9 Fourier series

9.1 Definitions

The goal of this section is to introduce the notion of *Fourier series*, whose partial sums correspond to *trigonometric polynomials*.

9.1.1 Trigonometric polynomials

Definition 9.1.1 : Let $N \in \mathbb{N}_0$. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be a *trigonometric polynomial* (三角多項式) if it satisfies one of the following equivalent identities.

- There exists a finite sequence $(c_n)_{-N \leqslant n \leqslant N}$ of complex numbers such that

$$\forall x \in \mathbb{C}, \quad f(x) = \sum_{n=-N}^{N} c_n e^{\mathrm{i}\, nx}.$$

- There exists finite sequences $(a_n)_{0 \leqslant n \leqslant N}$ and $(b_n)_{1 \leqslant n \leqslant N}$ of complex numbers such that

$$\forall x \in \mathbb{C}, \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \big(a_n \cos(nx) + b_n \sin(nx)\big).$$

Remark 9.1.2 :

(1) In Definition 9.1.1, the coefficients $(a_n)_{0 \leqslant n \leqslant N}$, $(b_n)_{1 \leqslant n \leqslant N}$, and $(c_n)_{-N \leqslant n \leqslant N}$ are related by the following relations,

$$\forall m = 0, \dots, N, \quad a_m = c_m + c_{-m}, \quad \text{and} \quad \forall m = 1, \dots, N, \quad b_m = \mathrm{i}(c_m - c_{-m}). \tag{9.1}$$

This is due to the relation $e^{\mathrm{i}\,\theta} = \cos\theta + \mathrm{i}\sin\theta$ for $\theta \in \mathbb{R}$.

(2) It is not hard to see that a trigonometric polynomial $P(x) = \sum_{n=-N}^{N} c_n e^{\mathrm{i}\, nx}$ is continuous and 2π-periodic. Moreover, its coefficients can be recovered by

$$\forall n \in \mathbb{N}_0, \quad c_n = \frac{1}{2\pi} \int_0^{2\pi} P(x) e^{-\mathrm{i}\, nx}\, \mathrm{d}x.$$

Definition 9.1.3 :

- A *trigonometric series* (三角級數) is a series of functions in the variable $x \in \mathbb{R}$ of one of the following forms,

$$\sum_{n\in\mathbb{Z}} c_n e^{\mathrm{i}\,nx}, \quad \text{or} \quad \frac{a_0}{2} + \sum_{n\geqslant 1} \big(a_n\cos(nx) + b_n\sin(nx)\big).$$

- A trigonometric series is said to be convergent at $x \in \mathbb{R}$ if one of the following partial sums (so both) converges,

$$\left(\sum_{n=-N}^{N} c_n e^{\mathrm{i}\,nx}\right)_{N\geqslant 0}, \quad \text{or} \quad \left(\frac{a_0}{2} + \sum_{n=1}^{N} \big(a_n\cos(nx) + b_n\sin(nx)\big)\right)_{N\geqslant 0}. \tag{9.2}$$

Remark 9.1.4 :

(1) From the relations in Eq. (9.1), it is not hard to see that if one of the partial sums in Eq. (9.2) converges, then the other one converges.

(2) For a fixed $x \in \mathbb{R}$, the way the convergence of trigonometric series at x is defined is weaker than the existence of the double limit

$$\lim_{\substack{q\to+\infty \\ p\to-\infty}} \sum_{n=p}^{q} c_n e^{\mathrm{i}\,nx}.$$

Proposition 9.1.5 : *The following properties holds.*

- *If $\sum_{n\geqslant 1} c_n$ and $\sum_{n\geqslant 1} c_{-n}$ converge absolutely, then the trigonometric series $\sum_{n\in\mathbb{Z}} c_n e^{\mathrm{i}\,nx}$ converges normally on $\mathbb{R}$.*
- *If $\sum_{n\geqslant 1} a_n$ and $\sum_{n\geqslant 1} b_n$ converge absolutely, then the trigonometric series $\frac{a_0}{2} + \sum_{n\geqslant 1} \big(a_n\cos(nx) + b_n\sin(nx)\big)$ converges normally on $\mathbb{R}$.*

The corresponding trigonometric series defines a continuous and 2π-periodic function.

