W$ 。

那麼，所有的 f_n 和 f 在 I 上都是可積的，且我們有

$$\lim_{n \rightarrow \infty} \int_I \|f_n - f\| \xrightarrow{n \rightarrow \infty} 0, \quad \text{以及} \quad \lim_{n \rightarrow \infty} \int_I f_n = \int_I f.$$

$\int_I f_{n,p}$ is increasing. Since $I_{n,p} \leq \int_I \varphi$ for all $p \geq n$, the sequence $(I_{n,p})_{p \geq n}$ converges, so it satisfies Cauchy's property. We may find $p_n \geq 1$ such that

$$|I_{n,p} - I_{n,q}| \leq 2^{-n}, \quad \forall p, q \geq p_n.$$

It is possible to make a choice of $(p_n)_{n \geq 1}$ such that it is an extraction (strictly increasing sequence).

- For $n \geq 1$, let $g_n = f_{n,p_n}$. We note that g_n converges pointwise to 0 (Cauchy's criterion at each point of I). For any $n \geq 1$, we have

$$|g_{n+1} - g_n| + (g_{n+1} - g_n) = \begin{cases} 0 & \text{if } g_{n+1} - g_n \leq 0, \\ 2(g_{n+1} - g_n) & \text{otherwise.} \end{cases}$$

Additionally, for any $n \geq 1$, we also have $g_{n+1} - g_n = f_{n+1,p_{n+1}} - f_{n,p_n} \leq f_{n,p_{n+1}} - f_{n,p_n}$ and $0 \leq f_{n,p_{n+1}} - f_{n,p_n}$. Therefore, we find

$$\forall n \geq 1, \quad |g_{n+1} - g_n| \leq 2(f_{n,p_{n+1}} - f_{n,p_n}) - (g_{n+1} - g_n).$$

- For $n \geq 1$, let $u_n = g_n - g_{n+1}$. Then, we have

$$\forall n \geq 1, \quad \int_I |u_n| \leq 2|I_{n,p_{n+1}} - I_{n,p_n}| + \int_I g_n - \int_I g_{n+1} \leq 2^{1-n} + \int_I g_n - \int_I g_{n+1}.$$

By taking a summation, we find,

$$\forall p \geq n \geq 1, \quad \sum_{k=n}^p \int_I |u_k| \leq \sum_{k=n}^p 2^{1-k} + \int_I g_n - \int_I g_{p+1} \leq 2 + \int_I g_n.$$

In the above formula, we see that the upper bound does not depend on p . Since the left side contains only positive terms in the series, we deduce that the series $\sum_{k \geq n} \int_I |u_k|$ converges.

From what we have shown above, and the fact that g_n converges pointwise to 0, we have $\sum_{k \geq n} u_k = g_n$.

This allows us to apply Eq. (8.26),

$$\forall n \geq 1, \quad 0 \leq \int_I f_n \leq \int_I g_n = \int_I \left(\sum_{k \geq n} u_k \right) = \sum_{k \geq n} \int_I u_k.$$

The rightmost term in the above formula is the remainder of an absolutely convergent series, so its limit when n tends to ∞ is zero. This shows that $\int_I f_n \xrightarrow{n \rightarrow \infty} 0$. \square

證明：假設在 $(W, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ 、 $(f_n)_{n \geq 1}$ 是非負函數，且 $f \equiv 0$ 是個零函數的情況下會成立。

對於所有 $n \geq 1$ ，令 $h_n = \|f_n - f\|$ ，這還會是個在 I 上的片段連續函數。那麼 $h_n \leq 2\varphi$ 而且 $(h_n)_{n \geq 1}$ 會逐點收斂至零函數。所以我們得到

$$\left\| \int_I f_n - \int_I f \right\| \leq \int_I \|f_n - f\| = \int_I h_n \xrightarrow{n \rightarrow \infty} \int_I 0 = 0.$$

