8 Sequences and series of functions

Let A be a set, and (M, d) be a metric space. We denote by $\mathcal{F}(A, M)$ the space of functions from A to M, and by $\mathcal{B}(A, M)$ the space of bounded functions from A to M. Instead of a metric space, we may also consider a vector spaces W over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, so that we have the $+$ operation. This vector space is equipped with a norm that we denote by $\|\cdot\|$.

In this chapter, we are interested in sequences and series of functions, which can also be seen as sequences and series with terms in $\mathcal{F}(A, M)$ or $\mathcal{F}(A, W)$.

8.1 Notions of convergence

We discuss different notions of convergence for sequences of functions, then for series of functions.

8.1.1 Sequences of functions

For a sequence of functions, we have different notions of convergence. Below we are going to discuss the *pointwise convergence* (Definition 8.1.1), and a stronger notion of convergence, called *uniform convergence* (Definition 8.1.4).

Definition 8.1.1 : Let $(f_n)_{n\geqslant 1}$ be a sequence of functions from A to M, that is, they are elements of $\mathcal{F}(A, M)$.

- Let $f \in \mathcal{F}(A, M)$. We say that the sequence $(f_n)_{n\geqslant 1}$ *converges pointwise* (逐點收斂) to f if for every $x \in A$, we have $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ in (M, d).
- We say that the sequence $(f_n)_{n\geqslant 1}$ converges pointwise if there exists $f \in \mathcal{F}(A, M)$ such that $(f_n)_{n\geqslant 1}$ converges pointwise to f.
- Let $B \subseteq A$ be a subset. We say that $(f_n)_{n\geqslant 1}$ converges pointwise on B if $((f_n)_{|B})_{n\geqslant 1}$ converges pointwise.

Example 8.1.2 : Let us consider the sequence of functions $(f_n)_{n\geqslant 1}$ defined by

$$\forall n \geqslant 1, \quad f_n : \begin{array}{ccc} [0,1] & \to & \mathbb{R} \\ x & \mapsto & x^n. \end{array}$$

The sequence of functions $(f_n)_{n\geqslant 1}$ converges pointwise to the indicator function $f = \mathbb{1}_{\{1\}}$ on $[0,1]$.

Remark 8.1.3 :

(1) If a sequence $(f_n)_{n\geqslant 1}$ converges pointwise, then its limit function f is unique.

(2) Let $(f_n)_{n\geqslant 1}$ be a pointwise convergent sequence of functions. Suppose that these functions take values in a finite dimensional vector space $(W, \|\cdot\|)$, then the limit does not depend on the norm, because all the norms are equivalent in W.

(3) Properties such as linearity, product, inequality, monotonicity, etc., are preserved for the pointwise convergence of functions.

(4) We see that in Example 8.1.2, the continuity at 1 is not preserved in the limit. Indeed, for all $n \in \mathbb{N}$, the function f_n is continuous, but the limit function f is not continuous at 1. In other words, the following two iterated limits are different,

$$\lim_{x\to 1}\lim_{n\to\infty} f_n(x) = \lim_{x\to 1} f(x) = 0 \neq 1 = \lim_{n\to\infty} 1 = \lim_{n\to\infty}\lim_{x\to 1} f_n(x).$$

We have already encountered a similar example in Example 6.7.2.

(5) Analytic properties such as continuity and differentiability are not preserved for the pointwise convergence. We will define the notion of uniform convergence below (Definition 8.1.4), and will see that analytic properties can be preserved if this convergence occurs (Proposition 8.2.1).

Definition 8.1.4 : Let $(f_n)_{n\geqslant 1}$ be a sequence of functions from A to M.

- Let $f \in \mathcal{F}(A, M)$. We say that the sequence $(f_n)_{n\geqslant 1}$ *converges uniformly* (均匀收敛) to f if

$$\forall \varepsilon > 0,\ \exists N \geqslant 1,\ \forall n \geqslant N,\ \forall x \in A, \quad d(f_n(x), f(x)) \leqslant \varepsilon. \tag{8.1}$$

- We say that the sequence $(f_n)_{n\geqslant 1}$ converges uniformly if there exists $f \in \mathcal{F}(A, M)$ such that $(f_n)_{n\geqslant 1}$ converges uniformly to f.

- Let $B \subseteq A$ be a subset. We say that $(f_n)_{n\geqslant 1}$ converges uniformly on B if $((f_n)_{|B})_{n\geqslant 1}$ converges uniformly.

Remark 8.1.5 : We may rewrite the definition of pointwise convergence using quantifiers. We say that $(f_n)_{n\geqslant 1}$ converges pointwise to f if

$$\forall x \in A,\ \forall \varepsilon > 0,\ \exists N \geqslant 1,\ \forall n \geqslant N, \quad d(f_n(x), f(x)) \leqslant \varepsilon. \tag{8.2}$$

If we compare Eq. (8.1) and Eq. (8.2), we see that the choice of N depends on $x \in A$ in the case of pointwise convergence, but does not depend on $x \in A$ in the case of uniform convergence. This is the reason why the convergence characterized by the condition Eq. (8.1) is called *uniform* convergence. This remark easily leads to the following corollary.

Corollary 8.1.6 : *If the sequence of functions $(f_n)_{n\geqslant 1}$ converges uniformly to f, then it converges pointwise to f.*

Remark 8.1.7 : Due to the uniqueness of the pointwise limit (Remark 8.1.3), we deduce the uniqueness of the uniform limit of a sequence of functions. To show that a sequence of functions $(f_n)_{n\geqslant 1}$ converges uniformly, we may start by computing its pointwise limit f, then show that $(f_n)_{n\geqslant 1}$ converges uniformly to f.

Proposition 8.1.8 (Cauchy's criterion for uniform convergence) **:** *Suppose that (M, d) is a complete metric space. Let $(f_n)_{n\geqslant 1}$ be a sequence of functions in $\mathcal{F}(A, M)$. Then, $(f_n)_{n\geqslant 1}$ converges uniformly if and only if it satisfies the* uniform Cauchy condition, *that is*

$$\forall \varepsilon > 0, \exists N \geqslant 1, \forall m, n \geqslant N, \forall x \in A, \quad d(f_n(x), f_m(x)) \leqslant \varepsilon.$$

Proof : Given $\varepsilon > 0$. Let $N \geqslant 1$ such that the uniform Cauchy condition holds, that is

$$\forall m, n \geqslant N, \forall x \in A, \quad d(f_n(x), f_m(x)) \leqslant \varepsilon. \tag{8.3}$$

For each $x \in A$, we see that $(f_n(x))_{n\geqslant 1}$ is a Cauchy sequence, so it converges to some limit that we denote by $f(x)$. By taking the limit $m \to \infty$ in Eq. (8.3), we find

$$\forall n \geqslant N, \forall x \in A, \quad d(f_n(x), f(x)) \leqslant \varepsilon,$$

which is the characterization of $(f_n)_{n\geqslant 1}$ uniformly converging to f from Eq. (8.1). □

Definition 8.1.9 : The notion of uniform convergence can be described using a distance (or a norm).

- Let (M, d) be a metric space and $\mathcal{B}(A, M)$ be the set of bounded functions from A to M. We may equip $\mathcal{B}(A, M)$ with the following distance

$$\forall f, g \in \mathcal{B}(A, M), \quad d_\infty(f, g) = d_{\infty,A}(f, g) := \sup_{x\in A} d(f(x), g(x)), \tag{8.4}$$

  called the *distance of uniform convergence.* A sequence of bounded functions $(f_n)_{n\geqslant 1}$ converges uniformly to f is equivalent to the convergence of $(f_n)_{n\geqslant 1}$ to f with respect to the distance d_∞.

- Let $(W, \|\cdot\|)$ be a normed vector space and $\mathcal{B}(A, W)$ be the set of bounded functions from A to W. We may equip $\mathcal{B}(A, W)$ with the following norm

$$\forall f \in \mathcal{B}(A, W), \quad \|f\|_\infty = \|f\|_{\infty,A} := \sup_{x\in A} \|f(x)\|, \tag{8.5}$$

  called the *norm of uniform convergence.* A sequence of bounded functions $(f_n)_{n\geqslant 1}$ converges uniformly to f is equivalent to the convergence of $(f_n)_{n\geqslant 1}$ to f with respect to the norm $\|\cdot\|_\infty$.

Proposition 8.1.10 : *Let $(W, \|\cdot\|)$ be a Banach space. Then, the following properties hold.*

(1) The space of bounded functions $\mathcal{B}(A, W)$ equipped with the norm $\|\cdot\|_\infty$, defined in Eq. (8.5), is a Banach space.

(2) A sequence $(f_n)_{n\geqslant 1}$ of $\mathcal{B}(A, W)$ converges uniformly to $f \in \mathcal{B}(A, W)$ if and only if $(f_n)_{n\geqslant 1}$ converges to f under the norm $\|\cdot\|_\infty$ given in Eq. (8.5), that is $\|f_n - f\|_\infty \xrightarrow[n\to\infty]{} 0$.

Proof :

(1) It is not hard to check that $\|\cdot\|_\infty$ defines a norm on the vector space $\mathcal{B}(A, W)$. To check that it is complete, let us be given a sequence $(f_n)_{n\geqslant 1}$ in $\mathcal{B}(A, W)$, which is Cauchy with respect to

the norm $\|\cdot\|_\infty$. For every $x \in A$, we know that $(f_n(x))_{n\geqslant 1}$ is a Cauchy sequence in the Banach space $(W, \|\cdot\|)$, so it converges to some limit $f(x) := \lim_{n\to\infty} f_n(x)$. Since $(f_n)_{n\geqslant 1}$ is Cauchy in $(\mathcal{B}(A, W), \|\cdot\|_\infty)$, there exists $M > 0$ such that $\|f_n\|_\infty \leqslant M$ for all $n \geqslant 1$. Therefore, for every $x \in A$, we have $\|f(x)\| = \lim_{n\to\infty} \|f_n(x)\| \leqslant M$, so $\|f\|_\infty \leqslant M$, that is $f \in \mathcal{B}(A, W)$. In the end, it is not hard to check that $\|f_n - f\|_\infty \xrightarrow[n\to\infty]{} 0$, so we conclude that $(\mathcal{B}(A, W), \|\cdot\|_\infty)$ is complete.

(2) It is exactly a rewriting of Eq. (8.1) in the normed vector space $(W, \|\cdot\|)$ with help of the new norm defined in Eq. (8.5). □

Example 8.1.11 : Consider the sequence of functions $(f_n)_{n\geqslant 1}$ defined by

$$\forall n \in \mathbb{N}, \quad \forall x \in [0, 1], \quad f_n(x) = x^n(1 - x).$$

It is not hard to see that $(f_n)_{n\geqslant 1}$ converges pointwise to the zero function. For every $n \in \mathbb{N}$, the function f_n is of class $\mathcal{C}^\infty$, so we may take its derivative to find its extrema on $[0, 1]$. We have

$$\forall x \in [0, 1], \quad f_n'(x) = nx^{n-1}\Big(1 - \frac{n+1}{n}x\Big).$$

Therefore, the function f_n is increasing on $[0, \frac{n}{n+1}]$ and decreasing on $[\frac{n}{n+1}, 1]$ with maximum at $x_n = \frac{n}{n+1}$, that is

$$\forall x \in [0, 1], \quad f_n(x) \leqslant f_n(x_n) = \frac{1}{n+1}\Big(\frac{n}{n+1}\Big)^n \leqslant \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0.$$

Therefore, the sequence $(f_n)_{n\geqslant 1}$ converges uniformly to the zero function on $[0, 1]$.

Remark 8.1.12 : If a sequence of functions $(f_n)_{n\geqslant 1}$ converges pointwise to f, in order to show that this convergence is not uniform, we may look at the negation of Eq. (8.1), which writes

$$\exists \varepsilon > 0, \ \forall N \geqslant 1, \ \exists n \geqslant N \ \exists x \in A \quad d(f_n(x), f(x)) > \varepsilon.$$

In other words, we need to find a sequence $(x_n)_{n\geqslant 1}$ with values in A and an extraction $\varphi : \mathbb{N} \to \mathbb{N}$ such that the sequence $(d(f_{\varphi(n)}(x_n), f(x_n)))_{n\geqslant 1}$ is bounded away from 0.

Example 8.1.13 : Let us consider the following sequence of functions,

$$\forall n \in \mathbb{N}, \quad \forall x \geqslant 0, \quad f_n(x) = \frac{x + \sqrt{n}}{x + n}.$$

It is easy to see that the sequence of functions $(f_n)_{n\geqslant 1}$ converges pointwise to the zero function. To show that it does not converge uniformly, we follow Remark 8.1.12. Let $x_n = n$ for $n \geqslant 1$. Then, we have

$$\forall n \in \mathbb{N}, \quad f_n(x_n) - 0 = \frac{n + \sqrt{n}}{n + n} \xrightarrow[n\to\infty]{} \frac{1}{2} \neq 0.$$

We conclude that the convergence $f_n \xrightarrow[n\to\infty]{} f$ is pointwise but not uniform.

The following theorem tells us which additional assumptions we may add to upgrade a pointwise convergence to a uniform convergence.

Theorem 8.1.14 (Dini's theorem) **:** *Let (K, d) be a compact space, and $(f_n)_{n\geqslant 1}$ be a sequence of continuous functions from K to $\mathbb{R}$. Suppose that*

(i) the sequence is increasing, that is for every $x \in K$ and $n \in \mathbb{N}$, we have $f_n(x) \leqslant f_{n+1}(x)$;

(ii) the sequence $(f_n)_{n\geqslant 1}$ converges pointwise to a continuous function $f : K \to \mathbb{R}$.

Then, the sequence $(f_n)_{n\geqslant 1}$ converges uniformly to f.

Proof : For every $n \in \mathbb{N}$, let us define the continuous function $g_n = f - f_n \geqslant 0$. By the assumption (i), the sequence of functions $(g_n)_{n\geqslant 1}$ is decreasing. Given $\varepsilon > 0$, we define $E_n = \{x \in K : g_n(x) < \varepsilon\}$ for $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, since g_n is continuous, the set E_n is open; since the sequence $(g_n)_{n\geqslant 1}$ is decreasing, the sequence $(E_n)_{n\geqslant 1}$ is increasing. Due to the assumption (ii), we find that $\bigcup_{n\geqslant 1} E_n = K$. Since K is compact, by the Borel–Lebesgue property (Definition 3.1.3), there exists $N \geqslant 1$ such that $E_N = \bigcup_{n=1}^{N} E_n = K$. This means that for any $n \geqslant N$ and $x \in K$, we have $|f_n(x) - f(x)| < \varepsilon$. □

Remark 8.1.15 : There is another version of Dini's theorem, stated as below. Let $I = [a, b]$ be a segment and $(f_n)_{n\geqslant 1}$ be a sequence of (not necessarily continuous) functions from I to $\mathbb{R}$. Suppose that

(i) for each $n \geqslant 1$, the function f_n is increasing on I;

(ii) the sequence $(f_n)_{n\geqslant 1}$ converges pointwise to a continuous function $f : I \to \mathbb{R}$.

Then, the sequence $(f_n)_{n\geqslant 1}$ converges uniformly to f. See Exercise 8.7 for a proof.

8.1.2 Series of functions

In this section, let $(u_n)_{n\geqslant 1}$ be a sequence of functions from A to W, where $(W, \|\cdot\|)$ is a Banach space.

Definition 8.1.16 :

- We say that the series of functions $\sum u_n$ converges pointwise if for every $x \in A$, the series $\sum u_n(x)$ converges. We write

$$\begin{array}{rccc} \sum_{n\geqslant 1} u_n : & A & \to & W \\ & x & \mapsto & \sum_{n\geqslant 1} u_n(x). \end{array}$$

- The function defined by $S_n(x) = \sum_{k=1}^{n} u_k(x)$ for $x \in A$ is called the n*-th partial sum* of the series of functions $\sum u_n$.

- If the series of functions $\sum u_n$ converges pointwise, then the n*-th remainder* is given by $R_n(x) = \sum_{k=n+1}^{\infty} u_k(x)$ for $x \in A$.

- We say that the series of functions $\sum u_n$ converges uniformly if the partial sums $(S_n)_{n\geqslant 0}$ converges uniformly.

Proposition 8.1.17 : *The series of functions $\sum u_n$ converges uniformly if and only if*

(i) the series $\sum u_n$ converges pointwise, and

(ii) the sequence of remainders $(R_n)_{n\geqslant 0}$ converges uniformly to the zero function.

Proof : Let $\sum u_n$ be a series of functions, $(S_n)_{n\geqslant 0}$ be its partial sums, and $(R_n)_{n\geqslant 0}$ be its remainders.

- Suppose that $\sum u_n$ converges uniformly to u, which means that $(S_n)_{n\geqslant 0}$ converges uniformly to u, and it follows from Corollary 8.1.6 that this convergence takes place pointwise. The uniform convergence means that $\|S_n - u\|_\infty \xrightarrow[n\to\infty]{} 0$, since $u - S_n = R_n$, we see that it is equivalent to $\|R_n\|_\infty \xrightarrow[n\to\infty]{} 0$.

- Suppose that (i) and (ii) holds, and denote by u the pointwise limit of $\sum u_n$. Since $R_n = u - S_n$, from its uniform convergence to zero, we find $\|S_n - u\|_\infty \xrightarrow[n\to\infty]{} 0$, which is the uniform convergence of $(S_n)_{n\geqslant 0}$ to u. $\square$

Example 8.1.18 : Let us consider the series of functions $\sum \frac{(-1)^n}{n} x^n$ where each term is a function defined on $[0,1]$. We are going to show that this series of functions converges uniformly. For every $x \in [0,1]$, the sequence $(\frac{x^n}{n})_{n\geqslant 1}$ is non-increasing with limit zero. It follows from Theorem 6.4.2 that the series $\sum \frac{(-1)^n}{n} x^n$ converges, and the remainder $R_n(x)$ satisfies

$$\forall x \in [0,1], \quad |R_n(x)| \leqslant \frac{x^{n+1}}{n+1} \leqslant \frac{1}{n+1},$$

which does not depend on $x \in [0,1]$. This implies that the convergence of the series of functions is uniform.

Remark 8.1.19 : We note that saying that a sequence of functions $(f_n)_{n\geqslant 1}$ converges uniformly is equivalent to saying that the series of functions $\sum (f_{n+1} - f_n)$ converges uniformly.

Proposition 8.1.20 (Cauchy's condition) **:** *A series of functions $\sum u_n$ converges uniformly if and only if for every $\varepsilon > 0$, there exists $N \geqslant 1$ such that*

$$\forall n \geqslant N, \forall k \geqslant 1, \quad \|u_{n+1} + \cdots + u_{n+k}\|_\infty < \varepsilon.$$

This is the Cauchy's condition *in the case of a series of functions.*

Proof : This is very similar to Corollary 6.1.11. From Proposition 8.1.10 (1), we know that $(\mathcal{B}(A,W), \|\cdot\|_\infty)$ is a Banach space, in which a sequence converges if and only if it is Cauchy. $\square$

Definition 8.1.21 : Let $u_n \in \mathcal{B}(A,W)$ for every $n \geqslant 1$. We say that the series of functions $\sum u_n$ *converges normally* (正規收斂) on A if the series $\sum \|u_n\|_{\infty,A}$ converges.

Proposition 8.1.22 : *Suppose that $(W, \|\cdot\|)$ is a Banach space. Let $\sum u_n$ be a series of bounded functions from A to W that converges normally on A. Then, the following properties hold.*

(1) For every $a \in A$, the series $\sum u_n(a)$ converges absolutely.