Proof : For any $x \in \mathbb{R}$, we have

$$\forall n \in \mathbb{N}, \quad |c_n e^{\mathrm{i}\,nx}| = |c_n|, \ |a_n\cos(nx) + b_n\sin(nx)| \leqslant |a_n| + |b_n|.$$

Therefore, the normal convergence follows directly. Since each of the function in the series is continuous and 2π-periodic, the same properties also hold for the series of functions. □

Proposition 9.1.6 : *If the sequences $(c_n)_{n\geqslant 1}$ and $(c_{-n})_{n\geqslant 1}$ are real and decrease to 0, then*

- *the trigonometric series $\sum_{n\in\mathbb{Z}} c_n e^{\mathrm{i}\,nx}$ converges pointwise on $\mathbb{R}\backslash 2\pi\mathbb{Z}$; and*
- *uniformly on all the intervals $[2k\pi + \alpha, 2(k+1)\pi - \alpha]$ with $k \in \mathbb{Z}$ and $\alpha \in (0, \pi)$.*

Proof : This is a direct consequence of the Abel's transform and the Dirichlet's test, see Proposition 6.4.5 and Theorem 6.4.7. □

9.1.2 Fourier series

In what follows, we are interested in 2π-periodic functions. In particular, let us introduce the following two vector spaces,

- $\mathcal{C}_{\text{per}}(\mathbb{R},\mathbb{C})$ the space of 2π-periodic continuous functions from $\mathbb{R}$ to $\mathbb{C}$; and
- $\mathcal{PC}_{\text{per}}(\mathbb{R},\mathbb{C})$ the space of 2π-periodic piecewise continuous functions from $\mathbb{R}$ to $\mathbb{C}$.

Definition 9.1.7 : Let $f \in \mathcal{PC}_{\text{per}}(\mathbb{R},\mathbb{C})$ be a 2π-periodic and piecewise continuous function on $\mathbb{R}$. Its *Fourier coefficients* are defined as below,

$$\begin{aligned}
\forall n \in \mathbb{Z}, \quad c_n(f) &= \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-\mathrm{i}\,nt}\,\mathrm{d}t,\\
\forall n \in \mathbb{N}_0, \quad a_n(f) &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,\mathrm{d}t,\\
\forall n \in \mathbb{N}, \quad b_n(f) &= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,\mathrm{d}t.
\end{aligned}$$

The *Fourier series* corresponding to f is the trigonometric series given by

$$\sum_{n\in\mathbb{Z}} c_n(f)e^{\mathrm{i}\,nx}, \quad \text{or} \quad \frac{a_0(f)}{2} + \sum_{n\geqslant 1}\big(a_n(f)\cos(nx) + b_n(f)\sin(nx)\big).$$

In particular, we denote the n-th partial sum of the Fourier series of f by

$$S_n(f)(x) = \sum_{k=-n}^{n} c_k(f)e^{\mathrm{i}\,kx}, \quad \text{or} \quad S_n(f)(x) = \frac{a_0(f)}{2} + \sum_{k=1}^{n}\big(a_k(f)\cos(kx) + b_k(f)\sin(kx)\big).$$

Remark 9.1.8 :

(1) We note that the Fourier series of f and f do not necessarily define the same function. Indeed, we have not yet discussed the convergence of the Fourier series.

(2) The coefficients $(c_n(f))_{n\in\mathbb{Z}}$, $(a_n(f))_{n\geqslant 0}$, and $(b_n(f))_{n\geqslant 1}$ are related in the same way as in Eq. (9.1).

(3) Since f is 2π-periodic, we may change the domain of integration to any interval of length 2π.

(4) If f is an even function, then the coefficients $b_n(f)$ are zero; if f is an odd function, then the coefficients $a_n(f)$ are zero.

In what follows, we will write the trigonometric series and Fourier series using the exponential functions instead of trigonometric functions. The two writings are equivalent, but the former one is more compact and easier to write.

Proposition 9.1.9 : *If $f(x) = \sum c_n e^{\mathrm{i}\, nx}$ is a trigonometric series that converges uniformly on $\mathbb{R}$, then $c_n = c_n(f)$ for all $n \in \mathbb{Z}$.*

Proof : If a series of functions converges uniformly, then its integral on any segment also converges, and can be computed term by term, see Proposition 8.2.5. Additionally, we know that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}\, kx}\,\mathrm{d}x = \mathbb{1}_{k=0}, \quad \forall k \in \mathbb{Z}.$$

This allows us to conclude that

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k\in\mathbb{Z}} c_k e^{\mathrm{i}(k-n)x}\,\mathrm{d}x = \sum_{k\in\mathbb{Z}} c_k \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}(k-n)x}\,\mathrm{d}x = c_n.$$

□

9.1.3 Kernels and convolution

The partial sums of a Fourier series can be rewritten using a *convolution* between the function itself and the Dirichlet's kernel.

Definition 9.1.10 (Dirichlet's kernel) **:** For $n \in \mathbb{N}_0$, we define

$$\forall t \in \mathbb{R}\backslash 2\pi\mathbb{Z}, \quad D_n(t) = \sum_{k=-n}^{n} e^{\mathrm{i}\, kt} = \frac{\sin\left(\frac{(2n+1)t}{2}\right)}{\sin\left(\frac{t}{2}\right)}. \tag{9.3}$$

The sequence of functions $(D_n)_{n\geqslant 0}$ is called *Dirichlet's kernel.*

Remark 9.1.11 : The last equality in Eq. (9.3) can be obtained by a geometric summation,

$$\sum_{k=-n}^{n} e^{\mathrm{i}\, kt} = e^{-\mathrm{i}\, nt}\frac{e^{\mathrm{i}(2n+1)t}-1}{e^{\mathrm{i}\, t}-1} = \frac{e^{\mathrm{i}\,\frac{2n+1}{2}t} - e^{-\mathrm{i}\,\frac{2n+1}{2}t}}{e^{\mathrm{i}\,\frac{t}{2}} - e^{-\mathrm{i}\,\frac{t}{2}}} = \frac{\sin\left(\frac{(2n+1)t}{2}\right)}{\sin\left(\frac{t}{2}\right)}.$$

Below are some properties of the Dirichlet's kernel. They can be checked by direct computations.