現在，讓我們證明在 $(W, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ 、 $(f_n)_{n \geq 1}$ 都是非負函數，而且 f 是零函數的假設之下，來證明定理。對於每個 $n \geq 1$ 以及 $p \geq n$ ，我們令

$$f_{n,p} := \max\{f_n, f_{n+1}, \dots, f_p\},$$

這還是個片段連續函數，而且滿足 $f_{n,p} \leq \varphi$ 。

- 固定 $n \geq 1$ 。由於 $(f_{n,p})_{p \geq n}$ 是個遞增序列，序列 $(I_{n,p})_{p \geq n}$ 定義做 $I_{n,p} = \int_I f_{n,p}$ 是遞增的。由於對於所有 $p \geq n$ ，我們有 $I_{n,p} \leq \int_I \varphi$ ，序列 $(I_{n,p})_{p \geq n}$ 會收斂，所以他滿足柯西性質。我們能找到 $p_n \geq 1$ 使得

$$|I_{n,p} - I_{n,q}| \leq 2^{-n}, \quad \forall p, q \geq p_n.$$

我們可以選擇 $(p_n)_{n \geq 1}$ 使得他是個萃取函數（嚴格遞增序列）。

- 對於 $n \geq 1$ ，令 $g_n = f_{n,p_n}$ 。我們注意到 g_n 會逐點收斂到 0（在 I 上的每個點，柯西準則都會成立）。對於任意 $n \geq 1$ ，我們有

$$|g_{n+1} - g_n| + (g_{n+1} - g_n) = \begin{cases} 0 & \text{若 } g_{n+1} - g_n \leq 0, \\ 2(g_{n+1} - g_n) & \text{其他情況.} \end{cases}$$

此外，對於任意 $n \geq 1$ ，我們也有 $g_{n+1} - g_n = f_{n+1,p_{n+1}} - f_{n,p_n} \leq f_{n,p_{n+1}} - f_{n,p_n}$ 以及 $0 \leq f_{n,p_{n+1}} - f_{n,p_n}$ 。因此，我們得到

$$\forall n \geq 1, \quad |g_{n+1} - g_n| \leq 2(f_{n,p_{n+1}} - f_{n,p_n}) - (g_{n+1} - g_n).$$

Example 8.5.6 : For every $n \in \mathbb{N}$, consider the function

$$\begin{array}{ccc} f_n : (1, +\infty) & \rightarrow & \mathbb{R} \\ t & \mapsto & \frac{1+t^n}{1+t^{n+2}} \end{array} \quad \text{and} \quad I_n = \int_1^\infty f_n(t) dt.$$

We can check the following properties.

- For every $n \in \mathbb{N}$, the function f_n is piecewise continuous.
- For every $t > 1$, we have

$$f_n(t) = \frac{1+t^n}{1+t^{n+2}} \sim \frac{1}{t^2}, \quad \text{when } n \rightarrow \infty.$$

So the sequence f_n converges pointwise to the function $t \mapsto \frac{1}{t^2}$, which is piecewise continuous on $(1, +\infty)$.

- 對於 $n \geq 1$, 令 $u_n = g_n - g_{n+1}$ 。那麼我們有

$$\forall n \geq 1, \quad \int_I |u_n| \leq 2|I_{n,p_{n+1}} - I_{n,p_n}| + \int_I g_n - \int_I g_{n+1} \leq 2^{1-n} + \int_I g_n - \int_I g_{n+1}.$$

對上式取和，我們會得到

$$\forall p \geq n \geq 1, \quad \sum_{k=n}^p \int_I |u_k| \leq \sum_{k=n}^p 2^{1-k} + \int_I g_n - \int_I g_{p+1} \leq 2 + \int_I g_n.$$