(2) The series of functions $\sum u_n$ converges uniformly.

Proof :

(1) Let $a \in A$. For every $n \geqslant 1$, we have $\|u_n(a)\| \leqslant \|u_n\|_\infty$. Since $\sum \|u_n\|_\infty$ is convergent, we deduce that $\sum u_n(a)$ converges absolutely.

(2) For every $n, k \geqslant 1$ and $x \in A$, we have

$$\|u_n(x) + \cdots + u_{n+k}(x)\| \leqslant \|u_n(x)\| + \cdots + \|u_{n+k}(x)\| \leqslant \|u_n\|_\infty + \cdots + \|u_{n+k}\|_\infty .$$

Therefore, the Cauchy's condition for the series $\sum \|u_n\|_\infty$ implies the Cauchy's condition for the series $\sum u_n(x)$, uniformly for all $x \in A$. This means that the series of functions $\sum u_n$ converges uniformly. $\square$

Remark 8.1.23 : Let us assume that $(W, \|\cdot\|)$ is a Banach space, and $u_n \in \mathcal{B}(A, W)$ for all $n \geqslant 1$. A series of functions $\sum u_n$ can also be seen as a series with terms in the Banach space $(\mathcal{B}(A, W), \|\cdot\|_\infty)$, meaning that the normal convergence of the series of functions $\sum u_n$ is the same as the absolute convergence of the series $\sum u_n$ with terms $u_n \in \mathcal{B}(A, W)$. This allows us to find an alternative proof to (2), by noting that from Theorem 6.1.16, we deduce that the series $\sum u_n$ converges in $\mathcal{B}(A, W)$, that is the series of functions $\sum u_n$ converges uniformly.

Example 8.1.24 : Let us define a sequence of functions $(f_n)_{n \geqslant 1}$ on $[0, 1]$ as below,

$$f_1 \equiv 1 \quad \text{and} \quad \forall n \geqslant 1, \forall x \in [0, 1], \quad f_{n+1}(x) = 1 + \frac{1}{2} \int_0^x f_n(t)\, \mathrm{d}t.$$

For any $n \geqslant 1$ and $x \in [0, 1]$, we have

$$\begin{aligned} |f_{n+2}(x) - f_{n+1}(x)| &= \frac{1}{2} \left| \int_0^x (f_{n+1}(t) - f_n(t))\, \mathrm{d}t \right| \\ &\leqslant \frac{1}{2} \int_0^x \|f_{n+1} - f_n\|_\infty \, \mathrm{d}t \\ &\leqslant \frac{1}{2} \|f_{n+1} - f_n\|_\infty , \end{aligned}$$

implying $\|f_{n+2} - f_{n+1}\|_\infty \leqslant \frac{1}{2} \|f_{n+1} - f_n\|_\infty$. Therefore, by induction, we find

$$\forall n \geqslant 1, \quad \|f_{n+1} - f_n\|_\infty \leqslant \frac{1}{2^{n-1}} \|f_2 - f_1\|_\infty .$$

It follows that the series $\sum (f_{n+1} - f_n)$ converges normally, so uniformly, and the sequence $(f_n)_{n \geqslant 1}$

converges also uniformly.

Example 8.1.25 : Let us consider the series of functions $\sum \frac{(-1)^n}{n} x^n$ defined on $[0,1]$. We have seen that this series of functions converges uniformly on $[0,1]$ (Example 8.1.18).

- However, it does not converge normally on $[0,1]$, because $\|u_n\|_\infty = \frac{1}{n}$ for $n \geqslant 1$, and the series $\sum \frac{1}{n}$ diverges.
- It does converge normally on $[0,a]$ for any $a \in [0,1)$, because $\left\|(u_n)_{|[0,a]}\right\|_\infty = \frac{a^n}{n}$ for $n \geqslant 1$, and the series $\sum \frac{a^n}{n}$ converges.

8.2 Properties of the uniform limit

In this section, we are going to discuss some analytic properties of the limit of a convergent sequence of functions. We are going to consider metric spaces (X, d_X) and (M, d_M), and a sequence of functions $(f_n)_{n\geqslant 1}$ in $\mathcal{B}(X, M)$.

8.2.1 Continuity

Proposition 8.2.1 : *Suppose that $(f_n)_{n\geqslant 1}$ is a sequence of functions from X to M and converges uniformly to f. If f_n is continuous at a for every $n \geqslant 1$, then f is continuous at a.*

Proof : Let $\varepsilon > 0$. Due to the uniform convergence of $(f_n)_{n\geqslant 1}$ to f, we may find $N \geqslant 1$ such that

$$\forall n \geqslant N, \forall x \in X, \quad d_M(f_n(x), f(x)) \leqslant \varepsilon.$$

Since f_N is continuous at a, we may find $\delta > 0$ such that

$$\forall y \in X, \quad d_X(x,y) < \delta \quad \Rightarrow \quad d_M(f_N(x), f_N(y)) \leqslant \varepsilon.$$

Therefore, for any $y \in X$ such that $d_X(x,y) < \delta$, we have

$$d_M(f(x), f(y)) \leqslant d_M(f(x), f_N(x)) + d_M(f_N(x), f_N(y)) + d_M(f_N(y), f(y)) \leqslant 3\varepsilon.$$

This shows that f is continuous at a. $\square$

Corollary 8.2.2 : *Let $(f_n)_{n\geqslant 1}$ be a sequence of continuous functions from X to M. If $(f_n)_{n\geqslant 1}$ converges uniformly to f on X, then f is continuous on X.*

Proof : It is a direct consequence of Proposition 8.2.1. $\square$

Corollary 8.2.3 : *Let $\sum u_n$ be a series of continuous functions from $[a,b]$ to a Banach space $(W, \|\cdot\|)$. If the series $\sum u_n$ converges uniformly on $[a,b]$, then the limit function $\sum u_n$ is continuous on $[a,b]$.*

Proof : It is a direct consequence of Corollary 8.2.2 by taking $(X, d_X) = ([a,b], |\cdot|)$ and $(M, d_M) = (W, \|\cdot\|)$. $\square$

Example 8.2.4 : Let us consider the series of functions $\sum_{n\geqslant 0} u_n$ defined on $\mathbb{R}_+$ as below,

$$\forall x \geqslant 0, \quad u_n(x) = \frac{x^n}{n!}.$$

- For each $x \geqslant 0$, the series $\sum_{n\geqslant 0} u_n(x)$ converges, and we denote the limit by $u(x)$.
- The convergence of the series $\sum_{n\geqslant 0} u_n$ to u is not uniform. In fact, for every $N \geqslant 1$, we have

$$\left| \sum_{n\geqslant 0} u_n(x) - \sum_{n=0}^{N-1} u_n(x) \right| \geqslant \frac{x^N}{N!} \xrightarrow[x\to\infty]{} +\infty.$$

- For any $M > 0$, the convergence of the series $\sum_{n\geqslant 0} u_n$ to u on $[0, M]$ is uniform. To see this, we write, for any $x \in [0, M]$,

$$\left| \sum_{n\geqslant 0} u_n(x) - \sum_{n=0}^{N-1} u_n(x) \right| = \left| \sum_{n\geqslant N} u_n(x) \right| \leqslant \sum_{n\geqslant N} \frac{M^n}{n!} \xrightarrow[N\to\infty]{} 0,$$

  which gives us a uniform upper bound of the remainder which does not depend on x.
- In consequence, the limit function u is continuous on $[0, M]$ for every $M > 0$, so it is also continuous on $\mathbb{R}_+$.

This examples illustrates that to get the continuity of the limit function, we do not necessarily need the uniform convergence on the whole domain of definition. Since the continuity is a *local* regularity, it is sufficient to show the uniform convergence on, for example, all the segments.

8.2.2 Integration

Let $I \subseteq \mathbb{R}$ be an interval such that $\mathring{I} \neq \varnothing$. Consider a sequence $(f_n)_{n\geqslant 1}$ of functions from I to a Banach space $(W, \|\cdot\|)$.

Proposition 8.2.5 : *Let $(f_n)_{n\geqslant 1}$ be a sequence of continuous functions that converges uniformly to f on every segment of I. Let $a \in I$, and define the following primitives,*

$$\varphi(x) = \int_a^x f(t)\,\mathrm{d}t \quad \text{and} \quad \varphi_n(x) = \int_a^x f_n(t)\,\mathrm{d}t, \quad \forall n \geqslant 1.$$

Then, the sequence $(\varphi_n)_{n\geqslant 1}$ converges uniformly to φ on every segment of I.

Remark 8.2.6 : The conclusion of Proposition 8.2.5 menas that we may interchange the order of the limit and integration,

$$\lim_{n\to\infty} \int_a^x f_n(t)\,\mathrm{d}t = \int_a^x \lim_{n\to\infty} f_n(t)\,\mathrm{d}t.$$

Proof : Let $[c, d] \subseteq I$ be a segment of I containing a. Since $(f_n)_{n\geqslant 1}$ converges uniformly on $[c, d]$ to f, it follows from Corollary 8.2.2 that f is also continuous on $[c, d]$. Therefore, the primitives φ and φ_n with $n \geqslant 1$ are well defined on $[c, d]$. For every $n \geqslant 1$ and $x \in [c, d]$, we have

$$\begin{aligned}\|\varphi_n(x) - \varphi(x)\| &= \left\| \int_a^x (f_n(t) - f(t))\,\mathrm{d}t \right\| \\ &\leqslant |x - a|\,\|f_n - f\|_{\infty,[c,d]} \leqslant |d - c|\,\|f_n - f\|_{\infty,[c,d]} \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

The convergence to 0 in the above bound does not depend on $x \in [c, d]$, so we have established the uniform convergence of $(\varphi_n)_{n\geqslant 1}$ to φ on $[c, d]$. □

Example 8.2.7 : Let $(f_n)_{n\geqslant 1}$ be a sequence of real-valued continuous functions on $[0, 1]$ that converges uniformly to f. This means that $(f_n)_{n\geqslant 1}$ is bounded in $\mathcal{B}([0, 1], \mathbb{R})$, so we may find $M > 0$

such that $\|f_n\|_\infty \leqslant M$ for all $n \geqslant 1$. Then, we have

$$\forall x \in [0,1], \quad |f_n(x)^2 - f(x)^2| \leqslant 2M|f_n(x) - f(x)|.$$

This means that $(f_n^2)_{n\geqslant 1}$ converges uniformly to f^2, so we have

$$\int_0^1 f_n^2 \xrightarrow[n\to\infty]{} \int_0^1 f^2.$$

Example 8.2.8 : Let us consider the sequence of functions $(f_n)_{n\geqslant 1}$ on $[0,1]$, defined by

$$\forall x \in [0,1], \quad f_n(x) = x^n.$$

This sequence of functions converges pointwise to the indicator function $f = \mathbb{1}_1$ (Example 8.1.2) which is not continuous, so this convergence is not uniform (Proposition 8.2.1). However, the sequence of integrals converges,

$$\int_0^1 f_n(x)\,\mathrm{d}x = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 = \int_0^1 \mathbb{1}_1(x)\,\mathrm{d}x.$$

This shows that the notion of uniform convergence is much stronger than the convergence of integrals. Actually, later in Section 8.5, we will see in a more general context, how to obtain the convergence of integrals without having the uniform convergence.

Corollary 8.2.9 : *Let $\sum u_n$ be a series of continuous functions from $[a,b]$ to a Banach space $(W, \|\cdot\|)$. If the series $\sum u_n$ converges normally on $[a,b]$, then, for $x \in [a,b]$, we have*

$$\int_a^x \Big(\sum_{n\geqslant 1} u_n(t)\Big)\,\mathrm{d}t = \sum_{n\geqslant 1}\Big(\int_a^x u_n(t)\,\mathrm{d}t\Big) = \lim_{n\to\infty}\sum_{k=1}^n \Big(\int_a^x u_k(t)\,\mathrm{d}t\Big),$$

where the limit on the right side is uniform on $[a,b]$.

Remark 8.2.10 : Corollary 8.2.9 gives us conditions under which we are allowed to interchange the order of integration and series. In such a circumstance, sometimes we also say that “we may integrate the series term by term”.

We also have a more general statement for the behavior of a uniformly convergent sequence of functions in the context of Riemann–Stieltjes integration. The following theorem states that (1) the Riemann–Stieltjes integrability is preserved by the uniform convergence, and (2) the sequence of primitives also converges uniformly.

Theorem 8.2.11 : *Let $\alpha \in \mathcal{BV}([a, b])$. Let $(f_n)_{n\geqslant 1}$ be a sequence of bounded functions from $[a, b]$ to $\mathbb{R}$ such that $f_n \in R(\alpha; a, b)$ for all $n \geqslant 1$. Suppose that $(f_n)_{n\geqslant 1}$ converges uniformly to a function $f : [a, b] \to \mathbb{R}$, and define*

$$g(x) = \int_a^x f(t)\,\mathrm{d}\alpha(t) \quad \textit{and} \quad g_n(x) = \int_a^x f_n(t)\,\mathrm{d}\alpha(t), \quad \forall n \geqslant 1.$$

Then, the following properties hold.

(1) $f \in R(\alpha; a, b)$.

(2) The sequence $(g_n)_{n\geqslant 1}$ converges uniformly to g on $[a, b]$.

Proof : By the decomposition theorem of functions with bounded variation, see Theorem 5.1.17 and Corollary 5.3.16, it is enough to show the statement for a strictly increasing function α. We have seen a similar argument in the proof of Theorem 5.3.21.

(1) Let us prove that f satisfies Riemann's condition with resepct to α on $[a, b]$ (Definition 5.3.8). Let $\varepsilon > 0$. The uniform convergence of $(f_n)_{n\geqslant 1}$ to f allows us to find $N \geqslant 1$ such that

$$\|f(x) - f_n(x)\| \leqslant \frac{\varepsilon}{\alpha(b) - \alpha(a)}, \quad \forall x \in [a, b], \forall n \geqslant N.$$

This means that for any partition $P \in \mathcal{P}([a, b])$, we have

$$|U_P(f - f_N, \alpha)| \leqslant \varepsilon \quad \text{and} \quad |L_P(f - f_N, \alpha)| \leqslant \varepsilon \tag{8.6}$$

Since $f_N \in R(\alpha; a, b)$, we may find a partition $P_\varepsilon \in \mathcal{P}([a, b])$ such that

$$\forall P \supseteq P_\varepsilon, \quad U_P(f_N, \alpha) - L_P(f_N, \alpha) \leqslant \varepsilon. \tag{8.7}$$

Therefore, for any $P \supseteq P_\varepsilon$, we have

$$\begin{aligned} U_P(f,\alpha) - L_P(f,\alpha) &\leqslant U_P(f-f_N,\alpha) - L_P(f-f_N,\alpha) + U_P(f_N,\alpha) - L_P(f_N,\alpha) \\ &\leqslant |U_P(f-f_N,\alpha)| + |L_P(f-f_N,\alpha)| + [U_P(f_N,\alpha) - L_P(f_N,\alpha)] \\ &\leqslant 3\varepsilon \end{aligned}$$

from Eq. (8.6) and Eq. (8.7). This shows that $f \in R(\alpha; a, b)$.

(2) For $n \geqslant N$ and $x \in [a,b]$, we have

$$|g_n(x) - g(x)| \leqslant \int_a^x |f_n(t) - f(t)| \, \mathrm{d}\alpha(t) \leqslant \|f_n - f\|_\infty [\alpha(x) - \alpha(a)] \leqslant \|f_n - f\|_\infty [\alpha(b) - \alpha(a)],$$

where the upper bound does not depend on x, and converges to 0 when $n \to \infty$. □

Corollary 8.2.12 : *Let $\alpha \in \mathcal{BV}([a,b])$. Let $\sum u_n$ be a series of bounded functions from $[a,b]$ to $\mathbb{R}$ such that $u_n \in R(\alpha; a, b)$ for all $n \geqslant 1$. Suppose that the series $\sum_n u_n$ converges uniformly on $[a,b]$. Then, the following properties hold.*

(1) $\sum_n u_n \in R(\alpha; a, b)$.

(2) For $x \in [a,b]$, we have

$$\int_a^x \Big(\sum_{n \geqslant 1} u_n(t) \Big) \mathrm{d}\alpha(t) = \sum_{n \geqslant 1} \Big(\int_a^x u_n(t) \, \mathrm{d}\, \mathrm{d}\alpha(t) \Big) = \lim_{n \to \infty} \sum_{k=1}^n \Big(\int_a^x u_k(t) \, \mathrm{d}\, \mathrm{d}\alpha(t) \Big),$$

where the convergence on the right side is uniform in $x \in [a,b]$.

8.2.3 Derivatives

Let $I \subseteq \mathbb{R}$ be an interval such that $\mathring{I} \neq \varnothing$. Consider a sequence $(f_n)_{n \geqslant 1}$ of functions from I to a Banach space $(W, \|\cdot\|)$.

Theorem 8.2.13 : *Let us make the following assumptions.*

(i) For every $n \geqslant 1$, the function $f_n : I \to W$ is of class $\mathcal{C}^1$.

(ii) The sequence $(f_n)_{n \geqslant 1}$ converges pointwise to $f \in \mathcal{F}(I, W)$.

(iii) The sequence $(f_n')_{n\geqslant 1}$ converges uniformly to $g \in \mathcal{F}(I, W)$ on every segment of I.

Then, the following properties hold.

(1) The function f is of class $\mathcal{C}^1$ and $f' = g$.

(2) The sequence $(f_n)_{n\geqslant 1}$ converges uniformly on every segment of I.

Proof : Let $a \in I$. From (ii), we know that $f_n(a) \xrightarrow[n\to\infty]{} f(a)$.

(1) First, we note that since $(f_n')_{n\geqslant 1}$ converges uniformly to g on every segment of I, it follows from Corollary 8.2.2 that g is continuous on I. By Proposition 8.2.5, for $x \in I$, we have

$$\int_a^x g(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_a^x f_n'(t)\,\mathrm{d}t = \lim_{n\to\infty} (f_n(x) - f_n(a)) = f(x) - f(a).$$

This shows that

$$\forall x \in I, \quad f(x) = f(a) + \int_a^x g(t)\,\mathrm{d}t.$$

Since g is continuous, we deduce that f is of class $\mathcal{C}^1$ and $f' = g$.

(2) To show the uniform convergence of $(f_n)_{n\geqslant 1}$ to f, let us proceed as follows. For every $n \geqslant 1$ and $x \in I$, the fundamental theorem of calculus gives us

$$\|f_n(x) - f(x)\| \leqslant \left\| \int_a^x (f_n'(t) - f'(t))\,\mathrm{d}t \right\| + \|f_n(a) - f(a)\|.$$

The first term on the right side converges uniformly to 0 by Proposition 8.2.5, and the second term converges to 0 due to the assumption (ii). Therefore, the above rate of convergence does not depend on $x \in I$, so $(f_n)_{n\geqslant 1}$ converges uniformly to f. □

Remark 8.2.14 : From the above proof, we see that the assumption (ii) can be softened to

(ii') there exists $a \in I$ such that $f_n(a) \xrightarrow[n\to\infty]{} f(a)$.

Corollary 8.2.15 : *Let $p \geqslant 1$ be an integer, and $(f_n)_{n\geqslant 1}$ be a sequence of $\mathcal{C}^p$ functions from I to W. Suppose that*

(i) for every $0 \leqslant k \leqslant p-1$, the sequence $(f_n^{(k)})_{n\geqslant 1}$ converges pointwise;

(ii) the sequence $(f_n^{(p)})_{n\geqslant 1}$ *converges uniformly on every segment of* I.