Proposition 9.1.12 : *The Dirichlet's kernel $(D_n)_{n\geqslant 0}$ satisfies the following properties.*

(1) For each $n \geqslant 0$, the function D_n is even.

(2) For each $n \geqslant 0$, the function D_n is 2π-periodic.

(3) For each $n \geqslant 0$, we have $\frac{1}{2\pi}\int_0^{2\pi} D_n(t)\,\mathrm{d}t = 1$.

Definition 9.1.13 : For two 2π-periodic and piecewise continuous functions $f, g : \mathbb{R} \to \mathbb{C}$, we define their *convolution* (捲積), denoted $f \star g$, as below,

$$\forall x \in \mathbb{R}, \quad (f \star g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-t) g(t) \, \mathrm{d}t. \tag{9.4}$$

Below are some properties of the convolution that can be checked easily.

Proposition 9.1.14 : *For 2π-periodic and piecewise continuous functions $f, g, h : \mathbb{R} \to \mathbb{C}$, the following properties hold.*

(1) (Linearity) For $\lambda \in \mathbb{C}$, we have $f \star (g + \lambda h) = f \star g + \lambda (f \star h)$.

(2) (Commutativity) $(f \star g) \star h = f \star (g \star h)$.

(3) (Symmetry) $f \star g = g \star f$.

We may rewrite the partial sums of a Fourier series using convolution.

Proposition 9.1.15 : *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic and piecewise continuous function on $\mathbb{R}$. Then, $S_n(f) = D_n \star f$ for every $n \in \mathbb{N}_0$.*

Proof : Let $n \in \mathbb{N}_0$. We write the n-th partial sum of the Fourier series $\sum c_n(f) e^{\mathrm{i}\, nx}$ as below,

$$\begin{aligned} S_n(f)(x) &= \sum_{k=-n}^{n} c_k(f) e^{\mathrm{i}\, kx} = \frac{1}{2\pi} \sum_{k=-n}^{n} \left(\int_0^{2\pi} f(t) e^{-\mathrm{i}\, kt} \, \mathrm{d}t \right) e^{\mathrm{i}\, kx} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(t) \left(\sum_{k=-n}^{n} e^{\mathrm{i}\, k(x-t)} \right) \mathrm{d}t = (D_n \star f)(x). \end{aligned}$$

□

9.2 Quadratic properties

The most important result of this section is the Parseval's identity (Theorem 9.2.7). Before discussing this result, we are going to see the quadratic structure that arises naturally from the way the Fourier series is defined, which is a *pre-Hilbert space*.

9.2.1 Pre-Hilbert spaces

Pre-Hilbert spaces generalize the notion of Euclidean spaces from $\mathbb{R}$ to $\mathbb{C}$, and from finite-dimensional spaces to infinite-dimensional spaces.

Definition 9.2.1 : Let V be a $\mathbb{K}$-vector space where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ is an *inner product* (內積) if it satisfies

(i) (Positive definiteness) $\langle x, x \rangle \geqslant 0$ with equality if and only if $x = 0$.

(ii) (Conjugate symmetry) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in V$.

(iii) (Linearity) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $a, b \in \mathbb{R}$ and $x, y, z \in V$.

If $\langle \cdot, \cdot\rangle$ is an inner product, then it induces a norm (so a distance) given by

$$\|x\|_2 := \sqrt{\langle x, x\rangle}, \quad \forall x \in V.$$

Then, the normed space $(V, \langle \cdot, \cdot\rangle)$ is called a *pre-Hilbert space*. Additionally, if this normed space is complete, it is called a *Hilbert space*. This generalizes the notion of inner product and Euclidean spaces defined in Definition 2.1.10.

Definition 9.2.2 : Let us denote by $\mathcal{D}$ the vector space consisting of functions in $\mathcal{PC}_{\text{per}}(\mathbb{R}, \mathbb{C})$ satisfying

$$\forall x \in \mathbb{R}, \quad f(x) = \frac{1}{2}\big[f(x-) + f(x+)\big]. \tag{9.5}$$

We define the following inner product on $\mathcal{D}$,

$$(f, g) \mapsto \langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,\mathrm{d}t. \tag{9.6}$$

Then, $\mathcal{D}$ is a pre-Hilbert space equipped with the norm $\|f\|_2 := \sqrt{\langle f, f\rangle}$ for $f \in \mathcal{D}$. We note that $(e_n : x \mapsto e^{\mathrm{i}nx})_{n\in\mathbb{Z}}$ is a family of orthonormal functions in $\mathcal{D}$ with respect to the inner product $\langle \cdot, \cdot\rangle$ defined above.