在上面的式子中，我們看到上界並不取決於 p 。由於左方級數中只有正的項，我們推得級數 $\sum_{k \geq n} \int_I |u_k|$ 會收斂。

從我們上面所證明的，以及 g_n 會逐點收斂到 0 的性質，我們有 $\sum_{k \geq n} u_k = g_n$ 。這讓我們可以使用式 (8.26)：

$$\forall n \geq 1, \quad 0 \leq \int_I f_n \leq \int_I g_n = \int_I \left(\sum_{k \geq n} u_k \right) = \sum_{k \geq n} \int_I u_k.$$

上式中最右邊的項會是絕對收斂級數的餘項，所以當 n 趨近於 ∞ 時，他的極限會为零。這證明了 $\int_I f_n \xrightarrow{n \rightarrow \infty} 0$ 。□

範例 8.5.6 : 對於每個 $n \in \mathbb{N}$ ，考慮函數

$$\begin{array}{ccc} f_n : (1, +\infty) & \rightarrow & \mathbb{R} \\ t & \mapsto & \frac{1+t^n}{1+t^{n+2}} \end{array} \quad \text{以及} \quad I_n = \int_1^\infty f_n(t) dt.$$

我們可以檢查下列性質。

- 對於每個 $n \in \mathbb{N}$ ，函數 f_n 是片段連續的。
- 對於每個 $t > 1$ ，我們有

$$f_n(t) = \frac{1+t^n}{1+t^{n+2}} \sim \frac{1}{t^2}, \quad \text{當 } n \rightarrow \infty.$$

所以序列 f_n 會逐點收斂到函數 $t \mapsto \frac{1}{t^2}$ ，這是個在 $(1, +\infty)$ 上片段連續的函數。

- (Domination assumption) For every $n \in \mathbb{N}$ and $t > 1$, we have

$$|f_n(t)| = \frac{1+t^n}{1+t^{n+2}} \leq \frac{t^n+t^n}{t^{n+2}} = \frac{2}{t^2}.$$

The function $t \mapsto \frac{2}{t^2}$ is integrable on $(1, +\infty)$, so the domination assumption is satisfied. Therefore, we may apply the dominated convergence theorem from Theorem 8.5.5, giving us

$$I_n \xrightarrow{n \rightarrow \infty} \int_1^\infty \frac{dt}{t^2} = 1.$$

Example 8.5.7 : For every $n \in \mathbb{N}$, consider the function

$$\begin{array}{ccc} f_n : [0, 1) & \rightarrow & \mathbb{R} \\ t & \mapsto & n^2 t^{n-1} \end{array} \quad \text{and} \quad I_n = \int_0^1 f_n(t) dt.$$

For every $n \in \mathbb{N}$, the function f_n is continuous and integrable on $[0, 1)$. For every $t \in [0, 1)$, we have $f_n(t) \xrightarrow{n \rightarrow \infty} 0$, which implies that the sequence $(f_n)_{n \geq 1}$ converges pointwise on $[0, 1)$ to the zero function. However, we have

$$\forall n \in \mathbb{N}, \quad I_n = [nt^n]_0^1 = n.$$

This shows that the order of the limit and the integration procedure cannot be interchanged,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(t) dt \neq \int_0^1 \lim_{n \rightarrow \infty} f_n(t) dt = 0.$$

The reason is that the domination assumption is not satisfied.

To be more precise, if φ is a function that dominates all the f_n 's, then for $t \in [0, 1)$, we need to have $\varphi(t) \geq f_n(t)$ for all $n \geq 1$. In particular, for $t \in [0, 1)$, we may choose $n = \lfloor \frac{2}{|\ln t|} \rfloor$, then, for $t \rightarrow 1-$, we have the following relation,

$$\begin{aligned} \ln f_n(t) &= 2 \ln n + (n-1) \ln t \\ &\geq 2 \ln \left(\frac{2}{|\ln t|} - 1 \right) + \left(\frac{2}{|\ln t|} - 1 \right) \ln t \\ &= -\ln t - 2 \ln |\ln t| + \mathcal{O}(1). \end{aligned}$$

