Chapter 2: Topology on metric spaces and normed spaces

Exercise 2.1 : Let (M, d) be a metric space. Define

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}, \quad \forall x, y \in M.$$

Show that (M, d') is also a metric space. Note that the metric space (M, d') is bounded, since $0 \leqslant d'(x,y) < 1$ for all $x, y \in M$.

Exercise 2.2 : Let $n \geqslant 0$ be an integer and fix $a_0, \dots, a_n \in \mathbb{R}$ be pairwise distinct real numbers. Let $\mathbb{R}_n[X]$ be the vector space consisting of real polynomials of degree at most n, that is

$$\mathbb{R}_n[X] = \Big\{ \sum_{k=0}^{n} c_k X^k : c_k \in \mathbb{R}, 0 \leqslant k \leqslant n \Big\}.$$

Define

$$\|P\| := \max_{0 \leqslant i \leqslant n} |P(a_i)|, \quad \forall P \in \mathbb{R}_n[X].$$

Show that $\|\cdot\|$ is a norm on $\mathbb{R}_n[X]$.

Exercise 2.3 : Check that the maps $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ defined in Example 2.1.9 (a) and (b) are indeed norms on $\mathbb{K}[X]$.

Exercise 2.4 : On a Euclidean space $(V, \langle \cdot, \cdot \rangle)$, let us define

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad \forall x \in \mathbb{R}^n.$$

Fix $x, y \in V$.

(1) Explain why the function $\lambda \mapsto \|x + \lambda y\|^2$ is a non-negative function.

(2) Deduce the Cauchy-Schwarz inequality $|\langle x, y \rangle| \leqslant \|x\| \, \|y\|$.

(3) Show the triangular inequality $\|x + y\| \leqslant \|x\| + \|y\|$.

Exercise 2.5 : Let $(V, \|\cdot\|)$ be a normed space.

(1) Suppose that the norm $\|\cdot\|$ is induced by an inner product in the sense of Proposition 2.1.12. Show that the norm satisfies the parallelogram law,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \forall x, y \in V. \tag{2.1}$$

(2) Suppose that the norm $\|\cdot\|$ satisfies the parallelogram law (2.1). Show that V is Euclidean, that is, there exists an inner product $\langle \cdot, \cdot \rangle$ that induces the norm $\|\cdot\|$ in the sense of Proposition 2.1.12. We may want to consider

$$\langle x, y \rangle := \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \tfrac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2), \quad x, y \in V.$$

(3) Is the vector space $\mathcal{C}([0,1], \mathbb{R})$ equipped with the norm $\|\cdot\|_\infty$ a Euclidean space?

Exercise 2.6 : In $\mathbb{R}^2$, draw the centered unit closed balls $\overline{B}_{N_1}(0,1)$, $\overline{B}_{N_2}(0,1)$, and $\overline{B}_{N_\infty}(0,1)$, for the norms defined as below,

$$N_1(x_1,x_2) = |x_1| + |x_2|, \qquad N_2(x_1,x_2) = \sqrt{x_1^2 + x_2^2}, \qquad \text{and} \qquad N_\infty(x_1,x_2) = \max |x_1|, |x_2|.$$

Give the inclusion relations between these closed unit balls.

Exercise 2.7 : Show that in a normed space, closure of open ball is the closed ball. In other words, given a normed space V, show that $\overline{B(a,r)} = \overline{B}(a,r)$ for all $a \in V$ and $r > 0$.

Exercise 2.8 (Question 2.1.26) **:** Is any union of closed sets still a closed set? If yes, please prove it; otherwise, please give a counterexample.

Exercise 2.9 : Show that in $\mathbb{R}$, apart from the emptyset $\varnothing$ and the full space $\mathbb{R}$, any other subset cannot be open and close at the same time. Is there a similar statement for $\mathbb{R}^2$?

Exercise 2.10 : Let (M,d) be a metric space and $A \subseteq M$ be a closed subset. Show that A can be written as a countably infinite intersection of open sets.

Exercise 2.11 : Let $A \subseteq M$ and $x \in M$. Then, the following properties are equivalent.

(1) $x \in \mathring{A}$.

(2) There exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq A$.