Remark 9.2.3 :

(1) We note that $\mathcal{D}$ contains 2π-periodic continuous functions from $\mathbb{R}$ to $\mathbb{C}$, that is $\mathcal{C}_{\text{per}}(\mathbb{R}, \mathbb{C}) \subseteq \mathcal{D}$.

(2) The main reason why we require the condition Eq. (9.5) for the functions in the space $\mathcal{D}$ is to ensure that Eq. (9.6) satisfies the definiteness. Actually, it is not hard to see that a function $f \in \mathcal{PC}_{\text{per}}(\mathbb{R}, \mathbb{C})$ that only has finitely many non-zero points satisfies $\langle f, f\rangle = 0$. In order to talk about a normed vector space, we refer to the space $\mathcal{D}$; but in Section 9.2.2, we will see that the Parseval's identity holds for more general functions.

Proposition 9.2.4 : *For $n \in \mathbb{N}_0$, let $\mathcal{P}_n = \mathrm{Span}(e_k)_{-n\leqslant k\leqslant n}$ be the linear span of $(e_k)_{-n\leqslant k\leqslant n}$, and write p_n for the orthogonal projection on $\mathcal{P}_n$. For any fixed $n \in \mathbb{N}_0$, the following properties are satisfied.*

(1) We have $\mathcal{P}_n \oplus \mathcal{P}_n^{\perp} = \mathcal{D}$, and the orthogonal projection p_n gives the n-th partial sum of the Fourier series,

$$\forall f \in \mathcal{D}, \quad p_n(f) := \sum_{k=-n}^{n} c_k(f)e_k = S_n(f).$$

(2) We have

$$\inf_{g\in\mathcal{P}_n} \|f - g\|_2^2 = \|f - S_n(f)\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,\mathrm{d}t - \sum_{k=-n}^{n} |c_k(f)|^2. \tag{9.7}$$

Proof :

(1) Let us fix $f \in \mathcal{D}$ and $n \in \mathbb{N}_0$. We note that for any $-n \leqslant k \leqslant n$, we have $c_k(f) = \langle e_k, f\rangle = \langle e_k, S_n(f)\rangle$, so $\langle e_k, f - S_n(f)\rangle = 0$. Since $(e_k)_{-n\leqslant k\leqslant n}$ spans $\mathcal{P}_n$, it means that $f - S_n(f) \in \mathcal{P}_n^\perp$. We may write $f = S_n(f) + (f - S_n(f))$ with $S_n(f) \in \mathcal{P}_n$, so we conclude that $f \in \mathcal{P}_n + \mathcal{P}_n^\perp$. Therefore, we have $\mathcal{D} = \mathcal{P}_n + \mathcal{P}_n^\perp$. It remains to check that $\mathcal{P}_n \cap \mathcal{P}_n^\perp = \{0\}$. Let $g = \sum_{k=-n}^{n} g_k e_k \in \mathcal{P}_n^\perp$, then we find $g_k = \langle e_k, g\rangle = 0$ for all $-n \leqslant k \leqslant n$. This means that $g_k = 0$ for $-n \leqslant k \leqslant n$, that is $g = 0$.

(2) From above, we know that $S_n(f) = p_n(f)$. Since $S_n(f) \perp (f - S_n(f))$, we have $\|S_n(f)\|_2^2 + \|f - S_n(f)\|_2^2 = \|f\|_2^2$, which shows the second equality in Eq. (9.7). Moreover, for any $g \in \mathcal{P}_n$, we have

$$\|f-g\|_2^2 = \|(f - S_n(f)) + (S_n(f) - g)\|_2^2 = \|f - S_n(f)\|_2^2 + \|S_n(f) - g\|_2^2 \geqslant \|f - S_n(f)\|_2^2,$$

which shows the first equality in Eq. (9.7).

□

Remark 9.2.5 :

(1) The first equality in Eq. (9.7) shows that the partial sum $S_n(f)$ is the best approximation of f in terms of quadratic variation among the trigonometric polynomials of degree less than or equal to n.

(2) Eq. (9.7) shows that for $f \in \mathcal{D}$, we have

$$\forall n \in \mathbb{N}_0, \quad \sum_{k=-n}^{n} |c_k(f)|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \,\mathrm{d}t.$$

This implies the convergence of the series $\sum_{n\in\mathbb{Z}} |c_k(f)|^2$ and the following relation, called *Bessel's inequality*,

$$\sum_{n\in\mathbb{Z}} |c_n(f)|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \,\mathrm{d}t.$$

Corollary 9.2.6 : *Let $\mathcal{P} = \mathrm{Span}(e_n)_{n\in\mathbb{Z}}$ be the vector space of trigonometric polynomials. For $f \in \mathcal{D}$, we have*

$$\inf_{g\in\mathcal{P}} \|f-g\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \,\mathrm{d}t - \sum_{n\in\mathbb{Z}} |c_n(f)|^2.$$

9.2.2 Parseval's identity

Parseval's identity is a result about L^2-isometry, see Remark 9.2.8 for a more detailed discussion.