- 【控制假設】對於每個 $n \in \mathbb{N}$ 和 $t > 1$ ，我們有

$$|f_n(t)| = \frac{1+t^n}{1+t^{n+2}} \leq \frac{t^n+t^n}{t^{n+2}} = \frac{2}{t^2}.$$

函數 $t \mapsto \frac{2}{t^2}$ 在 $(1, +\infty)$ 上是可積的，所以控制假設會滿足。因此，我們可以使用定理 8.5.5 中的控制收斂定理，得到

$$I_n \xrightarrow{n \rightarrow \infty} \int_1^\infty \frac{dt}{t^2} = 1.$$

範例 8.5.7 : 對於每個 $n \in \mathbb{N}$ ，考慮函數

$$\begin{array}{ccc} f_n : [0, 1) & \rightarrow & \mathbb{R} \\ t & \mapsto & n^2 t^{n-1} \end{array} \quad \text{以及} \quad I_n = \int_0^1 f_n(t) dt.$$

對於每個 $n \in \mathbb{N}$ ，函數 f_n 會在 $[0, 1)$ 上連續且可積。對於每個 $t \in [0, 1)$ ，我們有 $f_n(t) \xrightarrow{n \rightarrow \infty} 0$ ，這蘊含序列 $(f_n)_{n \geq 1}$ 會在 $[0, 1)$ 上逐點收斂到零函數。然而，我們有

$$\forall n \in \mathbb{N}, \quad I_n = [nt^n]_0^1 = n.$$

這證明了極限和積分的順序是無法交換的。

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(t) dt \neq \int_0^1 \lim_{n \rightarrow \infty} f_n(t) dt = 0.$$

原因是因為控制假設沒有成立。

更確切來說，如果 φ 是個會比所有 f_n 都來得大的函數，那麼對於 $t \in [0, 1)$ ，我們會有 $\varphi(t) \geq f_n(t)$ 對於所有 $n \geq 1$ 。特別來說，對於 $t \in [0, 1)$ ，我們可以取 $n = \lfloor \frac{2}{|\ln t|} \rfloor$ ，那麼，當 $t \rightarrow 1-$ 時，我們會有下面這個關係式：

$$\begin{aligned} \ln f_n(t) &= 2 \ln n + (n-1) \ln t \\ &\geq 2 \ln \left(\frac{2}{|\ln t|} - 1 \right) + \left(\frac{2}{|\ln t|} - 1 \right) \ln t \\ &= -\ln t - 2 \ln |\ln t| + \mathcal{O}(1). \end{aligned}$$

This means that when $t \rightarrow 1-$, we have

$$f_n(t) \geq \frac{\text{cst}}{t|\ln t|^2},$$

which implies that φ is not integrable around $1-$.

這代表著當 $t \rightarrow 1-$ 時，我們有

$$f_n(t) \geq \frac{\text{cst}}{t|\ln t|^2},$$

這蘊含 φ 在 $1-$ 附近不可積。

8.5.3 Applications: integrals with an additional parameter

We give a few important applications of the dominated convergence theorem. Let us consider a general interval $I \subseteq \mathbb{R}$, with endpoints a and b satisfying $-\infty \leq a < b \leq +\infty$, and a Banach space $(W, \|\cdot\|)$.

Theorem 8.5.8 (Continuity under integration) : Let (M, d) be a metric space and a map $f : M \times I \rightarrow W$ satisfying the following conditions.

- (i) For every $x \in M$, the map $f(x, \cdot) : t \mapsto f(x, t)$ is piecewise continuous on I .
- (ii) For every $t \in I$, the map $f(\cdot, t) : x \mapsto f(x, t)$ is continuous on M .
- (iii) (Domination assumption) There exists a non-negative, piecewise continuous, and integrable function $\varphi : I \rightarrow \mathbb{R}_+$ such that $\|f(x, t)\| \leq \varphi(t)$ for all $x \in M$ and $t \in I$.