Exercise 2.12 : Let (M,d) be a metric space and $A \subseteq M$ be a subset. Use double inclusion to prove that

$$\operatorname{int} A = M \backslash \operatorname{cl}(M \backslash A) \quad \text{and} \quad \operatorname{int}(M \backslash A) = M \backslash \operatorname{cl}(A).$$

Exercise 2.13 : Let (M,d) be a metric space. Consider two subsets A and B of M such that $\operatorname{int} A = \operatorname{int} B = \varnothing$.

(1) Show that $\operatorname{int}(A \cup B) = \varnothing$ if A is closed in M.

(2) Given an example for which we have $\operatorname{int}(A \cup B) = M$.

Exercise 2.14 : Let (M,d) be a metric space and $A \subseteq M$ be a subset.

(1) Show that if A is open, then $\operatorname{int}(\partial A) = \varnothing$.

(2) When do we have $\operatorname{int}(\partial A) = M$?

Exercise 2.15 : Let (M,d) be a metric space and $A \subseteq M$ be a subset. Show that $\partial A = \overline{A} \cap \overline{M \backslash A}$ and deduce that $\partial A = \partial(M \backslash A)$.

Exercise 2.16 : Let A and B be subsets in a metric space (M,d). Suppose that $\overline{A} \cap \overline{B} = \varnothing$. Show that $\partial(A \cup B) = \partial A \cup \partial B$.

Exercise 2.17 : Let (M,d) be a metric space. Consider a set I, and a family $(A_i)_{i\in I}$ of subsets of M that are indexed by the elements of I.

(1) Suppose that I is finite, for example $I = \{1,\dots,n\}$ for some integer $n \geqslant 1$. Show that

$$\operatorname{int}\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I}(\operatorname{int} A_i).$$

(2) Suppose that I is infinite. Show that

$$\operatorname{int}\Big(\bigcap_{i\in I} A_i\Big) \subseteq \bigcap_{i\in I}(\operatorname{int} A_i).$$

(3) Find an example where the equality does not hold in (2).

(4) Without any additional assumption on I, show that

$$\bigcup_{i\in I}(\operatorname{int} A_i) \subseteq \operatorname{int}\Big(\bigcup_{i\in I} A_i\Big).$$

(5) Find an example with finite I such that the equality does not hold in (4).

Exercise 2.18 : Let (M,d) be a metric space and $S \subseteq T \subseteq U$ be subsets of M. Suppose that S is dense in T and T is dense in U, show that S is also dense in U.

Exercise 2.19 : A metric space (M,d) is said to be separable (可分) if there exists a countable subset $A \subseteq M$ that is dense in M. Show that every Euclidean space $\mathbb{R}^n$ is separable.

Exercise 2.20 : The Bolzano–Weierstraß theorem (Theorem 2.2.5) is proven for $\mathbb{R}^n$, equiped with the metric induced by its Euclidean norm. If we equip $\mathbb{R}^n$ with other distances which are also induced by norms, such as $\|\cdot\|_1$ or $\|\cdot\|_\infty$, does the theorem still hold? And how about with the discrete distance given by $d(x,y) = \mathbb{1}_{x\neq y}$?

Exercise 2.21 (Cantor set) **:** We define a sequence of subsets of $\mathbb{R}$ by induction,

$$C_0 = [0,1], \quad C_{n+1} = \tfrac{1}{3}C_n \cup (\tfrac{1}{3}C_n + \tfrac{2}{3}), \quad \forall n \geqslant 0.$$

Let $\mathcal{C} := \cap_{n\geqslant 0} C_n$.

(1) Show that the subset C_n is closed, and $C_{n+1} \subseteq C_n$ for every $n \geqslant 0$.

(2) Show that the countable intersection $\mathcal{C}$ is nonempty.

Given $x \in [0,1]$, we may define its ternary expansion,

$$x = 0.x_1x_2x_3\cdots = \sum_{k\geqslant 1} x_k 3^{-k}, \tag{2.2}$$

where we have $x_k \in \{0,1,2\}$ for all $k \geqslant 1$. Note that this expansion may not be unique.

(3) In this question, we want to show the uncountability of $\mathcal{C}$.

(a) Show that if $x \in \mathcal{C}$, then there exists a ternary expansion Eq. (2.2) of x for which we have $x_k = 0$ or 2 for all $k \geqslant 1$.

(b) Given $x \in [0,1]$ and suppose that the ternary expansion Eq. (2.2) of x is such that $x_k = 0$ or 2 for all $k \geqslant 1$. Show that $x \in \mathcal{C}$.