Theorem 9.2.7 (Parseval's identity) **:** *Let $f \in \mathcal{PC}_{per}(\mathbb{R}, \mathbb{C})$ be a 2π-periodic and piecewise continuous function. Then, the series $\sum_{n\in\mathbb{Z}} |c_n(f)|^2$, $\sum_{n\in\mathbb{Z}} |a_n(f)|^2$, and $\sum_{n\in\mathbb{Z}} |b_n(f)|^2$ converge and we have*

$$\sum_{n\in\mathbb{Z}} |c_n(f)|^2 = \frac{|a_0(f)|^2}{4} + \frac{1}{2}\sum_{n\geqslant 1} |a_n(f)|^2 + \frac{1}{2}\sum_{n\geqslant 1} |b_n(f)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \,\mathrm{d}t. \tag{9.8}$$

Proof : Let $f \in \mathcal{PC}_{\text{per}}(\mathbb{R},\mathbb{C})$. We may modify the value of f at finitely many points to make it become a function in $\mathcal{D}$, without changing its Fourier coefficients $(c_n(f))_{n\in\mathbb{Z}}$ and the integral $\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^2\,\mathrm{d}t$. Let us denote this modification by f.

From Corollary 9.2.6, it is sufficient to show that $\inf_{g\in\mathcal{P}}\|f-g\|_2^2 = 0$. Let $\varepsilon > 0$. A similar construction as in the proof of Lemma 8.5.2 gives us a 2π-periodic continuous function g such that $\|f-g\|_2 \leqslant \varepsilon$. Then, the Stone–Weierstraß theorem (Example 8.4.7 (3)) gives us a trigonometric polynomial $h \in \mathcal{P}$ such that $\|g-h\|_2 \leqslant \varepsilon$. As a consequence, we find $\|f-g\|_2 \leqslant 2\varepsilon$, so $\inf_{g\in\mathcal{P}}\|f-g\|_2^2 \leqslant 4\varepsilon^2$. Since $\varepsilon > 0$ can be taken to be arbitrarily small, we can conclude. □

Remark 9.2.8 :

(1) Let us define the space $\ell^2(\mathbb{Z},\mathbb{C})$ as below,

$$\ell^2(\mathbb{Z},\mathbb{C}) := \Big\{a = (a_n)_{n\in\mathbb{Z}} : \sqrt{\sum_{n\in\mathbb{Z}}|a_n|^2} < +\infty\Big\},$$

equipped with the inner product $\langle\cdot,\cdot\rangle$,

$$\forall a,b \in \ell^2(\mathbb{Z},\mathbb{C}), \quad \langle a,b\rangle := \sum_{n\in\mathbb{Z}} a_n\overline{b_n}.$$

It is not hard to check that $(\ell^2(\mathbb{Z},\mathbb{C}),\langle\cdot,\cdot\rangle)$ is a pre-Hilbert space. A similar approach as in Exercise 3.30 shows that this space is complete, so is a Hilbert space.

(2) The above Parseval's identity states that the Fourier mapping

$$\begin{array}{rccc} \mathcal{F}: & \mathcal{PC}_{\text{per}}(\mathbb{R},\mathbb{C}) & \to & \ell^2(\mathbb{Z},\mathbb{C}) \\ & f & \mapsto & (c_n(f))_{n\in\mathbb{Z}} \end{array}$$

is an isometry (Definition 2.5.16) when restricted on the image. More precisely, the space $\mathcal{PC}_{\text{per}}(\mathbb{R},\mathbb{C})$ is isometric to a subspace of $\ell^2(\mathbb{Z},\mathbb{C})$, which is given by the image of $\mathcal{PC}_{\text{per}}(\mathbb{R},\mathbb{C})$ under $\mathcal{F}$. This is the reason why we sometimes refer to this result as "L^2-isometry" property for the Fourier series.

(3) If we look at Eq. (9.8), its right side is well defined for any square-integrable functions, that is functions $f : [0,2\pi] \to \mathbb{C}$ such that the following integral exists in the sense of Lebesgue (not discussed in our lecture),

$$\int_0^{2\pi}|f(t)|^2\,\mathrm{d}t < +\infty.$$

The collection of such functions is denoted by $L^2([0,2\pi])$. In fact, the Parseval's identity holds for all such functions.

(4) Additionally, the Riesz–Fischer theorem shows that $L^2([0,2\pi])$ is complete, so is a Hilbert space. As a direct consequence, the Fourier mapping $\mathcal{F}$ defined on $L^2([0,2\pi])$ is a bijection, which implies that $\mathcal{F}$ is an isometry betwteen $L^2([0,2\pi])$ and $\ell^2(\mathbb{Z},\mathbb{C})$.