Then, the map

$$\begin{aligned} F : M &\rightarrow W \\ x &\mapsto \int_a^b f(x, t) dt \end{aligned}$$

is well-defined and continuous on M .

第三小節 應用：有額外參數的積分

下面是一些控制收斂定理的重要應用。讓我們考慮一般的區間 $I \subseteq \mathbb{R}$ ，他的端點記作 a 和 b 而且滿足 $-\infty \leq a < b \leq +\infty$ ，以及 Banach 空間 $(W, \|\cdot\|)$ 。

定理 8.5.8 【積分下的連續性】：令 (M, d) 為賦距空間，且 $f : M \times I \rightarrow W$ 為滿足下列條件的映射。

- (i) 對於每個 $x \in M$ ，映射 $f(x, \cdot) : t \mapsto f(x, t)$ 在 I 上會片段連續。
- (ii) 對於每個 $t \in I$ ，映射 $f(\cdot, t) : x \mapsto f(x, t)$ 在 M 上連續。
- (iii) 【控制假設】存在非負、片段連續且可積函數 $\varphi : I \rightarrow \mathbb{R}_+$ 使得對於所有 $x \in M$ 還有 $t \in I$ ，我們有 $\|f(x, t)\| \leq \varphi(t)$ 。

那麼，映射

$$\begin{aligned} F : M &\rightarrow W \\ x &\mapsto \int_a^b f(x, t) dt \end{aligned}$$

是定義良好的，且在 M 上連續。

Proof : The assumption (iii), the domination assumption, shows that the function $f(x, \cdot)$ is integrable for every $x \in M$, so the map F is well defined. For a given $x \in M$, to check that F is continuous at x , we need to check that for any sequence $(x_n)_{n \geq 1}$ with values in M ,

$$x_n \xrightarrow{n \rightarrow \infty} x \quad \Rightarrow \quad F(x_n) \xrightarrow{n \rightarrow \infty} F(x).$$

證明：假設 (iii) 也就是控制假設，告訴我們對於每個 $x \in M$ ，函數 $f(x, \cdot)$ 是可積的，所以 F 是定義良好的。對給定的 $x \in M$ 來說，如果要檢查 F 在 x 連續，我們需要檢查對於任意取值在

Let $x \in M$ and $(x_n)_{n \geq 1}$ be a sequence in M such that $x_n \xrightarrow{n \rightarrow \infty} x$. For every $n \geq 1$, we may define the function

$$\begin{aligned} f_n : I &\rightarrow V \\ t &\mapsto f(x_n, t). \end{aligned}$$

Due to the assumption (ii), we know that $f_n(t) = f(x_n, t) \xrightarrow{n \rightarrow \infty} f(x, t)$ for every $t \in I$, where $t \mapsto f(x, t)$ is a piecewise continuous function by the assumption (i). This means that the assumption (ii) in Theorem 8.5.5 is satisfied. Then, the assumption (iii) here corresponds to the assumption (i) in Theorem 8.5.5, so we can apply Theorem 8.5.5 to the sequence of functions $(f_n)_{n \geq 1}$. This shows that

$$\lim_{n \rightarrow \infty} F(x_n) = \lim_{n \rightarrow \infty} \int_I f(x_n, t) dt = \int_I f(x, t) dt = F(x),$$

which allows us to conclude. \square

Theorem 8.5.9 (Differentiability under integration): Let $M \subseteq \mathbb{R}$ be an interval and a map $f : M \times I \rightarrow W$ satisfying the following conditions.

- (i) For every $x \in M$, the map $f(x, \cdot) : t \mapsto f(x, t)$ is piecewise continuous and integrable on I .
- (ii) For every $t \in I$, the map $f(\cdot, t) : x \mapsto f(x, t)$ is of class C^1 on M .
- (iii) The partial derivative $\frac{\partial f}{\partial x}$ is well defined and satisfies the assumptions from Theorem 8.5.8.

