Chapter 1: Countable sets

Exercise 1.1 : Let $f : [0, \infty) \to [0, 1)$ be defined by $f(x) = \frac{x}{x+1}$ for $x \geqslant 0$.

(1) Show that f is bijective, and find its inverse f^{-1}.

(2) For a positive integer $n \geqslant 1$, compute the composition $f_n(x) = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(x)$ and find the image $f_n([0, +\infty))$.

Exercise 1.2 : Consider the following hyperbolic functions sinh, cosh, and tanh, defined by

$$\forall x \in \mathbb{R}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \text{and} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

(1) Find the image of each of the hyperbolic functions.

(2) Show that $\sinh : \mathbb{R} \to \mathbb{R}$, $\cosh : \mathbb{R}_{\geqslant 0} \to [1, \infty)$, and $\tanh : \mathbb{R} \to (-1, 1)$ are bijections.

(3) Find the inverse functions of the bijections in (2).

Exercise 1.3 : Let $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. For $a \in \mathbb{C} \backslash \mathbb{U}$, define

$$f_a(z) = \frac{z + \overline{a}}{1 + az}, \quad \forall z \in \mathbb{U}.$$

Let us fix $a \in \mathbb{C} \backslash \mathbb{U}$.

(1) Show that f_a is well defined.

(2) Show that f_a is a bijection and find its inverse.

Exercise 1.4 : Let $f : S \to T$ be a function. Show that f is bijective if and only if for any $A \in \mathcal{P}(S)$, we have $f(A^c) = f(A)^c$.

Exercise 1.5 : Let S be a set, and A, B be subsets of S. Define the map

$$\begin{array}{rccc} f : & \mathcal{P}(S) & \to & \mathcal{P}(A) \times \mathcal{P}(B) \\ & X & \mapsto & (X \cap A, X \cap B). \end{array}$$

(1) Show that f is injective if and only if $A \cup B = S$.

(2) Show that f is surjective if and only if $A \cap B = \varnothing$.

(3) Find a necessary and sufficient condition on A and B so that f is bijective. Find its inverse function.

Exercise 1.6 : Let S and T be two sets and $f : S \to T$ be a function.

(1) Show that $A \subseteq f^{-1}(f(A))$ for $A \in \mathcal{P}(S)$.

(2) Show that $f(f^{-1}(B)) \subseteq B$ for $B \in \mathcal{P}(T)$.

(3) Do we have equality in the above relations?

Exercise 1.7 (Cantor–Schröder–Bernstein Theorem) **:** Given two sets S and T. Suppose that there exists an injection $f : S \to T$ and an injection $g : T \to S$. Our goal is to construct a bijection between S and T. Let

$$A_0 = S \backslash g(T), \qquad A_{n+1} = (g \circ f)(A_n), \quad \forall n \geqslant 0,$$
$$B = \bigcup_{n \geqslant 0} A_n, \qquad C = S \backslash B.$$

(1) We are going to define a function $\varphi : S \to T$.
  (a) Show that for $x \in C$, there exists a unique $z \in T$ such that $x = g(z)$. We write $\varphi(x) = z$.
  (b) For $x \in B$, write $\varphi(x) = f(x)$. Show that $\varphi : S \to T$ is well defined.

(2) Let us show that φ is injective.
  (a) Show that $\varphi_{|B}$ and $\varphi_{|C}$ are injective.
  (b) Let $x \in C$ and $y \in B$ with $\varphi(x) = \varphi(y)$. Show that $x = (g \circ f)(y)$.
  (c) Deduce that φ is injective.

(3) Show that φ is surjective.

(4) Let $S = \mathbb{N}$ and $T = \mathbb{N} \cup \{0\}$. Consider $f : n \mapsto n$ and $g : n \mapsto n + 10$. Find the sets A_n for $n \geqslant 0$, B, C, and the map φ.

Exercise 1.8 : Let X_1, X_2, Y_1, Y_2 be sets such that $X_1 \sim Y_1$ and $X_2 \sim Y_2$. Show that $X_1 \times X_2 \sim Y_1 \times Y_2$.

Exercise 1.9 (Question 1.4.6) **:** Construct an explicit bijection between $\mathbb{N}$ and $\mathbb{N}^2$ using the enumeration shown in Figure 1.1.

Exercise 1.10 : Let $\mathbb{Z}^{(\mathbb{N})}$ denote the set of integer sequences with only finitely many nonzero terms. Mathematically, it writes

$$\mathbb{Z}^{(\mathbb{N})} = \{(a_n)_{n \geqslant 1} \in \mathbb{Z}^{\mathbb{N}} : \text{there exists } N \geqslant 1 \text{ such that } a_n = 0 \text{ for all } n \geqslant N\}.$$

Show that $\mathbb{Z}^{(\mathbb{N})}$ is countable.