(c) For which $x \in \mathcal{C}$, the ternary expansion is not unique?

(d) Conclude.

(4) Given $0 \leqslant a < b \leqslant 1$, is the subset $\mathcal{C} \cap [a,b]$ dense in $[a,b]$?

Given a segment $I = [a,b]$ in $[0,1]$, let us define its length to be $\ell(I) := b - a$. Given $n \geqslant 1$ pairwise disjoint segments $(I_i = [a_i, b_i])_{1 \leqslant i \leqslant n}$ in $[0,1]$, we define the length of $I := \cup_{i=1}^n I_i$ as $\ell(I) := \sum_{i=1}^n \ell(I_i)$. Given a subset A in $[0,1]$, we say that it is of length zero if

$$\inf\{\ell(I) : A \subseteq I = \sqcup_{i=1}^n I_i, n \geqslant 1, I_i\text{'s are segments}\} = 0,$$

where $\sqcup$ means that the segments form a pairwise disjoint union.

(5) Find $\ell(C_n)$ for all $n \geqslant 0$ and show that $\mathcal{C}$ is of length zero.

The set $\mathcal{C}$ we constructed above is called *Cantor set*, which is a nonempty uncountable closed set in $[0,1]$, of length 0, thus not containing any interval.

Exercise 2.22 : Let (M,d) be a metric space and $S \subseteq M$ be an open subset. For any subset $A \subseteq S$, show that A is open in S if and only if it is open in M.

Exercise 2.23 : On the space $(0,1]$, we may consider the topology induced by the metric space $(\mathbb{R}, |\cdot|)$ as mentioned in Example 2.3.2. Alternatively, we may also define a distance d on $(0,1]$, given by

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \qquad \forall x,y \in (0,1].$$

(1) Let $x \in (0,1]$ and $\varepsilon > 0$. Let $B = B_{|\cdot|}(x,\varepsilon)$ be the ball centered at x of radius ε for the metric $|\cdot|$. Show that for any $y \in B$, we may find $\varepsilon' > 0$ such that

$$B_d(y,\varepsilon') \subseteq B = B_{|\cdot|}(x,\varepsilon).$$

(2) Show that an open ball in $((0,1],d)$ is also an open ball in $((0,1],|\cdot|)$.

(3) Conclude that the metric spaces $((0,1],|\cdot|)$ and $((0,1],d)$ are topologically equivalent, that is, a set A is open in one space if and only if it is also open in the other one.

(4) Is $((0,1],d)$ a complete metric space? How about $((0,1],|\cdot|)$?

Exercise 2.24 : Let M be a set equipped with two distances d and d'. Suppose that the metric spaces (M,d) and (M,d') are topologically equivalent, i.e. the open sets in one space are also open in the other space. Show that if a sequence $(a_n)_{n\geqslant 1}$ converges in (M,d), then it also converges in (M,d').

Exercise 2.25 : Let M be a set that we equip with two uniformly equivalent distances d and d'. Show that a sequence is Cauchy in (M,d) if and only if it is also Cauchy in (M,d').

Exercise 2.26 : Let (M, d) be a metric space and $\mathcal{C}$ be the set of all the Cauchy sequences in M. Let us define the function $\delta : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$ as follows. For $U = (u_n)_{n\geqslant 1}$, $V = (v_n)_{n\geqslant 1} \in \mathcal{C}$, let

$$\delta(U, V) = \lim_{n\to\infty} d(u_n, v_n).$$

(1) Show that δ is well defined, symmetric, and satisfies the triangle inequality.

(2) If $U = (u_n)_{n\geqslant 1}$, $V = (v_n)_{n\geqslant 1}$, and $S = (s_n)_{n\geqslant 1}$ are in $\mathcal{C}$ and such that $\delta(U, V) = 0$ and $\delta(V, S) = 0$, check that $\delta(U, S) = 0$.

(3) Let $U = (u_n)_{n\geqslant 1}$, $U' = (u'_n)_{n\geqslant 1}$, $V = (v_n)_{n\geqslant 1}$, and $V' = (v'_n)_{n\geqslant 1}$ be sequences such that $\delta(U, U') = 0$ and $\delta(V, V') = 0$. Show that $\delta(U, V) = \delta(U', V')$.

We will see in Section 3.3 that the above steps are preliminary steps towards the *completion* of the metric space (M, d).