Then, the pointwise limit $f := \lim_{n\to\infty} f_n$ *is of class* $\mathcal{C}^p$*, and for* $0 \leqslant k \leqslant p$*, we have*

$$\forall x \in I, \quad f^{(k)}(x) = \lim_{n\to\infty} f_n^{(k)}(x).$$

Proof : This can be shown by induction using Theorem 8.2.13. □

Corollary 8.2.16 : *Let* $(u_n)_{n\geqslant 1}$ *be a sequence of* $\mathcal{C}^1$ *functions from* I *to* W. *Suppose that*

(i) the series $\sum u_n$ *converges pointwise;*

(ii) the series $\sum u_n'$ *converges uniformly on every segment of* I.

Then, the function $\sum_{n\geqslant 1} u_n$ *is of class* $\mathcal{C}^1$ *and*

$$\Big(\sum_{n\geqslant 1} u_n\Big)' = \sum_{n\geqslant 1} u_n'. \tag{8.8}$$

Example 8.2.17 : We claim that the Riemann zeta function $s \mapsto \zeta(s)$ is of class $\mathcal{C}^1$, and

$$\forall s > 1, \quad \zeta'(s) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^s}. \tag{8.9}$$

For every $n \geqslant 1$, let $u_n : s \mapsto n^{-s}$, which is a $\mathcal{C}^1$ function with derivative given by

$$\forall s > 1, \quad u_n'(s) = -\frac{\ln n}{n^s}.$$

The series of functions $\sum u_n$ converges pointwise to ζ. Fix $b > a > 1$, let us show that $\sum u_n'$ converges normally on $[a, b]$, so also uniformly. Let us choose $c \in (1, a)$. We have

$$\left\|(u_n')_{|[a,b]}\right\|_\infty = \frac{\ln n}{n^a} = \mathcal{O}\Big(\frac{1}{n^c}\Big).$$

Since $\sum n^{-c}$ converges (Proposition 6.2.6), we deduce that $\sum u_n$ converges normally on $[a, b]$. Therefore, Eq. (8.8) gives us Eq. (8.9).

Corollary 8.2.18 : *Let $p \geqslant 1$ be an integer, and $(u_n)_{n\geqslant 1}$ be a sequence of $\mathcal{C}^p$ functions from I to W. Suppose that*

(i) for every $0 \leqslant k \leqslant p-1$, the series $\sum u_n^{(k)}$ converges pointwise;

(ii) the series $\sum u_n^{(p)}$ converges uniformly on every segment of I.

Then, the function $\sum_{n\geqslant 1} u_n$ is of class $\mathcal{C}^p$ and for $0 \leqslant k \leqslant p$, we have

$$\left(\sum_{n\geqslant 1} u_n\right)^{(k)} = \sum_{n\geqslant 1} u_n^{(k)}. \tag{8.10}$$

Example 8.2.19 : We follow the same notations as in Example 8.2.17, we find, for every $n, p \geqslant 1$, that

$$\forall s > 1, \quad u_n^{(p)}(s) = (-1)^p \frac{(\ln n)^p}{n^s}.$$

Let us fix $b > a > 1$. We show in the same way that $\sum u_n^{(p)}$ converges normally on $[a, b]$ for all $p \geqslant 0$, so also converges uniformly and pointwise. We apply Corollary 8.2.18 to conclude that $s \mapsto \zeta(s)$ is of class $\mathcal{C}^p$ for all $p \geqslant 0$, so it is of class $\mathcal{C}^\infty$. Moreover, Eq. (8.10) gives us

$$\forall s > 1, \forall p \geqslant 1, \quad \zeta^{(p)}(s) = \sum_{n\geqslant 1} (-1)^p \frac{(\ln n)^p}{n^s}.$$

Example 8.2.20 : Let $(W, \|\cdot\|_W)$ be a Banach space. We have seen in Theorem 3.2.18 that $\mathcal{L}_c(W) := \mathcal{L}_c(W, W)$ equipped with the operator norm $|||\cdot|||$ is a Banach space, and is also a normed algebra (Definition 6.6.1), that is the operator norm satisfies the submultiplicative property. Given $u \in \mathcal{L}_c(W)$, we may define the following function

$$\begin{array}{rccl} \mathcal{E}_u : & \mathbb{R} & \to & \mathcal{L}_c(W) \\ & t & \mapsto & \displaystyle\sum_{n\geqslant 0} \frac{t^n}{n!} u^n. \end{array}$$

We may denote $u_n(t) = \frac{t^n}{n!} u^n$ for all $n \geqslant 0$ and $t \in \mathbb{R}$.

- It is straightforward to check that $\mathcal{E}_u(t)$ is well defined for all $t \in \mathbb{R}$, because

$$\forall t \in \mathbb{R}, \quad \sum_{n\geqslant 0} \frac{|t|^n}{n!} |||u^n||| \leqslant \sum_{n\geqslant 0} \frac{|t|^n}{n!} |||u|||^n = \exp\left(|t| |||u|||\right).$$

- A similar argument as in Example 8.2.4 shows that for any $M > 0$, the series of functions $\sum_{n\geqslant 0} u_n$ converges uniformly on $[-M, M]$ to $\mathcal{E}_u$.

- We have $u_0(t) = 1$ for all $t \in \mathbb{R}$. For every $n \in \mathbb{N}$, we have

$$\forall t \in \mathbb{R}, \quad u_n'(t) = \frac{t^{n-1}}{(n-1)!}u^n = u \cdot u_{n-1}(t).$$

  This shows that the series of functions $\sum_{n\geqslant 0} u_n' = \sum_{n\geqslant 1} u_n' = \sum_{n\geqslant 0} u \cdot u_n$ converges pointwise to $u \cdot \mathcal{E}_u(t)$. This convergence is also uniform on every $[-M, M]$ for $M > 0$.

- Let us fix $M > 0$ and apply the uniform convergence of $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} u_n'$ on $[-M, M]$ to conclude that $\mathcal{E}_u$ is of class $\mathcal{C}^1$ on $[-M, M]$ and $\mathcal{E}_u'(t) = u \cdot \mathcal{E}_u(t)$ for $t \in (-M, M)$. This allows us to conclude that $\mathcal{E}_u$ is of class $\mathcal{C}^1$ on $\mathbb{R}$ and $\mathcal{E}_u'(t) = u \cdot \mathcal{E}_u(t)$ for all $t \in \mathbb{R}$.

- From the relation $\mathcal{E}_u' = u \cdot \mathcal{E}_u$, we deduce that if $\mathcal{E}_u$ is of class $\mathcal{C}^k$ for some $k \geqslant 1$, then so is $\mathcal{E}_u'$, meaning that $\mathcal{E}_u$ needs to be of class $\mathcal{C}^{k+1}$. As a consequence, $\mathcal{E}_u$ is of class $\mathcal{C}^\infty$.

8.3 Power series

In this section, we are going to study a particular form of series of functions, called *power series*. We restrict ourselves to real-valued and complexed-valued power series, but you need to keep in mind that all the notions are still valid if we replace $(\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$ by a normed algebra.

8.3.1 Definitions and radius of convergence

We define a few topological notions in $(\mathbb{C}, |\cdot|)$. An open ball centered at c with radius $r > 0$ is also called an *open disk* centered at c with the same radius r, denoted $D(c, r) := B(c, r)$. We also define the notion of closed disks in the same way.

Definition 8.3.1 : Let $(a_n)_{n\geqslant 0}$ be a sequence of complex numbers and $c \in \mathbb{C}$.

- A series of functions of the form $\sum_{n\geqslant 0} a_n(z-c)^n$ is called a power series (**冪級數**) centered at c, where $z \in \mathbb{C}$ is the variable of the functions.

- If the sequence $(a_n)_{n\geqslant 0}$ is real-valued and $c \in \mathbb{R}$, we may use $x \in \mathbb{R}$ as the variable of the power series, and write $\sum_{n\geqslant 0} a_n(x-c)^n$. Then, this power series takes values in $\mathbb{R}$.

We are going to develop some theories for power series centered at $c = 0$. For a general power series centered at $c \in \mathbb{C}$, all the corresponding notions and properties can be obtained by a shift $z \mapsto z + c$. The properties and theorems are stated in terms of complex-valued power series, but you should also know that the exact same proofs apply to the real-valued power series.

Proposition 8.3.2 (Abel's lemma) **:** *Let $\sum a_n z^n$ be a power series and $z_0 \in \mathbb{C}$ be such that the sequence $(a_n z_0^n)_{n\geqslant 0}$ is bounded. Then, the following properties hold.*

(1) For every $z \in \mathbb{C}$ with $|z| < |z_0|$, the series $\sum a_n z^n$ is absolutely convergent.

(2) For every $r \in (0, |z_0|)$, the series of functions $\sum a_n z^n$ is normally convergent in the closed disk $\overline{D}(0, r) := \overline{B}(0, r)$.

Proof : Let $M > 0$ be such that $|a_n||z_0|^n \leqslant M$ for every $n \geqslant 0$. For $z \in \mathbb{C}$ such that $|z| < |z_0|$, we have

$$\forall n \geqslant 0, \quad |a_n z^n| = \Big|\frac{z}{z_0}\Big|^n |a_n||z_0|^n \leqslant M \Big|\frac{z}{z_0}\Big|^n,$$

where the right-hand side is a convergent series (geometric series with ratio strictly smaller than 1). □

Definition 8.3.3 : Let $\sum a_n z^n$ be a power series. The following quantity

$$R = R\Big(\sum a_n z^n\Big) := \sup\{r \geqslant 0 : (|a_n| r^n)_{n\geqslant 0} \text{ is bounded}\} \in [0, +\infty]$$

is called the *radius of convergence* (收斂半徑) of $\sum a_n z^n$.

Remark 8.3.4 : We note that if we add phases to the sequence $(a_n)_{n\geqslant 0}$ defining the power series $\sum a_n z^n$, its radius of convergence remains unchanged.

Proposition 8.3.5 : *Let $\sum a_n z^n$ be a power series and R be its radius of convergence. Then, we have the following properties.*

(1) For $z \in \mathbb{C}$ with $|z| < R$, the series $\sum a_n z^n$ converges absolutely.

(2) For $z \in \mathbb{C}$ with $|z| > R$, the series $\sum a_n z^n$ diverges.

(3) For $r \in [0, R)$, the series $\sum a_n z^n$ converges normally on the closed disk $\overline{D}(0, r)$.

And the open disk $D(0, R)$ is called the disk of convergence **(收斂圓盤)** *of the power series* $\sum a_n z^n$.

Remark 8.3.6 :

(1) When $R = +\infty$, the power series $\sum a_n z^n$ converges for every $z \in \mathbb{C}$, so it defines a function from $\mathbb{C}$ to $\mathbb{C}$. Such a function is called an entire function (整函數).

(2) When $R < +\infty$, on the boundary of the disk of convergence, that is when $z \in \partial D(0, R)$, the power series may have any possible behavior, see Example 8.3.9.

Proof :

(1) It is a direct consequence of Proposition 8.3.2 (1).

(2) For $z \in \mathbb{C} \backslash \overline{D}(0, R)$, since $(|a_n||z|^n)_{n \geqslant 0}$ is not bounded, we do not have $a_n z^n \xrightarrow[n \to \infty]{} 0$, so the series $\sum a_n z^n$ diverges.

(3) It is a direct consequence of Proposition 8.3.2 (2). □

Proposition 8.3.7 (D'Alembert's criterion, ratio test) **:** *Let* $\sum a_n z^n$ *be a power series, and* R *be its radius of convergence. Suppose that the following limit exists,*

$$\ell := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \in [0, +\infty].$$

Then, $R = \ell^{-1}$.

Proof : It is a direct consequence of Theorem 6.3.1. □

Proposition 8.3.8 (Cauchy's criterion, root test) **:** *Let* $\sum a_n z^n$ *be a power series, and* R *be its radius of convergence. Let*

$$\lambda := \limsup_{n \to \infty} |a_n|^{1/n} \in [0, +\infty].$$

Then, $R = \frac{1}{\lambda}$.

Proof : It is a direct consequence of Corollary 6.3.8. □

Example 8.3.9 : The following three series have the same radius of convergence 1, that can be obtained by either the ratio test or the root test. However, they have totally different behaviors on the *boundary* of the disk of convergence.

(1) The series $\sum z^n$ has radius of convergence 1. For $z \in \mathbb{C}$ with $|z| = 1$, the series $\sum z^n$ never converges.

(2) The series $\sum \frac{z^n}{n^2}$ has radius of convergence 1. For $z \in \mathbb{C}$ with $|z| = 1$, the series $\sum \frac{z^n}{n^2}$ converges normally, so converges.

(3) The series $\sum \frac{z^n}{n}$ has radius of convergence 1. For $z = 1$, the series $\sum \frac{z^n}{n}$ diverges. For $z \in \mathbb{C}$ such that $|z| = 1$ and $z \neq 1$, the series $\sum \frac{z^n}{n}$ converges by Example 6.4.9.

8.3.2 Operations on power series

Proposition 8.3.10 : *Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ be power series with radius of convergence R_f and R_g. Let R be the radius of convergence of $\sum (a_n + b_n) z^n$. Then,*

$$R \geqslant \min(R_f, R_g).$$

Moreover, if $R_f \neq R_g$, we have $R = \min(R_f, R_g)$. For any $z \in \mathbb{C}$ with $|z| < \min(R_f, R_g)$, we also have

$$\sum_{n \geqslant 0} (a_n + b_n) z^n = \sum_{n \geqslant 0} a_n z^n + \sum_{n \geqslant 0} b_n z^n. \tag{8.11}$$

Proof : Let $z \in \mathbb{C}$ such that $|z| < \min(R_f, R_g)$. It follows from Proposition 8.3.5 that both $\sum a_n z^n$ and $\sum b_n z^n$ converges absolutely, so the series $\sum (a_n + b_n) z^n$ also converges absolutely. This means that Eq. (8.11) holds. Moreover, this also implies that $R \geqslant \min(R_f, R_g)$.

Suppose that $R_f \neq R_g$, for example, $R_f < R_g$. Let $z \in \mathbb{C}$ such that $R_f < |z| < R_g$. Since $(b_n z^n)_{n \geqslant 1}$ is bounded and $(a_n z^n)_{n \geqslant 1}$ is unbounded, we deduce that $\big((a_n + b_n) z^n\big)_{n \geqslant 1}$ is unbounded, so $|z| \geqslant R$. By taking infimum over $z \in \mathbb{C}$ satisfying $R_f < |z| < R_g$, we find that $R_f \geqslant R$. □

Definition 8.3.11 : Let $\sum a_n z^n$ and $\sum b_n z^n$ be power series. Their *Cauchy product* is the power series $\sum c_n z^n$, where the coefficients $(c_n)_{n\geqslant 1}$ are given by

$$\forall n \geqslant 0, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Proposition 8.3.12 : *Let $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ be power series with radius of convergence R_f and R_g. Let $\sum c_n z^n$ be their Cauchy product. For every $z \in \mathbb{C}$ with $|z| < \min(R_f, R_g)$, we have*

$$f(z)g(z) = \left(\sum_{n\geqslant 0} a_n z^n\right)\left(\sum_{n\geqslant 0} b_n z^n\right) = \sum_{n\geqslant 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n = \sum_{n\geqslant 0} c_n z^n. \tag{8.12}$$

In particular, if R is the radius of convergence of $\sum c_n z^n$, then we have

$$R \geqslant \min(R_f, R_g).$$

Proof : Let $z \in \mathbb{C}$ such that $|z| < \min(R_f, R_g)$. From Proposition 8.3.5, we know that both $\sum a_n z^n$ and $\sum b_n z^n$ converges absolutely, then by Theorem 6.6.3, we know that their Cauchy product $\sum c_n z^n$ converges absolutely, and satisfies Eq. (8.12). Additionally, this implies that $R \geqslant \min(R_f, R_g)$. $\square$

8.3.3 Regularity

Here, let $f := \sum a_n z^n$ be a power series with radius of convegence $R > 0$. We have seen in Proposition 8.3.5 that f is well defined on $D(0, R)$.

Theorem 8.3.13 : *The function $f : z \mapsto \sum_{n\geqslant 0} a_n z^n$ is continuous on the disk of convergence $D(0, R)$.*

Proof : Fix $z \in D(0, R)$. Let us consider a closed disk $\overline{D}(z, r)$ centered at z with radius $r < R - |z|$. Then, for any $w \in \overline{D}(z, r)$, we have $|w| \leqslant |w - z| + |z| \leqslant |z| + r < R$, which means that $\overline{D}(z, r) \subseteq D(0, R)$. It follows Proposition 8.3.5 (3) that the power series $\sum a_n z^n$ converges normally on $\overline{D}(z, r)$. Since the partial sums defining f are continuous (polynomial functions), we use Proposition 8.2.1 to conclude that the limit f is continuous at z. $\square$

Theorem 8.3.14 (Abel's theorem) **:** *Let $\sum a_n z^n$ be a power series with radius of convergence $R > 0$. Suppose that the series $\sum a_n R^n$ converges. Then, the function $x \mapsto \sum_{n\geqslant 0} a_n x^n$ defined on $[0, R]$ is continuous. In other words, we have*

$$\sum_{n\geqslant 0} a_n x^n \xrightarrow[x\to R-]{} \sum_{n\geqslant 0} a_n R^n.$$

Proof : For every $n \in \mathbb{N}_0$, let $u_n : [0, R] \to \mathbb{C}$ be defined by

$$\forall x \in [0, R], \quad u_n(x) = a_n x^n, \quad \text{and} \quad R_n = \sum_{k\geqslant n+1} a_k R^k.$$

By the assumption, the series of functions $\sum u_n$ converges pointwise on $[0, R]$. We want to show that this covnergence is uniform, then we can conclude by Proposition 8.2.1. By rewriting each u_n as $u_n = a_n R^n \left(\frac{x}{R}\right)^n$, we may assume that $R = 1$.

Let $\varepsilon > 0$. Since $\sum a_n$ is convergent, we may find $N \geqslant 1$ such that $|R_n| \leqslant \varepsilon$ for all $n \geqslant N$. For $m, n \in \mathbb{N}$ with $m > n \geqslant N$, and $x \in [0, 1]$, we establish the Abel's transform using the remainders of the convergent series $\sum a_k$,

$$\begin{aligned}\sum_{k=n+1}^{m} a_k x^k &= \sum_{k=n+1}^{m} (R_{k-1} - R_k) x^k = \sum_{k=n}^{m-1} R_k x^{k+1} - \sum_{k=n+1}^{m} R_k x^k \\ &= R_n x^{n+1} - R_m x^m + \sum_{k=n+1}^{m-1} R_k (x^{k+1} - x^k).\end{aligned}$$

Since $R_m \xrightarrow[m\to\infty]{} 0$ and $(x_m)_{m\geqslant 0}$ is bounded, we have $R_m x^m \xrightarrow[m\to\infty]{} 0$. Moreover, we have $|R_k(x^{k+1} - x^k)| \leqslant \varepsilon(x^k - x^{k+1})$, and the series $\sum_k (x^k - x^{k+1})$ converges, so $\sum R_k(x^{k+1} - x^k)$ converges absolutely. Thus, for $n \in \mathbb{N}$ and $x \in [0, 1]$, the remainder of the power series writes

$$r_n(x) = R_n x^{n+1} + \sum_{k\geqslant n+1} R_k (x^{k+1} - x^k).$$

For $n \geqslant N$ and $x \in [0, 1]$, we have

$$\begin{aligned}|R_n x^{n+1}| &\leqslant |R_n| \leqslant \varepsilon, \\ \sum_{k\geqslant n+1} |R_k(x^{k+1} - x_k)| &\leqslant \sum_{k\geqslant n+1} \varepsilon(x^k - x^{k+1}) = \varepsilon x^{n+1} \leqslant \varepsilon.\end{aligned}$$

So $|r_n(x)| \leqslant 2\varepsilon$ for all $n \geqslant N$ and $x \in [0, 1]$. This means that $r_n \xrightarrow[n\to\infty]{} 0$ uniformly. By

Proposition 8.1.17, we have shown that $\sum u_n$ converges uniformly on $[0, R]$. $\square$

The following Tauber's theorem gives a converse of the above Abel's theorem.