Corollary 9.2.9 : *Let $f \in \mathcal{PC}_{per}(\mathbb{R},\mathbb{C})$ be a 2π-periodic and piecewise continuous function. The Parseval's identity (Theorem 9.2.7) gives readily following consequences.*

(1) We have $\lim_{|n|\to\infty} c_n(f) = 0$.

(2) If $c_n(f) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

Remark 9.2.10 : The Riemann–Lebesgue lemma states that,

$$\int_0^{2\pi} f(t)e^{\mathrm{i}\,nt}\,\mathrm{d}t \xrightarrow[|n|\to\infty]{} 0,$$

where the variable n is a real number. However, the result in Corollary 9.2.9 (1) needs n to be restricted on integers.

9.3 Convergence results

We remind that for a given function $f \in \mathcal{C}_{\mathrm{per}}(\mathbb{R}, \mathbb{C})$, its Fourier series is just a formal definition, and is not necessarily equal to the function f itself, see Remark 9.1.8. In this section, we are going to discuss two convergence results, the Jordan–Dirichlet theorem in Section 9.3.1 and the Fejér's theorem in Section 9.3.2.

9.3.1 Jordan–Dirichlet theorem

The Jordan–Dirichlet theorem gives a sufficient condition for the Fourier series to converge pointwise. In particular, it leads to a result for periodic piecewise $\mathcal{C}^1$ functions, see Corollary 9.3.2; and a stronger result for periodic, continuous, and piecewise $\mathcal{C}^1$ functions, see Theorem 9.3.4.

Theorem 9.3.1 (Jordan–Dirichlet theorem) **:** *Let $f \in \mathcal{PC}_{per}(\mathbb{R}, \mathbb{C})$ be a 2π-periodic and piecewise continuous function on $\mathbb{R}$. Let $t_0 \in \mathbb{R}$ be such that*

$$h \mapsto \frac{1}{h}\big[f(t_0+h) + f(t_0-h) - f(t_0+) - f(t_0+)\big] \tag{9.9}$$

is bounded around 0. Then the following series converges and satisfies

$$\sum_{n\in\mathbb{Z}} c_n(f)e^{\mathrm{i}\,nt_0} = \frac{1}{2}\big[f(t_0+) + f(t_0-)\big].$$

Proof : Up to the translation $t \mapsto t+t_0$, we may assume that $t_0 = 0$. For $n \in \mathbb{N}_0$, let $s_n = \sum_{k=-n}^{n} c_k(f)$ and $u_n = s_n - \frac{1}{2}\big[f(0+) + f(0-)\big]$. We need to show that $u_n \xrightarrow[n\to\infty]{} 0$.

For $n \in \mathbb{N}$, we have

$$2\pi s_n = \sum_{k=-n}^{n} \int_{-\pi}^{\pi} f(t)e^{\mathrm{i}\,kt}\,\mathrm{d}t = \int_{-\pi}^{\pi} f(t)D_n(t)\,\mathrm{d}t,$$

where D_n is the Dirichlet's kernel defined in Definition 9.1.10. Use the parity of D_n from Proposition 9.1.12 (1), we deduce that

$$2\pi s_n = \int_0^{\pi} \big[f(t) + f(-t)\big]D_n(t)\,\mathrm{d}t,$$

Moreover, Proposition 9.1.12 (3) allows us to write

$$\pi[f(0+)+f(0-)] = \int_0^\pi [f(0+)+f(0-)]D_n(t)\,\mathrm{d}t.$$

By defining

$$\forall t \in (0, 2\pi), \quad g(t) = \frac{1}{\sin\left(\frac{t}{2}\right)}[f(t)+f(-t)-f(0+)-f(0-)],$$

we find

$$2\pi u_n = \int_0^\pi g(t)\sin\left(\frac{(2n+1)t}{2}\right)\mathrm{d}t. \tag{9.10}$$

We note that g is piecewise continuous on $(0, 2\pi)$ and bounded around 0 by the assumption in Eq. (9.9), so g is integrable on $(0, 2\pi)$. Then, it is not hard to see[1] from Corollary 9.2.9 (1) that the right side of Eq. (9.10) goes to 0 when $n \to \infty$. □

Corollary 9.3.2 : *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic and piecewise $\mathcal{C}^1$ function on $\mathbb{R}$[2]. Then for every $x \in \mathbb{R}$,*

$$\sum_{n\in\mathbb{Z}} c_n(f)e^{\mathrm{i}nx} = \frac{1}{2}[f(x+)+f(x-)].$$

In particular, if f is continuous at x, then

$$\sum_{n\in\mathbb{Z}} c_n(f)e^{\mathrm{i}nx} = f(x).$$

Proof : Let $0 = x_0 < x_1 < \cdots < x_m = 2\pi$ such that for every $1 \leqslant k \leqslant m$, f can be extended by continuity into a $\mathcal{C}^1$ function on $[x_{k-1}, x_k]$, this means that f' is continuous on $[x_{k-1}, x_k]$. Therefore, the function defined in Eq. (9.9) is bounded for every $t_0 \in \mathbb{R}$. So the result follows directly from Theorem 9.3.1. □