Then, the map

$$\begin{aligned} F : M &\rightarrow W \\ x &\mapsto \int_a^b f(x, t) dt \end{aligned}$$

is of class C^1 on M , and we have

$$\forall x \in M, \quad F'(x) = \int_a^b \frac{\partial f}{\partial x}(x, t) dt. \quad (8.28)$$

M 中的序列 $(x_n)_{n \geq 1}$ ，我們有

$$x_n \xrightarrow{n \rightarrow \infty} x \Rightarrow F(x_n) \xrightarrow{n \rightarrow \infty} F(x).$$

令 $x \in M$ 以及 $(x_n)_{n \geq 1}$ 為 M 中的序列滿足 $x_n \xrightarrow{n \rightarrow \infty} x$ 。對於每個 $n \geq 1$ ，我們可以定義函數

$$\begin{aligned} f_n : I &\rightarrow V \\ t &\mapsto f(x_n, t). \end{aligned}$$

從假設 (ii)，我們知道對於每個 $t \in I$ ，我們有 $f_n(t) = f(x_n, t) \xrightarrow{n \rightarrow \infty} f(x, t)$ ，其中根據假設 (i)， $t \mapsto f(x, t)$ 是個片段連續函數。這代表定理 8.5.5 中的假設 (ii) 會滿足。再來，這裡的假設 (iii) 對應到的是定理 8.5.5 的假設 (i)，所以我們可以對函數序列 $(f_n)_{n \geq 1}$ 來使用定理 8.5.5。這讓我們得到

$$\lim_{n \rightarrow \infty} F(x_n) = \lim_{n \rightarrow \infty} \int_I f(x_n, t) dt = \int_I f(x, t) dt = F(x),$$

因此我們能總結。 \square

定理 8.5.9 【積分下的可微分性】：令 $M \subseteq \mathbb{R}$ 為區間，且 $f : M \times I \rightarrow W$ 為滿足下列條件的映射。

- (i) 對於每個 $x \in M$ ，映射 $f(x, \cdot) : t \mapsto f(x, t)$ 在 I 上會片段連續且可積。
- (ii) 對於每個 $t \in I$ ，映射 $f(\cdot, t) : x \mapsto f(x, t)$ 在 M 上是 C^1 類的。
- (iii) 偏微分 $\frac{\partial f}{\partial x}$ 定義良好，且滿足定理 8.5.8 中的假設。

那麼，映射

$$\begin{aligned} F : M &\rightarrow W \\ x &\mapsto \int_a^b f(x, t) dt \end{aligned}$$

在 M 上會是 C^1 類的，而且我們有

$$\forall x \in M, \quad F'(x) = \int_a^b \frac{\partial f}{\partial x}(x, t) dt. \quad (8.28)$$

Proof : The proof is similar to that of Theorem 8.5.8. Let $x \in M$ and $(x_n)_{n \geq 1}$ be a sequence with values in $M \setminus \{x\}$ that converges to x . For every $n \geq 1$, define

$$\begin{aligned} g_n : I &\rightarrow W, \\ t &\mapsto \frac{f(x_n, t) - f(x, t)}{x_n - x}, \end{aligned}$$

which is a piecewise continuous function. For each $n \geq 1$, g_n is also integrable on I , being a linear combination of integrable functions.

The sequence $(g_n)_{n \geq 1}$ of functions converges pointwise to $\frac{\partial f}{\partial x}(x, \cdot)$. Moreover, the mean-value theorem (Eq. (4.3)) tells us that for every $n \geq 1$ and $t \in I$, there exists $y_n = y_n(t)$ between x and x_n such that

$$g_n(t) = \frac{f(x_n, t) - f(x, t)}{x_n - x} = \frac{\partial f}{\partial x}(y_n, t) \quad \text{and} \quad \|g_n(t)\| = \left\| \frac{\partial f}{\partial x}(y_n, t) \right\| \leq \varphi(t),$$