Theorem 8.3.15 (Tauber's theorem) **:** *Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence $R > 0$. Suppose that $f(x) \xrightarrow[x \to R-]{} \ell$ and $na_n \xrightarrow[n\to\infty]{} 0$. Then, the series $\sum a_n R^n$ converges to ℓ.*

Proof : Without loss of generality, we may assume that $R = 1$. Let us denote by $(S_n)_{n\geqslant 0}$ the partials sums of the series $\sum a_n$. For any $n \in \mathbb{N}_0$ and $x \in (-1, 1)$, we have

$$S_n - f(x) = \sum_{k=1}^{n} a_k(1 - x^k) - \sum_{k\geqslant n+1} a_k x^k.$$

For $x \in (0, 1)$, we have

$$1 - x^k = (1 - x)(1 + x + \cdots + x^{k-1}) \leqslant k(1 - x).$$

Therefore, for any $n \in \mathbb{N}_0$ and $x \in (0, 1)$, we have

$$|S_n - f(x)| \leqslant (1 - x) \sum_{k=1}^{n} k|a_k| + \sum_{k\geqslant n+1} |a_k| x^k.$$

Given $\varepsilon > 0$ and choose $N \geqslant 1$ such that $n|a_n| \leqslant \varepsilon$ for all $n \geqslant N$. For any $n \geqslant N$, we have

$$\sum_{k\geqslant n+1} |a_k| x^k \leqslant \varepsilon \sum_{k\geqslant n+1} \frac{x^k}{k} \leqslant \frac{\varepsilon}{n} \sum_{k\geqslant n+1} x^k \leqslant \frac{\varepsilon}{n(1 - x)}.$$

For $n \geqslant N$, let us choose $x_n = 1 - \frac{1}{n}$. Then, we find

$$|S_n - f(x_n)| \leqslant \frac{1}{n} \sum_{k=1}^{n} k|a_k| + \varepsilon.$$

Since $n|a_n| \xrightarrow[n\to\infty]{} 0$, it follows from Exercise 6.1 that the first term[1] on the right side converges to 0. Therefore,

$$\limsup_{n\to\infty} |S_n - f(x_n)| \leqslant \varepsilon.$$

Since $\varepsilon > 0$ can be made arbitrarily small, we find

$$\lim_{n\to\infty} |S_n - f(x_n)| = 0.$$

That is, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} f(x_n) = \lim_{x\to 1-} f(x) = \ell$. □

The following is a generalization of Theorem 6.6.3 and Exercise 6.24.

Corollary 8.3.16 : *Let $\sum a_n$ and $\sum b_n$ be convergent series. For $n \in \mathbb{N}_0$, let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Suppose that $\sum c_n$ is convergent. Then,*

$$\sum_{n\geqslant 0} c_n = \left(\sum_{n\geqslant 0} a_n\right)\left(\sum_{n\geqslant 0} b_n\right).$$

Proof : Let $\sum a_n z^n$, $\sum b_n z^n$, and $\sum c_n z^n$ be power series. Their radii of convergence are at least 1, because both $(a_n|z|^n)_{n\geqslant 0}$ and $(b_n|z|^n)_{n\geqslant 0}$ are bounded for $z \in D(0,1)$. It follows from Proposition 8.3.12 that the radius of convergence of the power series $\sum c_n z^n$ is greater or equal to 1. By Theorem 8.3.14, we know that

$$\sum_{n\geqslant 0} a_n x^n \xrightarrow[x\to 1-]{} \sum_{n\geqslant 0} a_n, \quad \sum_{n\geqslant 0} b_n x^n \xrightarrow[x\to 1-]{} \sum_{n\geqslant 0} b_n, \quad \text{and} \quad \sum_{n\geqslant 0} c_n x^n \xrightarrow[x\to 1-]{} \sum_{n\geqslant 0} c_n.$$

Moreover, Proposition 8.3.12 gives the following identity,

$$\forall x \in (-1,1), \quad \sum_{n\geqslant 0} c_n x^n = \left(\sum_{n\geqslant 0} a_n x^n\right)\left(\sum_{n\geqslant 0} a_n x^n\right)$$

By taking the limit $x \to 1-$ in the above identity, we establish the identity we want. □

Let us also introduce the notion of *differentiability* in a complex variable.

Definition 8.3.17 : Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$. We say that f is $\mathbb{C}$-differentiable (or simply differentiable) at $z_0 \in \mathring{A}$ if the following limit exists,

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) = \frac{\mathrm{d}}{\mathrm{d}z} f(z_0) = f'(z_0) := \lim_{\substack{z\to z_0\\ z\in\mathbb{C}}} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C},$$

[1]The sum $\frac{1}{n}\sum_{k=1}^{n} k a_k$ is called the Cesarò sum of $(na_n)_{n\geqslant 1}$.

which is also called the $\mathbb{C}$-derivative of f at z_0.

Remark 8.3.18 : We may identify $\mathbb{C}$ as a two-dimensional real vector space. If we compare the notion of differential from Definition 4.1.1, we may notice that the $\mathbb{C}$-derivative introduced here is much stronger. In fact, if a function $f : A \to \mathbb{C}$ is differentiable at z_0 in the sense of Definition 4.1.1, its differential is a continuous linear map. However, if the same function is $\mathbb{C}$-differentiable at z_0, its $\mathbb{C}$-derivative is given by a complex number, which, seen as a differential, is a composition between a rotation and a dilation (in $\mathbb{R}^2$). It is not hard to see that a composition between a rotation and a dilation is a continuous linear map, but the converse fails to hold in general. In Complex Analysis, you will see that if a function is $\mathbb{C}$-differentiable in an open subset $A \subseteq \mathbb{C}$, then it can be differentiated as many times as we want in A. Such functions are called *holomorphic functions.*

A power series contains only polynomials functions, and it is not hard to check that the $\mathbb{C}$-derivative of a polynomial function is the same as its usual $\mathbb{R}$-derivative. In other words, we have

$$\forall n \in \mathbb{N}_0, \quad \frac{\mathrm{d}(z^n)}{\mathrm{d}z} = nz^{n-1}.$$

Theorem 8.3.19 : *The function $f : D(0,R) \to \mathbb{C}$, $z \mapsto \sum_{n\geqslant 0} a_n z^n$ is of class $\mathcal{C}^1$. The power series $\sum_{n\geqslant 1} n a_n z^{n-1}$ has the same radius of convergence as $\sum_{n\geqslant 0} a_n z^n$, that is*

$$R\Big(\sum_{n\geqslant 1} n a_n z^{n-1}\Big) = R\Big(\sum_{n\geqslant 0} a_n z^n\Big).$$

We also have

$$\forall z \in D(0,R), \quad f'(z) = \sum_{n\geqslant 1} n a_n z^{n-1}. \tag{8.13}$$

Remark 8.3.20 : This theorem is of particular interest. It means that we can *always* differentiate term by term a power series, which is not the case of a general series of functions, where additional assumptions are needed (Corollary 8.2.16).

Proof : Let R' be the radius of convergence of $\sum na_n z^{n-1}$. For any $r \in [0, R')$, we know from Definition 8.3.3 that $(na_n r^{n-1})_{n\geqslant 1}$ is bounded, so $(a_n r^n)_{n\geqslant 0}$ is also bounded, which implies that $r < R$. By taking the limit $r \to R'-$, we find $R' \leqslant R$. For the converse, let $r \in (0, R)$ and $r_0 \in (r, R)$. Again by Definition 8.3.3, we know that $(a_n r_0^n)_{n\geqslant 0}$ is bounded. We have

$$na_n r^{n-1} = n(a_n r_0^{n-1})\Big(\frac{r}{r_0}\Big)^{n-1} \xrightarrow[n\to\infty]{} 0,$$

so we also know that $(na_n r^{n-1})_{n\geqslant 1}$ is bounded, that is $r < R'$. When we take $r \to R-$, we find $R \leqslant R'$. Now, we can deduce Eq. (8.13) as a direct consequence of Corollary 8.2.16 and Proposition 8.3.5.
□

Corollary 8.3.21 : *The power series $f(z) = \sum_{n\geqslant 0} a_n z^n$ is of class $\mathcal{C}^\infty$ on $D(0, R)$. For every $p \in \mathbb{N}_0$, the p-th derivative of the power series has the same radius of convergence and writes*

$$\forall z \in D(0,R), \quad f^{(p)}(z) = \sum_{n\geqslant p} n(n-1)\cdots(n-p+1)a_n z^{n-p} = \sum_{n\geqslant p}\binom{n}{p} p! a_n z^{n-p}.$$

In particular, this gives

$$\forall p \in \mathbb{N}_0, \quad a_p = \frac{f^{(p)}(0)}{p!},$$

and

$$\forall z \in D(0,R), \quad f(z) = \sum_{p\geqslant 0} \frac{f^{(p)}(0)}{p!} z^p.$$

Proof : It is a direct consequence of Theorem 8.3.19 with an induction. □

Example 8.3.22 : We have the following identity,

$$\forall z \in D(0,1), \quad \frac{1}{1-z} = \sum_{n\geqslant 0} z^n.$$

Theorem 8.3.19 allows us to differentiate the identity, giving us

$$\forall z \in D(0,1), \quad \frac{1}{(1-z)^2} = \sum_{n\geqslant 1} nz^{n-1} = \sum_{n\geqslant 0}(n+1)z^n. \tag{8.14}$$

By taking higher-order derivatives, for every $p \in \mathbb{N}$, by Corollary 8.3.21, we find

$$\forall z \in D(0,1), \quad \frac{p!}{(1-z)^{p+1}} = \sum_{n \geqslant 0} (n+1)\dots(n+p) z^n \quad \text{or} \quad \frac{1}{(1-z)^{p+1}} = \sum_{n \geqslant 0} \binom{n+p}{p} z^n$$

If we multiply Eq. (8.14) by z then differentiate again, we find

$$\forall z \in D(0,1), \quad \frac{1+z}{(1-z)^3} = \sum_{n \geqslant 1} n^2 z^{n-1} = \sum_{n \geqslant 0} (n+1)^2 z^n.$$

In particular, when $z = \frac{1}{2}$, we find the following identity,

$$\sum_{n \geqslant 1} \frac{n^2}{2^n} = 6.$$

Corollary 8.3.21 gives us following direct consequences, which are very useful when we deal with power series.

Corollary 8.3.23 : *The power series*

$$\begin{array}{rccc} F: & D(0,R) & \to & \mathbb{C} \\ & z & \mapsto & \displaystyle\sum_{n \geqslant 1} \frac{a_n}{n+1} z^{n+1} \end{array}$$

has the same radius of convergence as $\sum a_n z^n$. *Moreover, we have* $F' = f$ *on* $D(0,R)$.

8.3.4 Coefficients of power series

Corollary 8.3.24 (Uniqueness of power series) **:** *Let* $f(z) = \sum_{n \geqslant 0} a_n z^n$ *and* $g(z) = \sum_{n \geqslant 0} b_n z^n$ *be two power series with radius of convergence*

$$R_f := R\Big(\sum_{n \geqslant 0} a_n z^n\Big) > 0, \quad \text{and} \quad R_g := R\Big(\sum_{n \geqslant 0} b_n z^n\Big) > 0.$$

Suppose that there exists $r > 0$ *and* $r \leqslant \min(R_f, R_g)$ *such that* $f \equiv g$ *on* $(-r, r) \subseteq \mathbb{R}$. *Then, we have* $a_n = b_n$ *for all* $n \in \mathbb{N}_0$.

Proof : Let $R = \min(R_f, R_g)$ and consider the following functions defined on $(-R, R)$,

$$\forall z \in (-R, R), \quad f(z) = \sum_{n \geqslant 0} a_n z^n, \quad \text{and} \quad g(z) = \sum_{n \geqslant 0} b_n z^n.$$

It follows from Corollary 8.3.21 that both f and g are $\mathcal{C}^\infty$ functions, and their coefficients are given by

$$\forall n \in \mathbb{N}_0, \quad a_n = \frac{f^{(n)}(0)}{n!}, \quad \text{and} \quad b_n = \frac{g^{(n)}(0)}{n!}.$$

By the assumption that $f \equiv g$ on $(-r, r)$ for some $r \in (0, R]$, we deduce that $f^{(n)}(0) = g^{(n)}(0)$ for all $n \geqslant 0$, so we also have $a_n = b_n$ for all $n \geqslant 0$. $\square$

Example 8.3.25 : Let $f : D(0, R) \to \mathbb{C}$, $z \mapsto \sum_{n \geqslant 0} a_n z^n$ be a power series with $R > 0$. Suppose that f is an even function, that is $f(z) = f(-z)$ for $z \in (-R, R)$. In other words,

$$\forall z \in (-R, R), \quad \sum_{n \geqslant 0} a_n (-z)^n = \sum_{n \geqslant 0} a_n z^n.$$

This implies that

$$\forall n \in \mathbb{N}_0, \quad (-1)^n a_n = a_n.$$

In other words, $a_n = 0$ if n is an odd integer.

Theorem 8.3.26 (Cauchy's formula) **:** *Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence $R > 0$. Then, for any $r \in (0, R)$ and $n \in \mathbb{N}_0$, we have*

$$r^n a_n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{\mathrm{i}\theta}) e^{-\mathrm{i} n\theta} \, \mathrm{d}\theta.$$

Proof : Let us fix $r \in (0, R)$ and $n \in \mathbb{N}_0$. We have

$$\int_0^{2\pi} f(re^{\mathrm{i}\theta}) e^{-\mathrm{i} n\theta} \, \mathrm{d}\theta = \int_0^{2\pi} \left(\sum_{p \geqslant 0} a_p r^p e^{\mathrm{i}(p-n)\theta} \right) \mathrm{d}\theta.$$

Since $\sum |a_p| r^p$ converges, the series of functions $\theta \mapsto \sum a_p r^p e^{\mathrm{i}(p-n)\theta}$ converges normally on $[0, 2\pi]$. We deduce from Corollary 8.2.9 that we may interchange the order between integration and summation. As a consequence,

$$\int_0^{2\pi} f(re^{\mathrm{i}\,\theta})e^{-\,\mathrm{i}\,n\theta}\,\mathrm{d}\theta = \sum_{p\geqslant 0} a_p r^p \int_0^{2\pi} e^{\mathrm{i}(p-n)\theta}\,\mathrm{d}\theta = \sum_{p\geqslant 0} a_p r^p (2\pi)\mathbb{1}_{p=n} = 2\pi r^n a_n. \qquad \square$$

Remark 8.3.27 : This provides another proof of Corollary 8.3.24 if, using its notations, $f \equiv g$ on $D(0,r)$ for some $r \in (0,R)$.

8.3.5 Expansion in power series

In the previous subsections, we were given power series and discussed their properties. In this subsection, we are going to see when and which functions can be written (or exapnded) as a power series.

Definition 8.3.28 : Let $A \subseteq \mathbb{C}$ be an open set and a function $f : A \to \mathbb{C}$.

- Let $R > 0$. If $0 \in A$ and there exists a power series $\sum a_n z^n$ such that

$$\forall z \in D(0,R), \quad f(z) = \sum_{n\geqslant 0} a_n z^n, \tag{8.15}$$

  then we say that f can be written (or expanded) as a power series around 0, or on $D(0,R)$. In particular, such a function needs to be $\mathcal{C}^\infty$ at 0, which is a direct consequence of Corollary 8.3.21.

- Let $z_0 \in A$. We say that f can be written (or expanded) as a power series around z_0 if $z \mapsto f(z+z_0)$ can be written as a power series around 0.

Proposition 8.3.29 : *Let $A \subseteq \mathbb{C}$ be an open set containing 0 and a function $f : A \to \mathbb{C}$. Then, the following properties are equivalent.*

(1) f can be written as a power series around 0.

(2) There exists $r > 0$ such that the series of remainders $(R_n)_{n\geqslant 0}$ converges pointwise to 0 on $D(0,r)$, where

$$\forall n \in \mathbb{N}_0, \forall z \in D(0,r), \quad R_n(z) = f(z) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k. \tag{8.16}$$

When (2) holds, it means that the power series $\sum \frac{f^{(n)}(0)}{n!} z^n$ has radius of convergence R satisfying $R \geqslant r$, and f is equal to the series on $D(0,r)$.

Remark 8.3.30 :

(1) To check Proposition 8.3.29 (2), we use Taylor–Lagrange or Taylor integral formula (Section 4.3.1) to write the remainder as

$$R_n(z) = \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(\theta z), \quad \theta \in (0,1), \quad \text{or} \quad R_n(z) = z^{n+1} \int_0^1 \frac{(1-t)^n}{n!} f^{(n+1)}(tz)\,\mathrm{d}t.$$

(2) We note that to check Proposition 8.3.29 (2), it is not sufficient to check that the radius of convergence of $\sum \frac{f^{(n)}(0)}{n!}$ is strictly positive. Actually, there are functions such that this power series has a strictly positive radius of convergence without Eq. (8.15) holds, see Example 8.3.32 for an example. However, if this radius of convergence is 0, it tells us that f cannot be written as a power series around 0.

Proof : There is nothing to show for (1) $\Rightarrow$ (2). Suppose that (2) holds, let us show (1). Let $r > 0$ satisfying Eq. (8.16). Let $z \in D(0,r)$. The condition $R_n(z) \xrightarrow[n\to\infty]{} 0$ implies that $f(z) = \sum_{n\geqslant 0} \frac{f^{(n)}(0)}{n!} z^n$. Therefore, the sequence $(\frac{f^{(n)}(0)}{n!} z^n)_{n\geqslant 0}$ tends to 0, so is bounded, so the radius of convergence R of the corresponding power series satisfies $R \geqslant |z|$ (Definition 8.3.3). By taking supremum over $z \in D(0,r)$, we find $R \geqslant r$. $\square$

Example 8.3.31 : The following functions can be written as a power series around 0.

(1) The exponential function $z \mapsto \exp(z)$,

$$\forall z \in \mathbb{C}, \quad e^z = \sum_{n\geqslant 0} \frac{z^n}{n!}.$$

In fact, for any $z \in \mathbb{C}$ and $n \geqslant 0$, the n-th remainder writes

$$|R_n(z)| = \frac{|z|^{n+1}}{(n+1)!} |f^{(n+1)}(\theta z)| = \frac{|z|^{n+1}}{(n+1)!} e^{\theta \operatorname{Re}(z)} \xrightarrow[n\to\infty]{} 0.$$

(2) The function $z \mapsto \frac{1}{1-z}$ is defined on $\mathbb{C}\backslash\{1\}$, and we have

$$\forall z \in D(0,1), \quad \frac{1}{1-z} = \sum_{n\geqslant 0} z^n.$$

In fact, for any $z \in D(0,1)$ and $n \geqslant 0$, the n-th remainder writes

$$|R_n(z)| = \left|\frac{z^n}{1-z}\right| \leqslant \frac{|z|^n}{|1-z|} \xrightarrow[n\to\infty]{} 0.$$

(3) Any polynomial function $P \in \mathbb{C}[X]$ satisfies

$$\forall z \in \mathbb{C}, \quad P(z) = \sum_{n \geqslant 0} \frac{P^{(n)}(0)}{n!} z^n.$$

Actually, the above power series contains only finitely many terms.