Exercise 2.27 : Find the upper limit and the lower limit of the following sequences. Do not forget to justify how you obtain your results.

$$a_n = \frac{(-1)^n}{n}, \quad b_n = \cos(n), \quad c_n = \frac{1}{\sin(n)}, \quad d_n = \frac{n^2 + 2n + 1}{n(n + \cos(n))}.$$

Exercise 2.28 : Find a sequence $(a_n)_{n\geqslant 1}$ in $\mathbb{R}$ such that $\liminf_{n\to\infty} a_n = 0$, $\limsup_{n\to\infty} a_n = 1$, and that any number $x \in [0, 1]$ is an accmulation point of $\{a_n : n \geqslant 1\}$.

Exercise 2.29 : Given a bounded sequence $(x_n)_{n\geqslant 1}$ with $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$. Show that any point between $\ell := \liminf_{n\to\infty} a_n$ and $L := \limsup_{n\to\infty} a_n$ are accumulation points of $\{a_n : n \geqslant 1\}$.

Exercise 2.30 : Let $(a_n)_{n\geqslant 1}$ be a sequence of positive real numbers. Let $\varepsilon \in (0, 1)$ and suppose that there exists $N_0 \geqslant 1$ such that

$$\frac{1 + a_{n+1}}{a_n} \leqslant 1 + \frac{1 - \varepsilon}{n}, \quad \forall n \geqslant N_0. \tag{2.3}$$

(1) Show that for any positive integer $k \geqslant 1$, we have

$$a_{N_0+k} \leqslant \prod_{i=0}^{k-1} \Big(1 + \frac{1 - \varepsilon}{n + i}\Big) a_{N_0} - k.$$

(2) Show that for $x \geqslant 0$, we have $1 + x \leqslant e^x$.

(3) Show that $a_n \xrightarrow[n\to\infty]{} -\infty$, and conclude that Eq. (2.3) cannot hold.

(4) Deduce the following inequality,

$$\limsup_{n\to\infty} \Big(\frac{1 + a_{n+1}}{a_n}\Big)^n \geqslant e.$$

(5) Does there exist a sequence $(a_n)_{n\geqslant 1}$ of positive real numbers for which the above inequality becomes an equality?

Exercise 2.31 : Let $(a_n)_{n\geqslant 1}$ be a sequence of positive real numbers.

(1) Prove the following inequality

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leqslant \liminf_{n\to\infty} (a_n)^{1/n} \leqslant \limsup_{n\to\infty} (a_n)^{1/n} \leqslant \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

(2) Find a sequence $(a_n)_{n\geqslant 1}$ that makes all the three above inequalities strict.

(3) Show that if the limit of $\frac{a_{n+1}}{a_n}$ exists, then the limit of $(a_n)^{1/n}$ also exists, and we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (a_n)^{1/n}.$$

(4) Does the sequence $a_n = \frac{(n!)^{1/n}}{n}$ have a limit? If yes, find its value.

Exercise 2.32 : Let $a = (a_n)_{n\geqslant 1}$ be a sequence with values in $\mathbb{R}$. Show that the following properties are equivalent.

(i) a does not have any subsequential limit in $\mathbb{R}$.

(ii) Any subsequence of a is unbounded.

(iii) $|a_n| \xrightarrow[n\to\infty]{} +\infty$ when $n \to \infty$.

Exercise 2.33 : Let $(V, \|\cdot\|)$ be a normed vector space and $f : E \to E$ be the function

$$\forall x \in V, \quad f(x) = \frac{x}{1 + \|x\|}.$$

(1) Show that f is a bijection between V and $B(0,1)$.

(2) Show that f and f^{-1} are continuous.

Exercise 2.34 : Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be a function. Show that the following properties are equivalent.

(i) f is continuous.

(ii) For any subset $B \subseteq Y$, we have $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

(iii) For any subset $B \subseteq Y$, we have $f^{-1}(\operatorname{int}(B)) \subseteq \operatorname{int}(f^{-1}(B))$.

(iv) For any subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

Exercise 2.35 : Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Show that the following two maps are continuous

$$\begin{array}{ccc} E \times E & \to & E \\ (x,y) & \mapsto & x+y \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{K} \times E & \to & E \\ (\lambda, x) & \mapsto & \lambda x \end{array}.$$

Exercise 2.36 : Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\forall x, y \in \mathbb{R}, \quad f\Big(\frac{x+y}{2}\Big) = \frac{1}{2}(f(x) + f(y)).$$

(1) Show that the set $\mathcal{D}\{p \cdot 2^{-n} : p \in \mathbb{Z}, n \in \mathbb{N}_0\}$ is dense in $\mathbb{R}$.