where φ is the domination function given by the assumption (iii) for $\frac{\partial f}{\partial x}$ from Theorem 8.5.8. Then, we may apply Theorem 8.5.5 to conclude that

$$\lim_{n \rightarrow \infty} \int_I g_n(t) dt = \int_I \frac{\partial f}{\partial x}(x, t) dt,$$

and the left side of the above formula rewrite,

$$\lim_{n \rightarrow \infty} \int_I g_n(t) dt = \lim_{n \rightarrow \infty} \frac{F(x_n) - F(x)}{x_n - x}.$$

This shows that F is differentiable at x and its derivative does satisfy Eq. (8.28). To conclude, we note that the assumption (iii) guarantees that the right side of Eq. (8.28) is continuous, so F is of class C^1 . \square

Example 8.5.10 (Gamma function) : We recall the Gamma function defined in Example 7.1.21,

$$\forall x > 0, \quad \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

By applying Theorem 8.5.8 and Theorem 8.5.9, we can check that Γ is a function of class C^∞ , and its derivative writes

$$\forall n \in \mathbb{N}_0, \forall x > 0, \quad \Gamma^{(n)}(x) = \int_0^\infty (\log t)^n e^{-t} t^{x-1} dt.$$

證明：這個證明與定理 8.5.8 的證明相似。令 $x \in M$ 以及 $(x_n)_{n \geq 1}$ 為取值在 $M \setminus \{x\}$ 中的序列，且會收斂到 x 。對於每個 $n \geq 1$ ，定義

$$\begin{aligned} g_n : I &\rightarrow W, \\ t &\mapsto \frac{f(x_n, t) - f(x, t)}{x_n - x}, \end{aligned}$$

這會是個片段連續函數。對於每個 $n \geq 1$ ， g_n 在 I 上也會是可積的，因為他是可積函數的線性組合。

函數序列 $(g_n)_{n \geq 1}$ 會逐點收斂至 $\frac{\partial f}{\partial x}(x, \cdot)$ 。此外，中間值定理（式 (4.3)）告訴我們，對於每個 $n \geq 1$ 還有 $t \in I$ ，存在 $y_n = y_n(t)$ 介於 x 與 x_n 之間，滿足

$$g_n(t) = \frac{f(x_n, t) - f(x, t)}{x_n - x} = \frac{\partial f}{\partial x}(y_n, t) \quad \text{以及} \quad \|g_n(t)\| = \left\| \frac{\partial f}{\partial x}(y_n, t) \right\| \leq \varphi(t),$$

其中 φ 是由 $\frac{\partial f}{\partial x}$ 滿足定理 8.5.8 當中的假設 (iii) 所給定的。這樣一來，我們可以使用定理 8.5.5 來總結

$$\lim_{n \rightarrow \infty} \int_I g_n(t) dt = \int_I \frac{\partial f}{\partial x}(x, t) dt,$$

然後上式左方可以重新寫成：

$$\lim_{n \rightarrow \infty} \int_I g_n(t) dt = \lim_{n \rightarrow \infty} \frac{F(x_n) - F(x)}{x_n - x}.$$

這證明了 F 在 x 可微，且為分會滿足式 (8.28)。在最後的結論中，我們注意到假設 (iii) 保證式 (8.28) 的右方是連續的，所以 F 是 C^1 類的。 \square

範例 8.5.10 【Gamma 函數】：我們回顧範例 7.1.21 中定義的 Γ 函數：

$$\forall x > 0, \quad \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

透過定理 8.5.8 和定理 8.5.9，我們可以檢查 Γ 是個 C^∞ 類的函數，而且他的微分寫做：

$$\forall n \in \mathbb{N}_0, \forall x > 0, \quad \Gamma^{(n)}(x) = \int_0^\infty (\log t)^n e^{-t} t^{x-1} dt.$$

More precisely, let us consider the function

$$f : \mathbb{R}_+^* \times \mathbb{R}_+^* \mapsto \mathbb{R}, (x, t) \mapsto t^{x-1}e^{-t}.$$

We can check the following properties.