Example 8.3.32 : Let us consider the function f defined as below,

$$\begin{aligned} f: \quad \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0. \end{cases} \end{aligned}$$

For $k \in \mathbb{N}_0$, we may compute the k-th derivative of f on $(0, +\infty)$,

$$\forall x > 0, \quad f^{(k)}(x) = P_k\Big(\frac{1}{x}\Big) e^{-1/x}, \tag{8.17}$$

where P_k is a polynomial satisfying $\deg(P_k) \leqslant 2k$. Therefore, for each $k \geqslant 0$, we may extend $f^{(k)}$ continuously to 0 by the value 0, so f is a $\mathcal{C}^\infty$ function on $\mathbb{R}$. Therefore, the power series $\sum_{n \geqslant 0} \frac{f^{(n)}(0)}{n!} z^n$ is the zero function. Its radius of convergence is $+\infty$, and is not equal to f on $(0, r)$ for any $r > 0$.

Proposition 8.3.33 : *If f can be written as a power series in $D(0, R)$ for some $R > 0$, then for any $z_0 \in D(0, R)$, f can also be written as a power series around z_0.*

Proof : Let f be a function, $R > 0$, and a power series $\sum a_n z^n$ such that

$$\forall z \in D(0, R), \quad f(z) = \sum_{n \geqslant 0} a_n z^n.$$

Let $z_0 \in D(0, R)$ and $r = R - |z_0|$. It is not hard to see that $D(0, r) \subseteq D(0, R)$. Let $z \in D(z_0, r)$, we write

$$\begin{aligned} \sum_{n \geqslant 0} a_n z^n &= \sum_{n \geqslant 0} a_n \big(z_0 + (z - z_0)\big)^n = \sum_{n \geqslant 0} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^k \\ &= \sum_{n \geqslant 0} \sum_{k \geqslant 0} a_n \mathbb{1}_{n \geqslant k} \binom{n}{k} z_0^{n-k} (z - z_0)^k \end{aligned}$$

We may check that for every $n \geqslant 0$, the series $\sum_{k\geqslant 0} a_n \mathbb{1}_{n\geqslant k}\binom{n}{k} z_0^{n-k}(z-z_0)^k$ converges absolutely (finite series). Additionally, we have

$$\sum_{n\geqslant 0}\sum_{k\geqslant 0} |a_n| \mathbb{1}_{n\geqslant k}\binom{n}{k}|z_0|^{n-k}|z-z_0|^k = \sum_{n\geqslant 0}|a_n|\big(|z_0|+|z-z_0|\big)^n$$

which converges because $|z_0| + |z - z_0| < |z_0| + r = R$. Therefore, Theorem 6.7.4 allows us to interchange the order of summations. We find,

$$\sum_{n\geqslant 0} a_n z^n = \sum_{k\geqslant 0}\sum_{n\geqslant 0} a_n \mathbb{1}_{n\geqslant k}\binom{n}{k} z_0^{n-k}(z-z_0)^k = \sum_{k\geqslant 0}\left(\sum_{n\geqslant k} a_n \binom{n}{k} z_0^{n-k}\right)(z-z_0)^k,$$

which is a power series centered at z_0. □

8.3.6 Applications to ODEs

Power series can be used to solve linear ordinary differential equations with polynomial coefficients. We have two cases.

- We know that the solution can be written as a power series, and we look for recurrence relations between coefficients of the power series. Then, the uniqueness of the cofficients (Corollary 8.3.24) allows us to find this unique solution. See Example 8.3.34.
- We do not know whether the solution can be written as a power series and want to show that there exists such a solution. We apply the same method as in the previous point, and show that the corresponding power series has a strictly positive radius of convergence. This gives us the unique solution that can be written as a power series, see Example 8.3.35. Note that this does not prove any result about the uniqueness of the solution.

Example 8.3.34 : We want to look for a power series expansion of the following function around 0,

$$\begin{aligned} f: \quad \mathbb{R} &\to \mathbb{R} \\ x &\mapsto e^{x^2}\int_0^x e^{-t^2}\,\mathrm{d}t. \end{aligned}$$

The function f can be written as a power series centered at 0 with radius of convergence equal to $+\infty$, because it consists of multiplication and integration of such functions. Additionally, by the fundamental theorem of calculus, we have

$$\forall x \in \mathbb{R}, \quad f'(x) = 2xf(x) + 1, \quad \text{and} \quad f(0) = 0.$$

Suppose that $f(x) = \sum_{n \geqslant 0} a_n x^n$. Then, we have

$$\forall x \in \mathbb{R}, \quad f'(x) = \sum_{n \geqslant 1} n a_n x^{n-1}, \quad \text{and} \quad xf(x) = \sum_{n \geqslant 0} a_n x^{n+1} = \sum_{n \geqslant 2} a_{n-2} x^{n-1}.$$

Therefore,

$$\forall x \in \mathbb{R}, \quad f'(x) - 2xf(x) = a_1 + \sum_{n \geqslant 2} (n a_n - 2a_{n-2}) x^{n-1}.$$

The initial condition $f(0) = 0$ gives $a_0 = 0$. By Corollary 8.3.24, we know that

$$a_1 = 1, \quad \text{and} \quad \forall n \geqslant 2, \quad a_n = \frac{2}{n} a_{n-2}.$$

Thus, by induction, we find that

$$\forall n \geqslant 0, \quad a_{2n} = 0, \quad \text{and} \quad a_{2n+1} = \frac{4^n n!}{(2n+1)!}.$$

We check again (even though not necessary in this example) that the power series define by this sequence of $(a_n)_{n \geqslant 0}$ indeed has radius of convergence equal to $+\infty$, so

$$\forall x \in \mathbb{R}, \quad f(x) = \sum_{n \geqslant 0} \frac{4^n n!}{(2n+1)!} x^{2n+1}.$$

Note that this solution can also be expanded around every $a \in \mathbb{R}$ as a power series.

Example 8.3.35 : Let $\alpha \in \mathbb{C}$. We want to look for a power series expansion of the following function around 0,

$$\begin{array}{rccc} f: & (-1,1) & \to & \mathbb{C} \\ & x & \mapsto & (1+x)^\alpha. \end{array}$$

This function f satisfies the following first-order linear ordinary differential equation,

$$\forall x \in (-1,1), \quad (1+x)f'(x) = \alpha f(x), \quad \text{and} \quad f(0) = 1.$$

Such a differential equation has a unique solution (Theorem 8.4.17). Suppose that $f(x) = \sum_{n \geqslant 0} a_n x^n$

with radius of convergence $R > 0$. Then, we have

$$\forall x \in (-R, R), \quad f'(x) = \sum_{n \geqslant 1} n a_n x^{n-1} = \sum_{n \geqslant 0} (n+1) a_{n+1} x^n, \quad \text{and} \quad x f'(x) = \sum_{n \geqslant 1} n a_n x^n.$$

Therefore,

$$\forall x \in (-R, R), \quad (1+x) f'(x) - \alpha f(x) = \sum_{n \geqslant 0} ((n+1) a_{n+1} + n a_n - \alpha a_n) x^n.$$

From the initial condition $f(0) = 1$, we have $a_0 = 1$. By the uniqueness of the coefficients (Corollary 8.3.24), we find

$$\forall n \in \mathbb{N}_0, \quad a_{n+1} = \frac{\alpha - n}{n+1} a_n.$$

By induction, we deduce that

$$\forall n \in \mathbb{N}_0, \quad a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} = \binom{\alpha}{n}. \tag{8.18}$$

By d'Alembert's criterion, we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\alpha - n}{n+1} \right| \xrightarrow[n \to \infty]{} 1$$

Therefore, the power series $\sum a_n x^n$ defined by the cofficients in Eq. (8.18) has radius of convergence equal to 1, and we conclude that

$$\forall x \in (-1, 1), \quad (1+x)^\alpha = \sum_{n \geqslant 0} \binom{\alpha}{n} x^n = \sum_{n \geqslant 0} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n.$$

This generalizes the binomial expansion to the case with a complex-valued exponent.

8.4 Advanced theorems on uniform convergence

8.4.1 Arzelà-Ascoli theorem

Arzelà-Ascoli theorem is an important theorem in functional analysis, and it allows us to characterize when a subset of continuous functions is compact. In particular, it turns out to be useful to show the existence of solution for some differential equations, see Theorem 8.4.14. First, let us introduce the notion of *equicontinuity*.

Definition 8.4.1 : Let (K, d) be a metric space. In addition, if K is a compact space, the space of continuous functions $\mathcal{C}(K, \mathbb{R})$ is a subset of $\mathcal{B}(K, \mathbb{R})$. We have equipped $\mathcal{B}(K, \mathbb{R})$ with the supremum norm in Definition 8.1.9, which we may induce on the subspace $\mathcal{C}(K, \mathbb{R})$. A subset $\mathcal{F} \subseteq \mathcal{C}(K, \mathbb{R})$ is said to be *equicontinuous* (等度連續) if

$$\forall \varepsilon > 0, \forall x \in M, \exists \delta > 0, \forall f \in \mathcal{F}, \quad y \in B(x, \delta) \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon. \tag{8.19}$$

Remark 8.4.2 : We note that the definition in Eq. (8.19) is much stronger than just requiring that all the functions $f \in \mathcal{F}$ are continuous. Once $\varepsilon > 0$ and $x \in M$ are fixed, this condition needs the choice of $\delta > 0$ to be *uniform* in $f \in \mathcal{F}$.

Example 8.4.3 :

(1) A subset of finitely many continuous functions is equicontinuous.

(2) For every $L > 0$, the set of all the L-Lipschitz continuous functions is equicontinuous.

Theorem 8.4.4 (Arzelà–Ascoli theorem) **:** *Let (K, d) be a compact metric space and $\mathcal{F} \subseteq \mathcal{C}(K, \mathbb{R})$ be a subset. Then, we have the following properties.*

(1) $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is bounded, closed, and equicontinuous.

(2) $\mathcal{F}$ is precompact if and only if $\mathcal{F}$ is bounded and equicontinuous.

Remark 8.4.5 :

(1) We recall that a compact space is necessarily bounded and closed (Proposition 3.1.6), and a bounded and closed set may not be compact (Remark 3.1.34), except that we are in a finite-dimensional normed vector space (Corollary 3.2.24). If the compact metric space K is consisted of a finite number of points, it is clear that $\mathcal{C}(K, \mathbb{R})$ is isomorphic to $\mathbb{R}^n$ for $n = \mathrm{Card}(K)$, which is a finite-dimensional normed vector space, and the theorem becomes trivial. However, for a generic compact metric space K, the space of continuous functions $\mathcal{C}(K, \mathbb{R})$ is not of finite-dimensional.

(2) From Exercise 3.21, we know that a metric space is compact if and only if it is precompact and complete. Moreover, in Exercise 8.30, we can check that if $\mathcal{F}$ is equicontinuous, then so is $\overline{\mathcal{F}}$. Moreover, since $\mathcal{C}(K, \mathbb{R})$ is a Banach space, we see that (2) is a direct consequence of (1).

(3) We also note that $\mathbb{R}$ can be replaced by any Banach space, and the following proof can be adapted accordingly.

Proof :

- Suppose that $\mathcal{F}$ is compact. We already know that it is bounded and closed, so we only need to show that it is equicontinuous. A compact set is also relatively compact (or precompact), see Lemma 3.1.22. Let $\varepsilon > 0$. We may find $N \geqslant 1$ and $f_1, \dots, f_N \in \mathcal{F}$ such that $\mathcal{F} \subseteq \bigcup_{i=1}^{N} B(f_i, \varepsilon)$. Additionally, the finite set of functions $\{f_1, \dots, f_N\}$ is equicontinuous.

  Let $x \in M$. We may find $\delta > 0$ such that

  $$\forall i = 1, \dots, N, \quad y \in B(x, \delta) \quad \Rightarrow \quad |f_i(x) - f_i(y)| \leqslant \varepsilon.$$

  For any given $f \in \mathcal{F}$, we may find $1 \leqslant i \leqslant N$ such that $f \in B(f_i, \varepsilon)$. Then, for any $y \in B(x, \delta)$, we have

  $$|f(x) - f(y)| \leqslant |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \leqslant 3\varepsilon.$$

  This allows us to conclude that $\mathcal{F}$ is equicontinuous.

- Suppose that $\mathcal{F}$ is bounded, closed, and equicontinuous. In order to show that $\mathcal{F}$ is compact, it is sufficient to show that it satisfies the Bolzano–Weierstraß property (Definition 3.1.19), see Theorem 3.1.20.

  Let $(f_n)_{n\geqslant 1}$ be a sequence in $\mathcal{F}$. Since K is compact, we may find a dense sequence in K, that we denote by $(x_n)_{n\geqslant 1}$[2]. We are going to use a diagonal argument to extract a subsequence of $(f_n)_{n\geqslant 1}$ which converges at every x_k for $k \geqslant 1$.

  – The sequence $(f_n(x_1))_{n\geqslant 1}$ is bounded in $\mathbb{R}$, so by the Bolzano–Weierstraß theorem (Theorem 2.2.5), we may find a convergent subsequence, that we denote by $(f_{\varphi_1(n)}(x_1))_{n\geqslant 1}$, where $\varphi_1 : \mathbb{N} \to \mathbb{N}$ is an extraction.

  – Let $m \geqslant 1$. Suppose that we have already constructed extractions $\varphi_1, \dots, \varphi_m$ such that $(f_{\psi_m(n)}(x_k))_{n\geqslant 1}$ converges for all $1 \leqslant k \leqslant m$, where $\psi_m := \varphi_1 \circ \cdots \circ \varphi_m$. Then, the sequence $(f_{\psi_m(n)}(x_{m+1}))_{n\geqslant 1}$ is bounded, so we may find an extraction $\varphi_{m+1} : \mathbb{N} \to \mathbb{N}$ such that $(f_{\psi_m \circ \varphi_{m+1}(n)}(x_{m+1}))_{n\geqslant 1}$ converges. It is clear that for $1 \leqslant k \leqslant m$, the sequence $(f_{\psi_m \circ \varphi_{m+1}(n)}(x_k))_{n\geqslant 1}$ still converges, being a subsequence of a convergent sequence.

  – For $n \geqslant 1$, let $\psi(n) := \varphi_1 \circ \cdots \circ \varphi_n(n)$ and $g_n = f_{\psi(n)}$. Then, $(g_n)_{n\geqslant 1}$ is a subsequence of $(f_n)_{n\geqslant 1}$. From above, for every $k \geqslant 1$, the sequence $(g_n(x_k) = f_{\psi(n)}(x_k))_{n\geqslant k}$ is a subsequence of the convergent sequence $(f_{\psi_k(n)}(x_k))_{n\geqslant 1}$, so the sequence $(g_n(x_k))_{n\geqslant 1}$ converges. We may denote by $f(x_k)$ for the above limit for every $k \geqslant 1$.

Now, we need to show that this convergence can be extended to every $x \in K$, and that this convergence is uniform, so the limit is still in $\mathcal{C}(K, \mathbb{R})$.

Let us fix $\varepsilon > 0$.

- For every $k \geqslant 1$, from the convergence of the sequence $(g_n(x_k))_{n\geqslant 1}$, we may find $N(\varepsilon, x_k) \geqslant 1$ such that

$$\forall m, n \geqslant N(\varepsilon, x_k), \quad |g_m(x_k) - g_n(x_k)| \leqslant \varepsilon. \tag{8.20}$$

- By the equicontinuity of $\mathcal{F}$, for every $z \in K$, we may find $\delta_z > 0$ such that for every $n \geqslant 1$, we have

$$y \in B(z, \delta_z) \quad \Rightarrow \quad |g_n(z) - g_n(y)| \leqslant \varepsilon. \tag{8.21}$$

  The open balls $B(z, \delta_z)$ form an open covering of K, and by the compacity of K, we may find $L \geqslant 1$ and $z_1, \ldots, z_L \in K$ such that

$$K = \bigcup_{i=1}^{L} B(z_i, \delta_{z_i}).$$

  For every $1 \leqslant i \leqslant L$, we may also find $n_i \geqslant 1$ such that $x_{n_i} \in B(z_i, \delta_{z_i})$.

- We may take $N := \max\{N(\varepsilon, x_{n_1}), \ldots, N(\varepsilon, x_{n_L})\}$. This implies that we have a uniform Cauchy condition (Proposition 8.1.8) on $x_{n_1}, \ldots, x_{n_L}$,

$$\forall i = 1, \ldots, L, \forall m, n \geqslant N, \quad |g_m(x_{n_i}) - g_n(x_{n_i})| \leqslant \varepsilon.$$

- Let $x \in K$ and $1 \leqslant i \leqslant L$ such that $x \in B(z_i, \delta_{z_i})$. For $m, n \geqslant N$, we have

$$\begin{aligned}|g_m(x) - g_n(x)| &\leqslant |g_m(x) - g_m(z_i)| + |g_m(z_i) - g_m(x_{n_i})| + |g_m(x_{n_i}) - g_n(x_{n_i})| \\ &\quad + |g_n(x_{n_i}) - g_n(z_i)| + |g_n(z_i) - g_n(x)| \\ &\leqslant 5\varepsilon,\end{aligned}$$

[2] We use the precompactness of K. For every $n \geqslant 1$, we may find finitely many balls with radius $\frac{1}{n}$ that cover K. The union of the centers of these balls over all the integers $n \geqslant 1$ is a countable dense set in K.

where for the middle (thrid) term, we use Eq. (8.20); and for the other terms, we use Eq. (8.21) and the fact that $x, x_{n_i} \in B(z_i, \delta_{z_i})$.

Therefore, for every $x \in K$, the sequence $(g_n(x))_{n\geqslant 1}$ is Cauchy, and we saw from above that the choice of N is independent from the choice of $x \in K$. From this we can deduce that $(g_n(x))_{n\geqslant 1}$ converges for every $x \in K$, and this convergence is uniform, so the limit function is still an element of $\mathcal{C}(K, \mathbb{R})$. □

8.4.2 Stone–Weierstraß theorem

The following Stone–Weierstraß theorem allows us to find sets of functions that can approximate continuous functions uniformly on compact spaces.

Theorem 8.4.6 (Stone–Weierstraß theorem) **:** *Let X be a compact metric space and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The space of continuous functions $\mathcal{C}(X, \mathbb{K})$ equipped with the supremum norm $\|\cdot\|_\infty$ is a normed vector space and a normed algebra. Let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{K})$ be a subalgebra of $\mathcal{C}(X, \mathbb{K})$. Suppose that*

- $1 \in \mathcal{A}$;
- $\mathcal{A}$ separates points, *that is for any $x \neq y \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$;*
- *(in the case $\mathbb{K} = \mathbb{C}$) $f \in \mathcal{A}$ if and only if $\overline{f} \in \mathcal{A}$.*

Then, $\mathcal{A}$ is dense in $\mathcal{C}(X, \mathbb{K})$.

Example 8.4.7 : Below are some examples for which the Stone–Weierstraß theorem applies.

(1) Let $I = [a, b]$ be a segment with $\mathbb{K} = \mathbb{R}$. The set of polynomials $\mathbb{K}[X]$ viewed as functions defined on I is dense in $\mathcal{C}(I, \mathbb{R})$.

(2) Let $I = [a, b]$ be a segment with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The set of all the Lipschitz continuous functions is dense in $\mathcal{C}(I, \mathbb{K})$.