Lemma 9.3.3 : *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic, continuous, and piecewise $\mathcal{C}^1$ function on $\mathbb{R}$. Define $\varphi : \mathbb{R} \to \mathbb{C}$ by*

$$\forall t \in \mathbb{R}, \quad \varphi(t) = \begin{cases} f'(t) & \text{if } f \text{ is differentiable at } t, \\ \frac{1}{2}(f'(t+)+f'(t-)) & \text{otherwise.} \end{cases}$$

Then, the Fourier coefficients satisfy $c_n(\varphi) = \mathrm{i}nc_n(f)$ for all $n \in \mathbb{Z}$.

[1] We need the Riemann–Lebesgue lemma for half-integers. We can either apply a more general result from Remark 9.2.10 that we did not prove, or adapt the result from Corollary 9.2.9 (1) by writing $\sin\left(\frac{(2n+1)}{2}t\right) = \sin t\cos(\frac{t}{2}) + \cos t\sin(\frac{t}{2})$.

[2] The definition is similar to that of piecewise continuous functions in Definition 7.1.1. We say that $f : [a, b] \to \mathbb{R}$ is a piecewise $\mathcal{C}^1$ function if there exist $a = x_0 < x_1 < \cdots < x_m = b$ such that for every $1 \leqslant k \leqslant m$, f is $\mathcal{C}^1$ on (x_{k-1}, x_k), and can be extended by continuity to $[x_{k-1}, x_k]$ into a $\mathcal{C}^1$ function.

Proof : Let $0 = x_0 < x_1 < \cdots < x_m = 2\pi$ such that for every $1 \leqslant k \leqslant m$, f can be extended by continuity into a $\mathcal{C}^1$ function on $[x_{k-1}, x_k]$. Let us fix $n \in \mathbb{Z}$. For $1 \leqslant k \leqslant m$, an integration by parts gives

$$\int_{x_{k-1}}^{x_k} \varphi(t) e^{-\mathrm{i}nt}\,\mathrm{d}t = \left[f(t)e^{-\mathrm{i}nt}\right]_{t=x_{k-1}}^{x_k} + \mathrm{i}n \int_{x_{k-1}}^{x_k} f(t)e^{-\mathrm{i}nt}\,\mathrm{d}t.$$

By taking a summation over $1 \leqslant k \leqslant m$, we find

$$\int_0^{2\pi} \varphi(t) e^{-\mathrm{i}nt}\,\mathrm{d}t = \left[f(t)e^{-\mathrm{i}nt}\right]_{t=0}^{2\pi} + \mathrm{i}n \int_0^{2\pi} f(t)e^{-\mathrm{i}nt}\,\mathrm{d}t.$$

In other words,

$$c_n(\varphi) = \mathrm{i}n c_n(f).$$

□

Theorem 9.3.4 : *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic, continuous, and piecewise $\mathcal{C}^1$ function on $\mathbb{R}$. Then the Fourier series of f converges normally to f on $\mathbb{R}$.*

Proof : Let us define φ as in Lemma 9.3.3, then it follows that $c_n(\varphi) = \mathrm{i}nc_n(f)$ for all $n \in \mathbb{Z}$. Then, we may apply the AM–GM inequality to find

$$\forall n \in \mathbb{Z}\backslash\{0\}, \quad |c_n(f)| = \left|\frac{c_n(\varphi)}{n}\right| \leqslant \frac{1}{2}\left(|c_n(\varphi)|^2 + \frac{1}{n^2}\right).$$

The Parseval's identity (Theorem 9.2.7) implies the convergence of $\sum |c_n(\varphi)|^2$, from which we deduce that $\sum |c_n(f)|$ converges. Then, Proposition 9.1.5 implies that the Fourier series of f converges normally, and Corollary 9.3.2 shows that the Fourier series is equal to f. □

9.3.2 Fejér's theorem

Fejér's theorem states that a periodic continuous function can be approximate uniformly by the Cesàro means of the partial sums of its corresponding Fourier series.

Definition 9.3.5 (Fejér's kernel) **:** We define the Fejér's kernel via the Dirichlet's kernel introduced in Definition 9.1.10. For $n \in \mathbb{N}_0$, we define

$$\forall t \in \mathbb{R}\backslash 2\pi\mathbb{Z}, \quad F_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) = \frac{1}{n+1}\left(\frac{\sin\left(\frac{n+1}{2}\right)t}{\sin\frac{t}{2}}\right)^2. \tag{9.11}$$

The sequence $(F_n)_{n \geqslant 0}$ is called *Fejér's kernel*.