- For any fixed $t > 0$, the function $x \mapsto f(x, t)$ is \mathcal{C}^∞ , and we have

$$\forall k \in \mathbb{N}_0, \quad \forall x, t > 0, \quad \frac{\partial^k f}{\partial x^k}(x, t) = (\ln t)^k t^{x-1} e^{-t}.$$

- For any fixed $x > 0$ and $k \in \mathbb{N}_0$, the function $t \mapsto \frac{\partial^k f}{\partial x^k}(x, t)$ is piecewise continuous.
- (Domination assumption) Let $k \in \mathbb{N}_0$ and $[a, b] \subseteq (0, +\infty)$ be a segment. For all $x \in [a, b]$, we have

$$\begin{aligned} \forall t \in (0, 1], \quad \left| \frac{\partial^k f}{\partial x^k}(x, t) \right| &= |\ln t|^k t^{x-1} e^{-t} \leq |\ln t|^k t^{a-1} e^{-t}, \\ \forall t \in (1, +\infty), \quad \left| \frac{\partial^k f}{\partial x^k}(x, t) \right| &= |\ln t|^k t^{x-1} e^{-t} \leq |\ln t|^k t^{b-1} e^{-t}. \end{aligned}$$

Let φ be defined on \mathbb{R}_+^* by

$$\varphi(t) = |\ln t|^k t^{a-1} e^{-t} + |\ln t|^k t^{b-1} e^{-t},$$

which is an integrable function on \mathbb{R}_+^* . And we clearly have

$$\forall x \in [a, b], \quad \forall t > 0, \quad \left| \frac{\partial^k f}{\partial x^k}(x, t) \right| \leq \varphi(t).$$

更確切來說，我們考慮函數

$$f : \mathbb{R}_+^* \times \mathbb{R}_+^* \mapsto \mathbb{R}, (x, t) \mapsto t^{x-1}e^{-t}.$$

我們能檢查下面的性質。

- 對於任意固定的 $t > 0$ ，函數 $x \mapsto f(x, t)$ 是 \mathcal{C}^∞ 的，而且我們有

$$\forall k \in \mathbb{N}_0, \quad \forall x, t > 0, \quad \frac{\partial^k f}{\partial x^k}(x, t) = (\ln t)^k t^{x-1} e^{-t}.$$

- 對於任意固定的 $x > 0$ 還有 $k \in \mathbb{N}_0$ ，函數 $t \mapsto \frac{\partial^k f}{\partial x^k}(x, t)$ 是片段連續的。
- 【控制假設】令 $k \in \mathbb{N}_0$ 以及 $[a, b] \subseteq (0, +\infty)$ 為線段。對於所有的 $x \in [a, b]$ ，我們有

$$\begin{aligned} \forall t \in (0, 1], \quad \left| \frac{\partial^k f}{\partial x^k}(x, t) \right| &= |\ln t|^k t^{x-1} e^{-t} \leq |\ln t|^k t^{a-1} e^{-t}, \\ \forall t \in (1, +\infty), \quad \left| \frac{\partial^k f}{\partial x^k}(x, t) \right| &= |\ln t|^k t^{x-1} e^{-t} \leq |\ln t|^k t^{b-1} e^{-t}. \end{aligned}$$

令 φ 定義在 \mathbb{R}_+^* 上，如下：

$$\varphi(t) = |\ln t|^k t^{a-1} e^{-t} + |\ln t|^k t^{b-1} e^{-t},$$

這在 \mathbb{R}_+^* 上是個可積函數。而且我們顯然會有：

$$\forall x \in [a, b], \quad \forall t > 0, \quad \left| \frac{\partial^k f}{\partial x^k}(x, t) \right| \leq \varphi(t).$$