(3) Let $\mathcal{C}_{\text{per}}(\mathbb{R}, \mathbb{C})$ be the set of 2π-periodic continuous functions on $\mathbb{R}$. The set of trigonometric functions, which is spanned by the set $\{x \mapsto e^{\mathrm{i}nx} : n \in \mathbb{Z}\}$, is dense in $\mathcal{C}_{\text{per}}(\mathbb{R}, \mathbb{C})$.

The proof of the Stone–Weierstraß theorem is quite involved. We are going to state a particular example of this theorem, called *Weierstraß approximation theorem*, and prove it using a more elementary approach. After this, we need a few lemmas (Lemma 8.4.11 and Lemma 8.4.12) that allow us to prove the Stone–Weierstraß theorem.

Theorem 8.4.8 (Weierstraß approximation theorem) **:** *Let $I = [a, b]$ be a segment and $\mathcal{C}(I, \mathbb{R})$ be equipped with the supremum norm $\|\cdot\|_\infty$. Let $\mathcal{P}$ be the set of all polynomial functions. Then, $\mathcal{P}$ is dense in $\mathcal{C}(I, \mathbb{R})$. In other words, for any $f \in \mathcal{C}(I, \mathbb{R})$, we may find a sequence of polynomials $(P_n)_{n\geqslant 1}$ such that*

$$\|P_n - f\|_\infty \xrightarrow[n\to\infty]{} 0.$$

Remark 8.4.9 :

(1) It is not hard to check that the set of all polynomials $\mathcal{P}$ is a subalgebra of $\mathcal{C}(I, \mathbb{R})$ and it satisfies the conditions in Theorem 8.4.6. Thus, the Weierstraß approximation theorem can be seen as a special case of the Stone–Weierstraß theorem.

(2) It is important to take $I = [a, b]$ to be a segment. For example, in Exercise 8.6 we have seen that this theorem does not hold if $I = \mathbb{R}$.

The original proof from Weierstraß uses *convolution*, that we do not discuss in this class. The proof we give below is from Bernstein, which can be reformulated using a probabilistic language, in terms of the law of large numbers for Bernoulli random variables.

Proof : Without loss of generality, we may assume that $I = [0, 1]$. For every integer $0 \leqslant k \leqslant n$, let us define

$$\begin{array}{rccc} b_{n,k}: & I & \to & \mathbb{R} \\ & x & \mapsto & \displaystyle\binom{n}{k} x^k (1-x)^{n-k}, \end{array}$$

and for $n \in \mathbb{N}_0$, define

$$\begin{array}{rccc} B_n: & \mathcal{C}(I, \mathbb{R}) & \to & \mathbb{R}[x] \\ & f & \mapsto & \displaystyle\sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x). \end{array}$$

We are going to show that $B_n(f)$ converges to f uniformly.

Given $\varepsilon > 0$. Since f is continuous on the segment I, it is bounded. Let us take $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in I$. By the Heine–Cantor theorem (Theorem 3.1.17), we may find $\eta > 0$ such that

$$\forall x, y \in I, \quad |x - y| < \eta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Then, for any $n \in \mathbb{N}_0$ and $x \in I$, we have

$$\begin{aligned}
|B_n(f)(x) - f(x)| = |B_n(f)(x) - f(x)B_n(1)| &\leqslant \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) \\
&\leqslant \sum_{k \in K_1} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) + \sum_{k \in K_2} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x),
\end{aligned}$$

where

$$K_1 = \left\{ 0 \leqslant k \leqslant n : \left| \frac{k}{n} - x \right| \geqslant \eta \right\}, \quad \text{and} \quad K_2 = \left\{ 0 \leqslant k \leqslant n : \left| \frac{k}{n} - x \right| < \eta \right\}.$$

Using the uniform continuity, the second sum involving indices in K_2 can be bounded from above,

$$\sum_{k \in K_2} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) \leqslant \sum_{k \in K_2} \varepsilon b_{n,k}(x) \leqslant \sum_{k=0}^{n} \varepsilon b_{n,k}(x) = \varepsilon.$$

For the sum involving indices in K_1, we are going to use the following square trick,

$$\begin{aligned}
\sum_{k \in K_1} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_{n,k}(x) \leqslant 2M \sum_{k \in K_1} b_{n,k}(x) &\leqslant \frac{2M}{\eta^2} \sum_{k \in K_1} \left(\frac{k}{n} - x\right)^2 b_{n,k}(x) \\
&\leqslant \frac{2M}{\eta^2} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 b_{n,k}(x) \\
&= \frac{2M}{\eta^2} \left[B_n(x^2) - 2xB_n(x) + x^2 B_n(1) \right].
\end{aligned}$$

Consider the following identity,

$$F(a, b) = [a + (1 - b)]^n = \sum_{k=0}^{n} \binom{n}{k} a^k (1 - b)^{n-k}.$$

Then, we may compute $B_n(1)$, $B_n(x)$, and $B_n(x^2)$ as follow,

$$\begin{aligned}
B_n(1) &= \sum_{k=0}^{n} b_{n,k}(x) = F(x, x) = 1, \\
B_n(x) &= \sum_{k=0}^{n} \frac{k}{n} b_{n,k}(x) = \frac{x}{n} \sum_{k=1}^{n} k \binom{n}{k} x^{k-1} (1 - x)^{n-k}
\end{aligned}$$

$$= \frac{x}{n}\frac{\partial}{\partial a}F(x,x) = \frac{x}{n}n[x+(1-x)]^{n-1} = x,$$

$$B_n(x^2) = \sum_{k=0}^{n}\left(\frac{k}{n}\right)^2 b_{n,k}(x) = \sum_{k=0}^{n}\left(\frac{k(k-1)}{n^2}+\frac{k}{n^2}\right)b_{n,k}(x)$$

$$= \frac{x^2}{n^2}\frac{\partial^2}{\partial a^2}F(x,x) + \frac{x}{n^2}\frac{\partial}{\partial a}F(x,x)$$

$$= \frac{x^2}{n^2}[n(n-1)(x+(1-x))^{n-2}] + \frac{x}{n^2}n[x+(1-x)]^{n-1}$$

$$= x^2 + \frac{x(1-x)}{n}.$$

Therefore, we find

$$\sum_{k\in K_1}\left|f\left(\frac{k}{n}\right)-f(x)\right|b_{n,k}(x) \leqslant \frac{2M}{\eta^2}\frac{x(1-x)}{n} \leqslant \frac{M}{2n\eta^2}.$$

Putting all the inequalities together, we obtain

$$|B_n(f)(x)-f(x)| \leqslant \varepsilon + \frac{M}{2n\eta^2}.$$

By taking the supremum norm then $\limsup$ over n, we find

$$\limsup_{n\to\infty}\|B_n(f)-f\|_\infty \leqslant \varepsilon.$$

Since the above holds for any arbitrary $\varepsilon > 0$, we deduce that $\limsup_{n\to\infty}\|B_n(f)-f\|_\infty = 0$. □

We need to introduce the notion of *lattice*, and state the lattice version of the Stone–Weierstraß theorem. This will allow us to recover the original version in Theorem 8.4.6.

Definition 8.4.10 : Let X be a compact metric space and $\mathcal{L} \subseteq \mathcal{C}(X,\mathbb{R})$ be a subset. We say that $\mathcal{L}$ is a *lattice* if

$$\forall f,g \in \mathcal{L}, \quad \max\{f,g\}, \min\{f,g\} \in \mathcal{L}.$$

Lemma 8.4.11 : *For any $a > 0$, there exists a sequence of polynomials that converges uniformly on $[-a,a]$ to the function $x \mapsto |x|$.*

Proof : There are two ways to prove this lemma. It can either be seen as a direct consequence of the Weierstraß approximation theorem (Theorem 8.4.8), or be proven by construction.

By scaling, we may assume that $a = 1$. We note that for $x \in [-1,1]$, and $u = 1-x^2 \in [0,1]$, we have

$$|x| = \sqrt{x^2} = \sqrt{1-(1-x^2)} = \sqrt{1-u}$$

If $|u| < 1$, we have

$$\sqrt{1-u} = \sum_{n\geqslant 0} a_n(-u)^n, \quad \text{where } a_n = \binom{1/2}{n}, \tag{8.22}$$

where the power series comes from Example 8.3.35, and it has radius of convergence equal to 1. We want to show that this power series converges uniformly for $u \in [0,1]$. We may check that it converges normally, then the uniform convergence follows, see Proposition 8.1.22. For this, it suffices to check that $\sum a_n$ converges absolutely. For $n \in \mathbb{N}_0$, we have

$$\begin{aligned} a_n = \frac{\frac{1}{2}(-\frac{1}{2})\dots(\frac{1}{2}-n+1)}{n!} &= \frac{(-1)^{n-1}}{2^n}\frac{(2n-3)!!}{n!} \\ &= \frac{(-1)^{n-1}}{2^n}\frac{(2n-3)!!(2n-2)!!}{n!(2n-2)!!} = \frac{(-1)^{n-1}}{2^{2n-1}}\frac{(2n-2)!}{n!(n-1)!}, \end{aligned}$$

and the Stirling's formula gives us $|a_n| \sim \text{cst} \cdot n^{-3/2}$. This means that $\sum a_n$ converges absolutely. □

Lemma 8.4.12 : *Any closed subalgebra $\mathcal{A} \subseteq \mathcal{C}(X,\mathbb{R})$ is a lattice.*

Proof : Let $\mathcal{A} \subseteq \mathcal{C}(X,\mathbb{R})$ be a subalgebra. Given $f, g \in \mathcal{A}$, we have

$$\max\{f,g\} = \frac{f+g}{2} + \frac{|f-g|}{2}, \quad \text{and} \quad \min\{f,g\} = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

Therefore, it is sufficient to show that for $h \in \mathcal{A}$, we also have $|h| \in \mathcal{A}$ to conclude. Let $h \in \mathcal{A}$. Due to the continuity of h and the compacity of X, we can define $a := \max_{x\in X} |h(x)| < \infty$, see Proposition 3.1.12. By Lemma 8.4.11, we may find a sequence of polynomials $(P_n)_{n\geqslant 1}$ that converges uniformly to the absolute value function on $[-a,a]$. For every $n \geqslant 1$, define $h_n = P_n(h) \in \mathcal{A}$. Therefore, $(h_n)_{n\geqslant 1}$ is a sequence of functions that converges uniformly to $|h|$ on X. Since $\mathcal{A}$ is closed, we conclude that $|h| \in \mathcal{A}$. □

Theorem 8.4.13 : *Let X be a compact metric space with at least two points and $\mathcal{L} \subseteq \mathcal{C}(X,\mathbb{R})$ be a lattice. Suppose that for any $x \neq y \in X$ and $a, b \in \mathbb{R}$, there exists $f \in \mathcal{L}$ with $f(x) = a$ and $f(y) = b$. Then, $\mathcal{L}$ is dense in $\mathcal{C}(X,\mathbb{R})$.*

Proof : Let $\mathcal{L} \subseteq \mathcal{C}(X, \mathbb{R})$ be a lattice. Let $g \in \mathcal{C}(X, \mathbb{R})$ and $\varepsilon > 0$. We want to construct a function $f \in \mathcal{L}$ such that $\|f - g\|_\infty \leqslant \varepsilon$.

For any $a, b \in X$, we may find $f_{a,b} \in \mathcal{L}$ such that $f_{a,b}(a) = g(a)$ and $f_{a,b}(b) = g(b)$. By the continuity of $f_{a,b}$ and g, we know that there exists an open set $U_{a,b}$ containing b such that $f_{a,b} \geqslant g - \varepsilon$ on $U_{a,b}$. Since $(U_{a,b})_{b \in X}$ is an open covering of the compact space X, by the Borel–Lebesgue property (Definition 3.1.3), we may find $b_1, \ldots, b_m \in X$ such that $(U_{a,b_i})_{1 \leqslant i \leqslant m}$ covers X. Let $f_a := \sup_{1 \leqslant i \leqslant m} f_{a,b_i} \in \mathcal{L}$. Then, we have $f_a(a) = a$ and $f_a \geqslant g - \varepsilon$ on X. Similarly, by the continuity of f_a and g, there exists an open set V_a containing a such that $f_a \leqslant g + \varepsilon$ on V_a. Since $(V_a)_{a \in X}$ is an open covering of the compact space X, again by the Borel–Lebesgue property (Definition 3.1.3), we may find $a_1, \ldots, a_n \in X$ such that $(V_{a_j})_{1 \leqslant j \leqslant n}$ covers X. Let $f := \inf_{1 \leqslant j \leqslant n} f_{a_j}$. Then, we may easily check that $g - \varepsilon \leqslant f \leqslant g + \varepsilon$ on X, so $\|f - g\|_\infty \leqslant \varepsilon$. This concludes that $\mathcal{L}$ is dense in $\mathcal{C}(X, \mathbb{R})$. $\square$

Proof of Proof of Theorem 8.4.6: Let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{R})$ be a subalgebra satisfying the assumptions in Theorem 8.4.6. We write $\mathcal{L} = \overline{\mathcal{A}}$, which is still a subalgebra, because addition, multiplication, and scalar multiplication are continuous. It follows from Lemma 8.4.12 that $\mathcal{L}$ is a lattice. Now, let us check that the assumptions in Theorem 8.4.13 are satisfied.

Let $x \neq y \in X$ and $a, b \in \mathbb{R}$. By the assumptions in Theorem 8.4.6, we may find $p \in \mathcal{A}$ such that $p(x) \neq p(y)$. Since $1 \in \mathcal{A}$, we may also add $c \times 1 \in \mathcal{A}$ to p, to make $p(x) + c \neq 0$ and $p(y) + c \neq 0$. Without loss of generality, let us assume that $p(x) \neq p(y)$, $p(x) \neq 0$, and $p(y) \neq 0$ for some $p \in \mathcal{A}$. Then, we may look for $f \in \mathcal{A}$ in the form $f = \alpha p + \beta p^2$, where $\alpha, \beta \in \mathbb{R}$ can be chosen properly so that $f(x) = a$ and $f(y) = b$. Therefore, Theorem 8.4.13 tells us that $\overline{\mathcal{L}} = \mathcal{C}(X, \mathbb{R})$, that is $\overline{\mathcal{A}} = \mathcal{C}(X, \mathbb{R})$.

For the complex version of the theorem, we proceed as follows. Let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{C})$ be a subalgebra satisfying the assumptions in Theorem 8.4.6. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be the set of real-valued functions in $\mathcal{A}$, which is a $\mathbb{R}$-subalgebra of $\mathcal{C}(X, \mathbb{R})$. We want to check that $\overline{\mathcal{A}_0} = \mathcal{C}(X, \mathbb{R})$. First, it is not hard to check that $1 \in \mathcal{A}_0$. Then, for any $f \in \mathcal{A}$, since $\overline{f} \in \mathcal{A}$, we deduce that $\mathrm{Re}(f), \mathrm{Im}(f) \in \mathcal{A}_0$. For any $x \neq y \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$, so we need to have $\mathrm{Re}(f)(x) \neq \mathrm{Re}(f)(y)$ or $\mathrm{Im}(f)(x) \neq \mathrm{Im}(f)(y)$. This means that $\mathcal{A}_0$ separates points. By the real version of the theorem, we conclude that $\overline{\mathcal{A}_0} = \mathcal{C}(X, \mathbb{R})$. For any function $f \in \mathcal{C}(X, \mathbb{C})$ and $\varepsilon > 0$, we may find $g_1, g_2 \in \mathcal{A}_0$ such that

$$\|\mathrm{Re}(f) - g_1\|_\infty \leqslant \varepsilon, \quad \text{and} \quad \|\mathrm{Im}(f) - g_2\|_\infty \leqslant \varepsilon.$$

Since $\mathcal{A}$ is a $\mathbb{C}$-algebra, we know that $g_1 + \mathrm{i}\, g_2 \in \mathcal{A}$. Moreover,

$$\|f - (g_1 + \mathrm{i}\, g_2)\|_\infty \leqslant \|\mathrm{Re}(f) - g_1\|_\infty + \|\mathrm{Im}(f) - g_2\|_\infty \leqslant 2\varepsilon.$$

This shows that $\mathcal{A}$ is dense in $\mathcal{C}(X, \mathbb{C})$. $\square$

8.4.3 Peano existence theorem

As an application of the Arzelà–Ascoli theorem and the Stone–Weierstraß theorem, we have the following Peano existence theorem, which gives us the existence of solution for differential equations.

Theorem 8.4.14 (Peano existence theorem) **:** *Fix an integer $n \geqslant 1$. Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$ be a non-empty open subset, and $F : \Omega \to \mathbb{R}^n$ be a continuous function. Let $t_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^n$ such that $(t_0, y_0) \in \Omega$. Let $a, b > 0$ such that*

$$\mathcal{R} := \{(t, y) : |t - t_0| \leqslant a, \|y - y_0\| \leqslant b\} \subseteq \Omega.$$

Let $M > 0$ and suppose that $\|F(t, y)\| \leqslant M$ for $(t, y) \in \mathcal{R}$. Then, the following differential equation

$$\begin{cases} y'(t) = F(t, y(t)), & \forall t \in \mathring{I}, \\ y(t_0) = y_0, \end{cases}$$

has a solution $t \mapsto y(t)$ defined on $I := [t_0 - a', t_0 + a']$ with $a' = \min\{a, \frac{b}{M}\}$.

Remark 8.4.15 : It is important to note that the Peano existence theorem does not guarantee uniqueness, see Example 8.4.16. In order to have a unique solution, the function F needs to satisfy stronger properties, as stated in the Picard–Lindelöf theorem, also known as the Cauchy–Lipschitz theorem, see Theorem 8.4.17.

Proof : The proof consists of three parts: (1) We reformulate the solution to the differential equation as a fixed-point problem; (2) we show the existence of the solution in the case that F is a Lipschitz continuous function; (3) we show the existence in the general setting.

Without loss of generality, we may assume that $t = 0$ and $y_0 = 0 \in \mathbb{R}^n$ by a translation in time and in space.

(1) First, let us reformulate this as a solution to some fixed-point problem. Let us write $\mathcal{X} = \mathcal{C}(I, \overline{B}(0, b))$. Consider the following operator,

$$\begin{aligned} T : \quad \mathcal{X} &\to \quad \mathcal{X} \\ f &\mapsto \int_0^t F(s, f(s))\, \mathrm{d}s. \end{aligned}$$

Let us check that for $f \in \mathcal{X}$, the image $T(f)$ is well defined. We first note that $(s, f(s)) \in \mathcal{R}$ for any $s \in I$, so for any $t \in I$, we have

$$\|T(f)(t)\| = \left\| \int_0^t F(s, f(s))\, \mathrm{d}s \right\| \leqslant |t| M \leqslant b.$$

In other words, $T(f)$ is a function from I to $\overline{B}(0, b)$. Moreover, it follows from the fundamental theorem of calculus that $T(f)$ is of class $\mathcal{C}^1$, so we do have $T(f) \in \mathcal{X}$. As a consequence, if y is a fixed point of T, that is $T(y) = y$, we deduce that y is of class $\mathcal{C}^\infty$. Moreover, if y is a fixed point, by taking the derivative at $t \in I$, we find

$$y'(t) = (T(y))'(t) = F(t, y(t)).$$

We may also check easily that $y(0) = T(y)(0) = 0$. Therefore, the conclusion of Theorem 8.4.14 is equivalent to showing that T has at least one fixed point.

(2) Let us assume that F is an L-Lipschitz continuous function on $\mathcal{R}$. In this case, we can easily check that T is an (La')-Lipschitz continuous function, so it is continuous.