Exercise 1.11 : Let A and B be two sets.

(1) Show that A is an infinite set if and only if A contains a countably infinite subset.

(2) Suppose that A is countable and B be infinite, show that there exists a bijection between $A \cup B$ and B. Why is not there necessarily a bijection between $A \cup B$ and A?

Exercise 1.12 : If $S \subseteq \mathbb{R}$ is countable, show that there exists a real number $a \in \mathbb{R}$ such that $(a+S)\cap S = \varnothing$.

Exercise 1.13 : Show that $(0,1)$ and $(0,1]$ are equinumerous and construct a bijection between them. (Hint: you may get inspiration from the steps in Exercise 1.7 for instance.)

Exercise 1.14 : Let $(f_n)_{n\geqslant 1}$ be a sequence of functions from $\mathbb{N}$ to $\mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{N}$ be defined as below,

$$f(n) = f_n(n) + 1, \quad \forall n \geqslant 1.$$

(1) Show that $f_n \neq f$ for every $n \geqslant 1$.

(2) Deduce that the set of functions from $\mathbb{N}$ to $\mathbb{N}$ is not countable.

Exercise 1.15 : Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. We write D for the set of discontinuous points of f.

(1) Let $I = (a,b)$ with $a < b$. Show that $D \cap (a,b)$ is countable.

(2) Deduce that D is countable.

Exercise 1.16 : Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, that is, for any $x \in \mathbb{R}$, the function

$$y \mapsto \frac{f(y) - f(x)}{y - x}$$

is non-decreasing on $\mathbb{R}\backslash\{x\}$. Let D be the set of points where f is differentiable. Show that $\mathbb{R}\backslash D$ is countable. (Hint: use Exercise 1.15.)

Exercise 1.17 : Given a function $f : \mathbb{R} \to \mathbb{R}$. We say that f attains a strict local maximum at $x \in \mathbb{R}$ if there exists $\varepsilon > 0$ such that

$$\forall y \in \mathbb{R}, \quad |y - x| < \varepsilon \quad \Rightarrow \quad f(y) > f(x).$$

Show that the following set is countable,

$$\{x \in \mathbb{R} : f \text{ attains a strict local maximum at } x\}.$$

Exercise 1.18 : Let V be a vector space over the rationals $\mathbb{Q}$.

(1) Suppose that V is finite dimensional. Show that V is countably infinite.

(2) Let $\mathbb{Q}[X]$ be the set of polynomials with rational coefficients, that is,

$$\mathbb{Q}[X] = \Big\{ \sum_{n=0}^{N} a_n X^n : a_n \in \mathbb{Q}, 0 \leqslant n \leqslant N, N \geqslant 0 \Big\}.$$

Show that $\mathbb{Q}[X]$ is countably infinite.

A real number $x \in \mathbb{R}$ is said to be an algebraic number (代數數) if it is a root of some polynomial $P \in \mathbb{Q}[X]$. Otherwise, we say that it is a transcendental number (超越數).

(3) Show that there exists $x \in \mathbb{R}$ that is transcendental.

Exercise 1.19 : Given a sequence $(u_n)_{n\geqslant 1}$ of real numbers. Let $a_1 = u_1 + 1$ and $b_1 = u_1 + 2$, so that we have $u_1 \notin [a_1, b_1]$.

(1) Suppose that for an integer $n \geqslant 1$, we have $a_n < b_n$ with $u_n \notin [a_n, b_n]$. Consider $c_n = \frac{1}{3}(2a_n+b_n)$ and $d_n = \frac{1}{3}(a_n + 2b_n)$. Show that we can choose a_{n+1} and b_{n+1} from the set $\{a_n, c_n, d_n, b_n\}$ such that

$$b_{n+1} - a_{n+1} = \tfrac{1}{3}(b_n - a_n) \quad \text{and} \quad u_{n+1} \notin [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n].$$

(2) Consider the sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ that are constructed in the previous question. Show that $(a_n)_{n\geqslant 1}$ is non-decreasing and that $(b_n)_{n\geqslant 1}$ is non-increasing.

(3) Deduce that the sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ converge to the same limit, denoted ℓ.

(4) What can we say about the countability of the set of real numbers $\mathbb{R}$?

Exercise 1.20 : Show that $\{0,1\}^{\mathbb{N}} \sim \mathcal{P}(\mathbb{N})$ and deduce that $\mathbb{R}$ is not countable.