Proposition 9.3.6 : *The Fejér's kernel satisfies the following properties.*

(1) For $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$, we have $F_n(x) \geqslant 0$.

(2) For $n \in \mathbb{N}_0$, we have $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(x)\,\mathrm{d}x = 1$.

(3) *For any fixed $\delta > 0$, we have*

$$\lim_{n\to\infty} \int_{\delta\leqslant|x|\leqslant\pi} F_n(x)\,\mathrm{d}x = 0.$$

Proof : The first two properties are easy to check. Indeed, they follow directly from the properties of the Dirichlet's kernel, see Proposition 9.1.12. Let us check (3). For a fixed $\delta > 0$, we observe that

$$\int_{\delta\leqslant|x|\leqslant\pi} F_n(x)\,\mathrm{d}x \leqslant \frac{1}{n+1}\int_{\delta\leqslant|x|\leqslant\pi} \frac{\mathrm{d}t}{\sin^2\left(\frac{t}{2}\right)} \xrightarrow[n\to\infty]{} 0.$$

□

Theorem 9.3.7 (Fejér's theorem) **:** *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic and continuous function on $\mathbb{R}$. For $n \in \mathbb{N}_0$, let*

$$S_n(f) : x \mapsto \sum_{k=-n}^{n} c_k(f)e^{\mathrm{i}\,kx}, \quad \sigma_n(f) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(f).$$

Then, the sequence of functions $(\sigma_n(f))_{n\geqslant 0}$ converges uniformly to f on $\mathbb{R}$.

Remark 9.3.8 :

(1) We note that the sequence $(\sigma_n(f))_{n\geqslant 1}$ consists of Cesàro means of the sequence $(S_n(f))_{n\geqslant 0}$.

(2) This is a constructive proof of the Stone–Weierstraß theorem in the case of trigonometric polynomials, see Example 8.4.7 (3).

Proof : We are going to rewrite $(\sigma_n(f))_{n\geqslant 0}$ using the Fejér's kernel, and the properties in Proposition 9.3.6. Given $n \geqslant 0$ and $x \in \mathbb{R}$. We may write

$$\sigma_n(f)(x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(f)(x) = \frac{1}{n+1}\sum_{k=0}^{n}(D_n \star f)(x) = (F_n \star f)(x).$$

Then,

$$\sigma_n(f)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)F_n(t)\,\mathrm{d}t - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x-t) - f(x)]F_n(t)\,\mathrm{d}t.$$

Let us fix $\varepsilon > 0$. Using the continuity of f on $[-\pi, \pi]$, we know that it is uniformly continuous and bounded. We choose $\delta > 0$ such that for $x, y \in [-\pi, \pi]$, the condition $|x - y| \leqslant \delta$ implies $|f(x) - f(y)| \leqslant \varepsilon$; and $M > 0$ such that $\|f\|_\infty \leqslant M$. Then, we find

$$\begin{aligned} |\sigma_n(f)(x) - f(x)| &\leqslant \frac{1}{2\pi}\int_{|t|\leqslant\delta} |f(x-t) - f(x)|F_n(t)\,\mathrm{d}t + \frac{1}{2\pi}\int_{\delta\leqslant|t|\leqslant\pi} |f(x-t) - f(x)|F_n(t)\,\mathrm{d}t \\ &\leqslant \frac{\varepsilon}{2\pi}\int_{|t|\leqslant\delta} F_n(t)\,\mathrm{d}t + \frac{M}{\pi}\int_{\delta\leqslant|t|\leqslant\pi} F_n(t)\,\mathrm{d}t \\ &\leqslant \varepsilon + \frac{M}{\pi}\int_{\delta\leqslant|t|\leqslant\pi} F_n(t)\,\mathrm{d}t. \end{aligned}$$

The above inequality does not depend on $x \in \mathbb{R}$, so we have

$$\|\sigma_n(f) - f\|_\infty \leqslant \varepsilon + \frac{M}{\pi} \int_{\delta \leqslant |t| \leqslant \pi} F_n(t) \, \mathrm{d}t.$$

By taking $\limsup$ for $n \to \infty$, we find

$$\limsup_{n \to \infty} \|\sigma_n(f) - f\|_\infty \leqslant \varepsilon.$$

Since this inequality holds for any arbitrary $\varepsilon > 0$, we deduce that $\lim_{n \to \infty} \|\sigma_n(f) - f\|_\infty = 0$. □

Remark 9.3.9 : We note that in the proof of Féjer's theorem, we used the properties of the Fejér's kernel stated in Proposition 9.3.6, and did not rely on the exact form of $(F_n)_{n \geqslant 0}$. In particular, a kernel $(F_n)_{n \geqslant 0}$ satisfying the properties in Proposition 9.3.6 is called an *approximate identity*, and for such a kernel, we can apply the same proof to show that $F_n \star f$ converges uniformly to f on $\mathbb{R}$.