We are going to define a sequence of functions $(y_n)_{n\geqslant 1}$ which are elements of $\mathcal{X}$. First, let y_1 be the constant zero function, which is indeed in $\mathcal{X}$. For $n \geqslant 1$, we define $y_{n+1} = T(y_n)$, which is in $\mathcal{X}$ from (1). By induction, we establish a sequence $(y_n)_{n\geqslant 1}$ in $\mathcal{X}$. Moreover, for any $t, t' \in I$ and $n \geqslant 1$, we have

$$\|y_n(t) - y_n(t')\| = \left\| \int_{t'}^t F(s, y_{n-1}(s))\, \mathrm{d}s \right\| \leqslant M|t - t'|. \tag{8.23}$$

This means that $(y_n)_{n\geqslant 1}$ is a sequence of equicontinuous functions. The Arzelà–Ascoli theorem[3] allows us to find a convergent subsequence $(y_{\varphi(n)})_{n\geqslant 1}$ with limit $y \in \mathcal{X}$. We want to check that $T(y) = y$.

Let us denote $I_+ = I \cap \mathbb{R}_+ = [0, a']$. For every $n \geqslant 1$ and $t \in I_+$, let us define

$$M_n(t) := \sup_{0\leqslant s\leqslant t} \|T(y_n)(s) - y_n(s)\| = \sup_{0\leqslant s\leqslant t} \|y_{n+1}(s) - y_n(s)\|.$$

For $n \geqslant 2$ and $s \in I_+$, we have

$$\begin{aligned}\|T(y_n)(s) - y_n(s)\| &= \|T(y_n)(s) - T(y_{n-1})(s)\| \\ &= \left\|\int_0^s (F(u, y_n(u)) - F(u, y_{n-1}(u)))\,\mathrm{d}u\right\| \\ &\leqslant \int_0^s LM_{n-1}(u)\,\mathrm{d}u,\end{aligned}$$

which implies that

$$\forall t \in I_+, \quad M_n(t) \leqslant L\int_0^t M_{n-1}(u)\,\mathrm{d}u. \tag{8.24}$$

We may compute M_1 as below,

$$\forall t \in I_+, \quad M_1(t) = \sup_{0\leqslant s\leqslant t} \|y_2(s)\| = \sup_{0\leqslant s\leqslant t} \left\|\int_0^s F(u,0)\,\mathrm{d}u\right\| \leqslant tM.$$

Then, for M_2, we apply Eq. (8.24) and find

$$\forall t \in I_+, \quad M_2(t) \leqslant L\int_0^t M_1(u)\,\mathrm{d}u = \frac{t^2}{2}LM.$$

By induction, we find, for every $n \geqslant 1$,

$$\forall t \in I_+, \quad M_n(t) \leqslant \frac{t^n}{n!}L^{n-1}M \leqslant \frac{(a')^n}{n!}L^{n-1}M \xrightarrow[n\to\infty]{} 0.$$

Therefore, this allows us to conclude that $(T(y_{\varphi(n)}) - y_{\varphi(n)})_{n\geqslant 1}$ uniformly converges to 0 on I_+. Then, a similar argument allows us to get the uniform convergence to 0 on $I_- := I \cap \mathbb{R}_-$, so this convergence is uniform on I. Since $y_{\varphi(n)}$ uniformly converges to y and T is continuous, we deduce that $T(y_{\varphi(n)})$ uniformly converges to $T(y)$, giving us $T(y) = y$.

(3) If F is only continuous, by the Stone–Weierstraß theorem (Theorem 8.4.6), we may find a sequence of Lipschitz continuous functions $(F_n)_{n\geqslant 1}$ that converges uniformly to F on $\mathcal{R}$. For every $n \geqslant 1$, let y_n be the corresponding solution to the differential equation with F replaced by F_n. Then, $(y_n)_{n\geqslant 1}$ is a sequence in $\mathcal{X}$. Since $(F_n)_{n\geqslant 1}$ converges to F uniformly on $\mathcal{R}$, we know that $(F_n)_{n\geqslant 1}$ can be uniformly bounded by a constant $M' > 0$. This implies that the sequence of functions $(y_n)_{n\geqslant 1}$ is equicontinuous due to the same Eq. (8.23), with M replaced by M'. Therefore, the Arzelà–Ascoli theorem gives us a subsequence $(y_{\varphi(n)})_{n\geqslant 1}$ that converges uniformly to $y \in \mathcal{X}$, and we need to check that $T(y) = y$. To achieve this, we start by checking that the

[3] Theorem 8.4.4 (2) tells us that the set $\{y_n : n \geqslant 1\}$ is a precompact subset. It can be shown that there exists a subsequence of $(y_n)_{n\geqslant 1}$ which is a Cauchy sequence, see Exercise 8.31. Then, this subsequence converges by the completeness of $\mathcal{X}$.

functions in the sequence $(s \mapsto F_{\varphi(n)}(s, y_{\varphi(n)}(s)))_{n\geqslant 1}$ are equicontinuous.

Let $\varepsilon > 0$. For $n \geqslant 1$ and $s, t \in I$, we write

$$\begin{aligned}
&\|F_n(s, y_n(s)) - F_n(t, y_n(t))\| \\
&\leqslant \|F_n(s, y_n(s)) - F(s, y_n(s))\| + \|F(s, y_n(s)) - F(s, y(s))\| + \|F(s, y(s)) - F(t, y(t))\| \\
&\quad + \|F(t, y(t)) - F(t, y_n(t))\| + \|F(t, y_n(t)) - F_n(t, y_n(t))\|
\end{aligned}$$

Since $s \mapsto F(s, y(s))$ is continuous on the segment I, it is uniformly continuous. Similarly, the map $(t, y) \mapsto F(t, y)$ is also uniformly continuous on $\mathcal{R}$. We may take $\eta > 0$ such that

$$\begin{aligned}
|t - s| \leqslant \eta \quad &\Rightarrow \quad \|F(s, y(s)) - F(t, y(t))\| \leqslant \varepsilon, \\
\|(t, y) - (s, x)\| \leqslant \eta \quad &\Rightarrow \quad \|F(t, y) - F(s, x)\| \leqslant \varepsilon.
\end{aligned}$$

Since $y_{\varphi(n)} \xrightarrow[n\to\infty]{} y$ uniformly and $F_{\varphi(n)} \xrightarrow[n\to\infty]{} F$ uniformly, there exists $N \geqslant 1$ such that

$$\forall n \geqslant N, \quad \left\|y_{\varphi(n)} - y\right\|_\infty \leqslant \eta, \quad \text{and} \quad \left\|F_{\varphi(n)} - F\right\|_\infty \leqslant \varepsilon.$$

Therefore, for $n \geqslant N$, and $s, t \in I$ such that $|s - t| \leqslant \eta$, we find

$$\left\|F_{\varphi(n)}(s, y_{\varphi(n)}(s)) - F_{\varphi(n)}(t, y_{\varphi(n)}(t))\right\| \leqslant 5\varepsilon$$

This means that $(s \mapsto F_{\varphi(n)}(s, y_{\varphi(n)}(s)))_{n\geqslant 1}$ is equicontinuous, so has a convergent subsequence, and we denote the corresponding extraction by ψ. Therefore, for $t \in I$, we have

$$T(y_{\varphi\circ\psi(n)})(t) = \int_0^t F_{\varphi\circ\psi(n)}(s, y_{\varphi\circ\psi(n)}(s))\,\mathrm{d}s \xrightarrow[n\to\infty]{} \int_0^t F(s, y(s))\,\mathrm{d}s = T(y)(t),$$

which is uniform in $t \in I$ by Proposition 8.2.5. We conclude that $T(y) = y$. □

Example 8.4.16 : Let us take $n = 1$, and $F(t, y) = \sqrt{|y|}$ with initial condition $(t_0, y_0) = (0, 0)$. In other words, the differential equation we are looking at is

$$y'(t) = \sqrt{|y(t)|} \quad \text{and} \quad y(0) = 0. \tag{8.25}$$

We have many different solutions to Eq. (8.25),

- $y(t) = 0$ for $t \in \mathbb{R}$;

- $y(t) = \frac{t|t|}{4}$ for $t \in \mathbb{R}$;
- for any $a > 0$, $y(t) = \frac{(t-a)^2}{4}$ for $t \geqslant a$ and $y(t) = 0$ for $t \leqslant a$.

Indeed, the function $x \mapsto \sqrt{|x|}$ is not locally Lipschitz continuous at 0, so does not satisfy the assumptions of the Picard–Lindelöf theorem (Theorem 8.4.17).

The following Picard–Lindelöf theorem, also known as Cauchy–Lipschitz theorem, gives sufficient conditions for the solution to an ordinary differential equation to be unique.

Theorem 8.4.17 (Picard–Lindelöf theorem or Cauchy–Lipschitz theorem) **:** *Let us fix the same notations as in the statement of Theorem 8.4.14. In addition, suppose that F is L-Lipschitz continuous in the second variable in $\mathcal{R}$. Then, apart from the existence provided in Theorem 8.4.14, we also have uniqueness of the solution, in the sense that if J is an interval containing t_0 and $\varphi : J \to \mathbb{R}^n$ is a solution, then y and φ conincide on $I \cap J$.*

Proof : We keep the notations from the proof of Theorem 8.4.14. In particular, we want to show that the map T defined therein has a unique fixed point. More precisely, we want to show that there exists an integer $m \in \mathbb{N}$ such that T^m is a contraction, then we may conclude by Exercise 3.24.

Let $f, g \in \mathcal{X}$. We proceed in a similar way as in (2) in the proof of Theorem 8.4.14. For $n \geqslant 1$ and $t \in I_+$, let us define

$$K_n(t) := \sup_{0 \leqslant s \leqslant t} \|(T^n f)(s) - (T^n g)(s)\| .$$

For $n \geqslant 2$ and $s \in I_+$, we have

$$\begin{aligned} \|T^n(f)(s) - T^n(g)(s)\| &= \left\| \int_0^s F(u, T^{n-1}(f)(u)) - F(u, T^{n-1}(g)(u)) \,\mathrm{d}u \right\| \\ &\leqslant \int_0^s L \left\| T^{n-1}(f)(u) - T^{n-1}(g)(u) \right\| \mathrm{d}u \\ &\leqslant \int_0^s L K_{n-1}(u) \,\mathrm{d}u, \end{aligned}$$

which implies that

$$\forall t \in I_+, \quad K_n(t) \leqslant L \int_0^t K_{n-1}(u) \,\mathrm{d}u.$$

We may compute K_1 as below,

$$\forall t \in I_+, \quad K_1(t) = \sup_{0 \leqslant s \leqslant t} \left\| \int_0^t F(u, f(u)) - F(u, g(u)) \,\mathrm{d}u \right\| \leqslant Lt \|f - g\|_\infty .$$

By induction, we find, for every $n \geqslant 1$,

$$\forall t \in I, \quad K_n(t) \leqslant \frac{t^n}{n!} L^n \|f - g\|_\infty \leqslant \frac{(a')^n}{n!} L^n \|f - g\|_\infty \xrightarrow[n \to \infty]{} 0,$$

which tells us that T^n is a contraction map for large enough n. □

8.5 Theorems on convergence of integrals

In Proposition 8.2.5, we saw that the uniform convergence of a sequence of functions implies the uniform convergence of their primitives. As a consequence, the sequence of integrals also converges. In practice, however, we are more interested in the convergence of integrals. We have already seen in Example 8.2.8 that a sequence of integrals may converge without the sequence of integrands converges uniformly. Below we are going to prove the monotone convergence theorem (Theorem 8.5.3) and the dominated convergence theorem (Theorem 8.5.5), which are consequences of Eq. (8.26).

8.5.1 Monotone convergence theorem

We start with the following key lemma.

Lemma 8.5.1 : *Let $I \subseteq \mathbb{R}$ be an interval. Let $(u_n)_{n\geqslant 1}$ be a sequence of piecewise continuous functions from I to a Banach space $(W, \|\cdot\|)$. Suppose that*

(i) for each $n \geqslant 1$, u_n is integrable on I;

(ii) the series of functions $\sum u_n$ converges pointwise to a piecewise continuous function $f : I \to W$;

(iii) the series $\sum_n \int_I \|u_n\|$ converges.

Then, f is integrable on I and

$$\int_I \|f\| \leqslant \sum_{n\geqslant 1} \int_I \|u_n\|, \quad \text{and} \quad \int_I f = \sum_{n\geqslant 1} \int_I u_n. \tag{8.26}$$

Proof : We are going to prove this in three steps: (1) I is a segment and all the functions are continuous; (2) I is a segment and all the functions are piecewise continuous; (3) I is an interval and all the functions are piecewise continuous.

(1) If $I = [a, b]$ is a segment, and all the u_n's and f are continuous functions, the proof is similar to the Dini's theorem (Theorem 8.1.14).

Let $\varepsilon > 0$ and define

$$\forall n \geqslant 1, \quad E_n = \{x \in [a, b] : \|f(x)\| - \sum_{k=1}^{n} \|u_k(x)\| < \varepsilon\}. \tag{8.27}$$

The continuity implies that E_n is open for every $n \geqslant 1$. The pointwise convergence of $\sum u_n$ to f implies that $\bigcup_{n\geqslant 1} E_n = [a, b]$. Since $[a, b]$ is a compact set, by the Borel–Lebesgue property, we may find $N \geqslant 1$ such that $\bigcup_{n=1}^{N} E_n = [a, b]$. Therefore, we have

$$\int_{[a,b]} \|f\| \leqslant \int_{[a,b]} \Big(\sum_{k=1}^{N} \|u_k\| + \varepsilon \Big) = \sum_{k=1}^{N} \int_{[a,b]} \|u_k\| + \varepsilon(b-a) \leqslant \sum_{n\geqslant 1} \int_{[a,b]} \|u_n\| + \varepsilon(b-a).$$

The above inequality holds for any arbitrary $\varepsilon > 0$, so we deduce that

$$\int_{[a,b]} \|f\| \leqslant \sum_{n\geqslant 1} \int_{[a,b]} \|u_n\| .$$

(2) Next, we suppose that $I = [a, b]$ is a segment, and all the u_n's and f are piecewise continuous. Let $\varepsilon > 0$. From Lemma 8.5.2, we may find continuous functions g and $(v_n)_{n\geqslant 1}$ such that

$$\begin{aligned} g \leqslant \|f\| \quad &\text{such that} \quad \int_I \|f\| \leqslant \varepsilon + \int_I g, \\ \forall n \geqslant 1, \|u_n\| \leqslant v_n \quad &\text{such that} \quad \int_I v_n \leqslant \frac{\varepsilon}{2^n} + \int_I \|u_n\| . \end{aligned}$$

Define the following subsets as in Eq. (8.27), but for the continuous functions g and $(v_n)_{n\geqslant 1}$,

$$\forall n \geqslant 1, \quad G_n = \{x \in [a, b] : g(x) - \sum_{k=1}^{n} v_k(x) < \varepsilon\}.$$

Similarly, we know that there exists $N \geqslant 1$ such that $\bigcup_{n=1}^{N} G_n = [a, b]$. Therefore, we find

$$\begin{aligned} \int_I \|f\| \leqslant \varepsilon + \int_I g \leqslant \varepsilon + \int_I \Big(\sum_{k=1}^{N} v_k + \varepsilon \Big) &= (b-a+1)\varepsilon + \sum_{k=1}^{N} \int_I v_k \\ &\leqslant (b-a+1)\varepsilon + \sum_{k=1}^{N} \Big(\frac{\varepsilon}{2^k} + \int_I \|u_k\| \Big) \leqslant (b-a+2)\varepsilon + \sum_{k=1}^{N} \int_I \|u_k\| \\ &\leqslant (b-a+2)\varepsilon + \sum_{n\geqslant 1} \int_I \|u_n\| . \end{aligned}$$

Then we conclude as in the previous point.

(3) For any subsegment $J \subseteq I$, from above, we have

$$\int_J \|f\| \leqslant \sum_{n\geqslant 1} \int_J \|u_n\| \leqslant \sum_{n\geqslant 1} \int_I \|u_n\| < \infty.$$

Therefore, f is integrable on I and satisfies

$$\int_I \|f\| \leqslant \sum_{n\geqslant 1} \int_I \|u_n\| < \infty,$$

which is the first part of Eq. (8.26).

For the second part of Eq. (8.26), let us apply the first part to the remainder $\sum_{k\geqslant n+1} u_k = f - \sum_{k=1}^n u_k$, and we find

$$\int_I \left\| f - \sum_{k=1}^n u_k \right\| \leqslant \sum_{k\geqslant n+1} \int_I \|u_k\| \xrightarrow[n\to\infty]{} 0,$$

since the right side in the above relation is the remainder of the convergent series $\sum \int_I \|u_n\|$. Then, it follows that

$$\left\| \int_I f - \sum_{k=1}^n \int_I u_k \right\| = \left\| \int_I \Big(f - \sum_{k=1}^n u_k\Big) \right\| \leqslant \int_I \left\| f - \sum_{k=1}^n u_k \right\| \xrightarrow[n\to\infty]{} 0,$$

which gives us the relation

$$\int_I f = \lim_{n\to\infty} \sum_{k=1}^n \int_I u_k = \sum_{n\geqslant 1} \int_I u_n.$$

□

Lemma 8.5.2 : *Let $J = [a,b]$ be a segment of $\mathbb{R}$ and $f \in \mathcal{PC}(J,\mathbb{R})$. For every $\varepsilon > 0$, there exists continuous functions f_- and f_+ on J such that*

$$f_- \leqslant f \leqslant f_+ \quad \textit{and} \quad \Big(\int_J f_+\Big) - \varepsilon \leqslant \int_J f \leqslant \Big(\int_J f_-\Big) + \varepsilon.$$

Proof : If f is continuous, then there is nothing to prove. Suppose that f has discontinuities. Let $P = (x_k)_{0\leqslant k\leqslant n}$ be a partition of $[a,b]$ such that f restricted on (x_{k-1}, x_k) can be extended to a continuous function on $[x_{k-1}, x_k]$ for every $1 \leqslant k \leqslant n$. From Proposition 7.1.3, we know that f is bounded on

$[a, b]$, so we may take

$$M > \sup_{x \in J} f(x) \quad \text{and} \quad m < \inf_{x \in J} f(x).$$

Let $\delta > 0$ with $\delta < \frac{1}{2} \|P\|$, so that we may define disjoint intervals $J_i := B(x_i, \delta) \cap J$ for $0 \leqslant i \leqslant n$. We define a continuous function φ_- on J as below,

$$\varphi_-(x) = \begin{cases} m + (M-m)\frac{|x-x_i|}{\delta} & \text{if } x \in J_i, \\ M & \text{otherwise.} \end{cases}$$

Then, the function $f_- := \min(f, \varphi_-)$ satisfies $f_- \leqslant f$ on J is continuous. In fact, we can see that

- if $x \neq x_i$ for all i, then f is continuous at x, and $f_- = \frac{1}{2}(f + \varphi_- - |f - \varphi_-|)$ is also continuous at x;
- if $x = x_i$ for some i, then $\varphi_-(x) = m < \inf_{x \in J} f(x)$, so we may find $\varepsilon > 0$ such that φ_- stays strictly below f on $B(x, \varepsilon)$. This means that $f_- = \varphi_-$ on $B(x, \varepsilon)$, so we get the continuity of f_- at x.

Then, let us compute the following integral,

$$\int_J (f - f_-) = \sum_{i=1}^n \int_{J_i} (f - f_-) \leqslant \sum_{i=1}^n \int_{J_i} (M - m) \leqslant 2\delta n(M - m),$$

where the equality is obtained from the fact that when $x \notin J_i$ for all i, $\varphi_-(x) = M > f(x)$, so $f_-(x) = f(x)$. To conclude, for $\varepsilon > 0$, we may choose $\delta \leqslant \min\{\frac{\varepsilon}{2(M-m)n}, \frac{1}{4} \|P\|\}$, which will give us

$$\int_J (f - f_-) \leqslant \varepsilon \quad \Leftrightarrow \quad \int_J f \leqslant \Big(\int_J f_-\Big) + \varepsilon.$$

For the construction of f_+, we proceed in a similar way. We consider the following continuous function φ_+ on J,

$$\varphi_+(x) = \begin{cases} M - (M-m)\frac{|x-x_i|}{\delta} & \text{if } x \in J_i, \\ m & \text{otherwise.} \end{cases}$$

Then, we define $f_+ := \max(f, \varphi_+)$. □

Theorem 8.5.3 (Monotone convergence theorem) **:** *Let $I \subseteq \mathbb{R}$ be an interval. Let $(f_n)_{n \geqslant 1}$ be a sequence of non-negative, piecewise continuous, and integrable functions from I to $\mathbb{R}_+$. Suppose that* *(i) for every $x \in I$ and $n \geqslant 1$, we have $f_n(x) \leqslant f_{n+1}(x)$;*

(ii) $(f_n)_{n\geqslant 1}$ converges pointwise to a piecewise continuous function f;

(iii) $\int_I f_n$ converges when $n \to \infty$.

Then,

$$\int_I |f_n - f| \xrightarrow[n\to\infty]{} 0, \quad \text{and} \quad \int_I f_n \xrightarrow[n\to\infty]{} \int_I f.$$

Remark 8.5.4 : We note that this theorem is very similar to Dini's theorem (Theorem 8.1.14), with the followins differences.

(1) We make a weaker assumption in Theorem 8.5.3, which is piecewise continuity.

(2) We do not get the uniform convergence of the sequence of functions $(f_n)_{n\geqslant}$ to deduce the convergence of the integrals. Actually, we *do not* have the uniform convergence here in general, whereas the convergence of integrals still holds.

Proof : It is a special case of Eq. (8.26). For every $n \geqslant 1$, let $u_n = f_{n+1} - f_n \geqslant 0$. We may check the following properties.

(i) For every $n \geqslant 1$, u_n is integrable because both f_{n+1} and f_n are integrable.

(ii) $\sum u_n = \sum(f_{n+1} - f_n)$ converges pointwise to a piecewise continuous function because $(f_n)_{n\geqslant 1}$ converges pointwise to a piecewise continuous function.

(iii) We have
$$\sum_{n=1}^{N} \int_I |u_n| = \sum_{n=1}^{N} \int_I (f_{n+1} - f_n) = \int_I f_{N+1} - \int_I f_1,$$
where the right side can be uniformly bounded from above due to the convergence of $\int_I f_n$. This shows that $\sum \int_I |u_n|$ converges.

Therefore, we may apply Eq. (8.26) to conclude that f is integrable on I and

$$\int_I |f_n - f| = \int_I \Big|\sum_{k\geqslant n} u_k\Big| \leqslant \sum_{k\geqslant n} \int_I |u_k|.$$

The right side in the above inequality is the remainder of a convergent series, so goes to 0 when n goes to ∞. □

8.5.2 Dominated convergence theorem

Theorem 8.5.5 (Dominated convergence theorem) **:** *Let $I \subseteq \mathbb{R}$ be an interval and W be a Banach space. Let $(f_n)_{n\geqslant 1}$ be a sequence of piecewise continuous functions from I to W. Suppose that*

(i) There exists a piecewise continuous non-negative integrable function $\varphi : I \to \mathbb{R}_+$ such that $\|f_n\| \leqslant \varphi$ for every $n \geqslant 1$.

(ii) The seuqnece $(f_n)_{n\geqslant 1}$ converges pointwise to a piecewise continuous function $f : I \to W$.

Then, each f_n and f are integrable on I and we have

$$\lim_{n\to\infty} \int_I \|f_n - f\| \xrightarrow[n\to\infty]{} 0, \quad \textit{and} \quad \lim_{n\to\infty} \int_I f_n = \int_I f.$$

Proof : Suppose that the theorem holds when $(W, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $(f_n)_{n\geqslant 1}$ are non-negative functions, and $f \equiv 0$ is the zero function. For all $n \geqslant 1$, let $h_n = \|f_n - f\|$, which is still a piecewise continuous function on I. Then, $h_n \leqslant 2\varphi$ and $(h_n)_{n\geqslant 1}$ converges pointwise to the zero function. So we find

$$\left\| \int_I f_n - \int_I f \right\| \leqslant \int_I \|f_n - f\| = \int_I h_n \xrightarrow[n\to\infty]{} \int_I 0 = 0.$$

Now, let us prove the theorem with the assumption that $(W, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $(f_n)_{n\geqslant 1}$ are non-negative functions, and f is the zero function. For every $n \geqslant 1$ and $p \geqslant n$, let

$$f_{n,p} := \max\{f_n, f_{n+1}, \ldots, f_p\},$$

which is still a piecewise continuous function and satisfies $f_{n,p} \leqslant \varphi$.

- Fix $n \geqslant 1$. Since $(f_{n,p})_{p\geqslant n}$ is an increasing sequence, the sequence $(I_{n,p})_{p\geqslant n}$ defined by $I_{n,p} = \int_I f_{n,p}$ is increasing. Since $I_{n,p} \leqslant \int_I \varphi$ for all $p \geqslant n$, the sequence $(I_{n,p})_{p\geqslant n}$ converges, so it satisfies Cauchy's property. We may find $p_n \geqslant 1$ such that

  $$|I_{n,p} - I_{n,q}| \leqslant 2^{-n}, \quad \forall p, q \geqslant p_n.$$

  It is possible to make a choice of $(p_n)_{n\geqslant 1}$ such that it is an extraction (strictly increasing sequence).

- For $n \geqslant 1$, let $g_n = f_{n,p_n}$. We note that g_n converges pointwise to 0 (Cauchy's criterion at each point of I). For any $n \geqslant 1$, we have

$$|g_{n+1} - g_n| + (g_{n+1} - g_n) = \begin{cases} 0 & \text{if } g_{n+1} - g_n \leqslant 0, \\ 2(g_{n+1} - g_n) & \text{otherwise.} \end{cases}$$

Additionally, for any $n \geqslant 1$, we also have $g_{n+1} - g_n = f_{n+1,p_{n+1}} - f_{n,p_n} \leqslant f_{n,p_{n+1}} - f_{n,p_n}$ and $0 \leqslant f_{n,p_{n+1}} - f_{n,p_n}$. Therefore, we find

$$\forall n \geqslant 1, \quad |g_{n+1} - g_n| \leqslant 2(f_{n,p_{n+1}} - f_{n,p_n}) - (g_{n+1} - g_n).$$

- For $n \geqslant 1$, let $u_n = g_n - g_{n+1}$. Then, we have

$$\forall n \geqslant 1, \quad \int_I |u_n| \leqslant 2|I_{n,p_{n+1}} - I_{n,p_n}| + \int_I g_n - \int_I g_{n+1} \leqslant 2^{1-n} + \int_I g_n - \int_I g_{n+1}.$$

By taking a summation, we find,

$$\forall p \geqslant n \geqslant 1, \quad \sum_{k=n}^{p} \int_I |u_k| \leqslant \sum_{k=n}^{p} 2^{1-k} + \int_I g_n - \int_I g_{p+1} \leqslant 2 + \int_I g_n.$$

In the above formula, we see that the upper bound does not depend on p. Since the left side contains only positive terms in the series, we deduce that the series $\sum_{k\geqslant n} \int_I |u_k|$ converges.

From what we have shown above, and the fact that g_n converges pointwise to 0, we have $\sum_{k\geqslant n} u_k = g_n$. This allows us to apply Eq. (8.26),

$$\forall n \geqslant 1, \quad 0 \leqslant \int_I f_n \leqslant \int_I g_n = \int_I \Big(\sum_{k\geqslant n} u_k \Big) = \sum_{k\geqslant n} \int_I u_k.$$

The rightmost term in the above formula is the remainder of an absolutely convergent series, so its limit when n tends to ∞ is zero. This shows that $\int_I f_n \xrightarrow[n\to\infty]{} 0$. $\square$

Example 8.5.6 : For every $n \in \mathbb{N}$, consider the function

$$f_n : \begin{array}{ccc} (1, +\infty) & \to & \mathbb{R} \\ t & \mapsto & \frac{1+t^n}{1+t^{n+2}} \end{array} \quad \text{and} \quad I_n = \int_1^{\infty} f_n(t)\,\mathrm{d}t.$$

We can check the following properties.

- For every $n \in \mathbb{N}$, the function f_n is piecewise continuous.
- For every $t > 1$, we have

$$f_n(t) = \frac{1+t^n}{1+t^{n+2}} \sim \frac{1}{t^2}, \quad \text{when } n \to \infty.$$

So the sequence f_n converges pointwise to the function $t \mapsto \frac{1}{t^2}$, which is piecewise continuous on $(1, +\infty)$.

- (Domination assumption) For every $n \in \mathbb{N}$ and $t > 1$, we have

$$|f_n(t)| = \frac{1+t^n}{1+t^{n+2}} \leqslant \frac{t^n+t^n}{t^{n+2}} = \frac{2}{t^2}.$$

The function $t \mapsto \frac{2}{t^2}$ is integrable on $(1, +\infty)$, so the domination assumption is satisfied. Therefore, we may apply the dominated convergence theorem from Theorem 8.5.5, giving us

$$I_n \xrightarrow[n\to\infty]{} \int_1^\infty \frac{\mathrm{d}t}{t^2} = 1.$$

Example 8.5.7 : For every $n \in \mathbb{N}$, consider the function

$$f_n : \begin{array}{ccc} [0,1) & \to & \mathbb{R} \\ t & \mapsto & n^2 t^{n-1} \end{array} \quad \text{and} \quad I_n = \int_0^1 f_n(t)\,\mathrm{d}t.$$

For every $n \in \mathbb{N}$, the function f_n is continuous and integrable on $[0,1)$. For every $t \in [0,1)$, we have $f_n(t) \xrightarrow[n\to\infty]{} 0$, which implies that the sequence $(f_n)_{n\geqslant 1}$ converges pointwise on $[0,1)$ to the zero function. However, we have

$$\forall n \in \mathbb{N}, \quad I_n = \left[nt^n\right]_0^1 = n.$$

This shows that the order of the limit and the integration procedure cannot be interchanged,

$$\lim_{n\to\infty} \int_0^1 f_n(t)\,\mathrm{d}t \neq \int_0^1 \lim_{n\to\infty} f_n(t)\,\mathrm{d}t = 0.$$

The reason is that the domination assumption is not satisfied.

To be more precise, if φ is a function that dominates all the f_n's, then for $t \in [0,1)$, we need to have $\varphi(t) \geqslant f_n(t)$ for all $n \geqslant 1$. In particular, for $t \in [0,1)$, we may choose $n = \lfloor \frac{2}{|\ln t|} \rfloor$, then, for $t \to 1-$, we have the following relation,

$$\begin{aligned}\ln f_n(t) &= 2\ln n + (n-1)\ln t \\ &\geqslant 2\ln\Big(\frac{2}{|\ln t|} - 1\Big) + \Big(\frac{2}{|\ln t|} - 1\Big)\ln t \\ &= -\ln t - 2\ln|\ln t| + \mathcal{O}(1).\end{aligned}$$

This means that when $t \to 1-$, we have

$$f_n(t) \geqslant \frac{\text{cst}}{t|\ln t|^2},$$

which implies that φ is not integrable around $1-$.

8.5.3 Applications: integrals with an additional parameter

We give a few important applications of the dominated convergence theorem. Let us consider a general interval $I \subseteq \mathbb{R}$, with endpoints a and b satisfying $-\infty \leqslant a < b \leqslant +\infty$, and a Banach space $(W, \|\cdot\|)$.

Theorem 8.5.8 (Continuity under integration) **:** *Let (M, d) be a metric space and a map $f : M \times I \to W$ satisfying the following conditions.*

(i) For every $x \in M$, the map $f(x, \cdot) : t \mapsto f(x, t)$ is piecewise continuous on I.

(ii) For every $t \in I$, the map $f(\cdot, t) : x \mapsto f(x, t)$ is continuous on M.

(iii) (Domination assumption) There exists a non-negative, piecewise continuous, and integrable function $\varphi : I \to \mathbb{R}_+$ such that $\|f(x, t)\| \leqslant \varphi(t)$ for all $x \in M$ and $t \in I$.

Then, the map

$$\begin{array}{rccc} F: & M & \to & W \\ & x & \mapsto & \displaystyle\int_a^b f(x,t)\,\mathrm{d}t \end{array}$$

is well-defined and continuous on M.

Proof : The assumption (iii), the domination assumption, shows that the function $f(x, \cdot)$ is integrable for every $x \in M$, so the map F is well defined. For a given $x \in M$, to check that F is continuous at x, we need to check that for any sequence $(x_n)_{n\geqslant 1}$ with values in M,

$$x_n \xrightarrow[n\to\infty]{} x \quad \Rightarrow \quad F(x_n) \xrightarrow[n\to\infty]{} F(x).$$

Let $x \in M$ and $(x_n)_{n\geqslant 1}$ be a sequence in M such that $x_n \xrightarrow[n\to\infty]{} x$. For every $n \geqslant 1$, we may define the function

$$\begin{array}{rccc} f_n: & I & \to & V \\ & t & \mapsto & f(x_n, t). \end{array}$$

Due to the assumption (ii), we know that $f_n(t) = f(x_n, t) \xrightarrow[n\to\infty]{} f(x,t)$ for every $t \in I$, where $t \mapsto f(x,t)$ is a piecewise continuous function by the assumption (i). This means that the assumption (ii) in Theorem 8.5.5 is satisfied. Then, the assumption (iii) here corresponds to the assumption (i) in Theorem 8.5.5, so we can apply Theorem 8.5.5 to the sequence of functions $(f_n)_{n\geqslant 1}$. This shows that

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_I f(x_n, t)\,\mathrm{d}t = \int_I f(x,t)\,\mathrm{d}t = F(x),$$

which allows us to conclude. $\square$

Theorem 8.5.9 (Differentiability under integration) **:** *Let $M \subseteq \mathbb{R}$ be an interval and a map $f : M \times I \to W$ satisfying the following conditions.*

(i) For every $x \in M$, the map $f(x,\cdot) : t \mapsto f(x,t)$ is piecewise continuous and integrable on I.

(ii) For every $t \in I$, the map $f(\cdot, t) : x \mapsto f(x,t)$ is of class $\mathcal{C}^1$ on M.

(iii) The partial derivative $\frac{\partial f}{\partial x}$ is well defined and satisfies the assumptions from Theorem 8.5.8.

Then, the map

$$\begin{array}{rccc} F: & M & \to & W \\ & x & \mapsto & \displaystyle\int_a^b f(x,t)\,\mathrm{d}t \end{array}$$

is of class $\mathcal{C}^1$ on M, and we have

$$\forall x \in M, \quad F'(x) = \int_a^b \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t. \tag{8.28}$$

Proof : The proof is similar to that of Theorem 8.5.8. Let $x \in M$ and $(x_n)_{n\geqslant 1}$ be a sequence with values in $M\backslash\{x\}$ that converges to x. For every $n \geqslant 1$, define

$$\begin{array}{rccc} g_n: & I & \to & W, \\ & t & \mapsto & \dfrac{f(x_n,t)-f(x,t)}{x_n-x}, \end{array}$$

which is a piecewise continuous function. For each $n \geqslant 1$, g_n is also integrable on I, being a linear combination of integrable functions.

The sequence $(g_n)_{n\geqslant 1}$ of functions converges pointwise to $\frac{\partial f}{\partial x}(x,\cdot)$. Moreover, the mean-value theorem (Eq. (4.3)) tells us that for every $n \geqslant 1$ and $t \in I$, there exists $y_n = y_n(t)$ between x and x_n such that

$$g_n(t) = \frac{f(x_n,t)-f(x,t)}{x_n-x} = \frac{\partial f}{\partial x}(y_n,t) \quad \text{and} \quad \|g_n(t)\| = \left\|\frac{\partial f}{\partial x}(y_n,t)\right\| \leqslant \varphi(t),$$

where φ is the domination function given by the assumption (iii) for $\frac{\partial f}{\partial x}$ from Theorem 8.5.8. Then, we may apply Theorem 8.5.5 to conclude that

$$\lim_{n\to\infty}\int_I g_n(t)\,\mathrm{d}t = \int_I \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t,$$

and the left side of the above formula rewrite,

$$\lim_{n\to\infty}\int_I g_n(t)\,\mathrm{d}t = \lim_{n\to\infty}\frac{F(x_n)-F(x)}{x_n-x}.$$

This shows that F is differentiable at x and its derivative does satisfy Eq. (8.28). To conclude, we note that the assumption (iii) guarantees that the right side of Eq. (8.28) is continuous, so F is of class $\mathcal{C}^1$. □

Example 8.5.10 (Gamma function) **:** We recall the Gamma function defined in Example 7.1.21,

$$\forall x > 0, \quad \Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,\mathrm{d}t.$$

By applying Theorem 8.5.8 and Theorem 8.5.9, we can check that Γ is a function of class $\mathcal{C}^\infty$, and its derivative writes

$$\forall n \in \mathbb{N}_0, \forall x > 0, \quad \Gamma^{(n)}(x) = \int_0^{\infty} (\log t)^n e^{-t}t^{x-1}\,\mathrm{d}t.$$

More precisely, let us consider the function

$$f: \mathbb{R}_+^* \times \mathbb{R}_+^* \mapsto \mathbb{R}, (x,t) \mapsto t^{x-1}e^{-t}.$$

We can check the following properties.

- For any fixed $t > 0$, the function $x \mapsto f(x,t)$ is $\mathcal{C}^\infty$, and we have

$$\forall k \in \mathbb{N}_0, \quad \forall x, t > 0, \quad \frac{\partial^k f}{\partial x^k}(x,t) = (\ln t)^k t^{x-1} e^{-t}.$$

- For any fixed $x > 0$ and $k \in \mathbb{N}_0$, the function $t \mapsto \frac{\partial^k f}{\partial x^k}(x,t)$ is piecewise continuous.

- (Domination assumption) Let $k \in \mathbb{N}_0$ and $[a,b] \subseteq (0,+\infty)$ be a segment. For all $x \in [a,b]$, we have

$$\forall t \in (0,1], \quad \left|\frac{\partial^k f}{\partial x^k}(x,t)\right| = |\ln t|^k t^{x-1} e^{-t} \leqslant |\ln t|^k t^{a-1} e^{-t},$$
$$\forall t \in (1,+\infty), \quad \left|\frac{\partial^k f}{\partial x^k}(x,t)\right| = |\ln t|^k t^{x-1} e^{-t} \leqslant |\ln t|^k t^{b-1} e^{-t}.$$

  Let φ be defined on $\mathbb{R}_+^*$ by

$$\varphi(t) = |\ln t|^k t^{a-1} e^{-t} + |\ln t|^k t^{b-1} e^{-t},$$

  which is an integrable function on $\mathbb{R}_+^*$. And we clearly have

$$\forall x \in [a,b], \quad \forall t > 0, \quad \left|\frac{\partial^k f}{\partial x^k}(x,t)\right| \leqslant \varphi(t).$$