(2) Show that $f(0) = f(1) = 0$ implies that $f \equiv 0$.

(3) Conclude that there exists $a, b \in \mathbb{R}$ such that $f(x) = ax + b$.

Exercise 2.37 : Let (M, d) be a metric space and $A, B \subseteq M$ be two disjoint, closed, nonempty subsets. Consider the map

$$\begin{array}{rccc} \varphi : & M & \to & \mathbb{R} \\ & x & \mapsto & d(x, A) - d(x, B). \end{array}$$

(1) Show that φ is continuous on M.

(2) Show that $\varphi(x) < 0$ for $x \in A$ and $\varphi(x) > 0$ for $x \in B$.

(3) Deduce that there exist two disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Exercise 2.38 : Consider a closed set F and an open set U in a metric space (M, d) such that $F \subseteq U$. Show that there exists an open set V satisfying

$$F \subseteq V \subseteq \overline{V} \subseteq U.$$

Exercise 2.39 : Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subseteq \mathbb{R}\backslash\mathbb{Q}$ and $f(\mathbb{R}\backslash\mathbb{Q}) \subseteq \mathbb{Q}$. Hint: see below[1].

Exercise 2.40 : Consider the vector space $V = \mathcal{C}([0,1], \mathbb{R})$ equipped with the norm $\|\cdot\|_\infty$. Show that the function $f \mapsto \inf_{x \in [0,1]} f(x)$ is continuous.

Exercise 2.41 : Find an example of metric spaces (M_1, d_1), (M_2, d_2), and (M_2, d_2') such that

- (M_2, d_2) and (M_2, d_2') are topologically equivalent,
- there exists a function $f : (M_1, d_1) \to M_2$ such that f is uniformly continuous when M_2 is equipped with d_2, but not uniformly continuous when M_2 is equipped with d_2'.

You may get inspiration from Exercise 2.23.

[1]Use the intermediate-value theorem in $\mathbb{R}$, and the fact that any nonempty interval of $\mathbb{R}$ contains uncountably infinite elements.

Exercise 2.42 : Let X and Y be metric spaces and $f : X \to Y$. The graph (圖) of f is the set

$$\Gamma_f := \{(x, y) \in X \times Y : y = f(x)\}.$$

(1) Show that if f is continuous, then Γ_f is closed in $X \times Y$.

(2) Find an example for which Γ_f is closed in $X \times Y$ without f being continuous.

(3) Show that f is continuous if and only if the following map is an homeomorphism,

$$\begin{array}{ccc} X & \to & \Gamma_f \\ x & \mapsto & (x, f(x)) \end{array}.$$

Exercise 2.43 :

(1) Show that the open interval $(-1, 1) \subseteq \mathbb{R}$ and $\mathbb{R}$ are homeomorphic, and write $f : (-1, 1) \to \mathbb{R}$ be a homeomorphism.

(2) Let $(a_n)_{n \geqslant 1}$ be a converging sequence in $(-1, 1)$. Does the sequence $(f(a_n))_{n \geqslant 1}$ converge?

(3) Let $(a_n)_{n \geqslant 1}$ be a Cauchy sequence in $(-1, 1)$. Is the sequence $(f(a_n))_{n \geqslant 1}$ Cauchy?

Exercise 2.44 : Let $n \geqslant 1$ and $(V_1, \varphi_1), \ldots, (V_n, \varphi_n)$ be normed spaces. We consider the product space $V = V_1 \times \cdots \times V_n$, and define the following maps on V,

$$N_1(x) = \sum_{i=1}^{n} \varphi_i(x_i), \quad \text{and} \quad N_2(x) = \sqrt{\sum_{i=1}^{n} \varphi_i(x_i)^2}, \quad \forall x = (x_1, \ldots, x_n) \in V.$$

Show that N_1 and N_2 are norms on V.

Exercise 2.45 : Check that the maps D_1 and D_2 defined in Remark 2.6.3 are indeed distances on the product space.

Exercise 2.46 : Given countably infinitely many metric spaces $((M_n, d_n))_{n \geqslant 1}$ which are assumed to be uniformly bounded, i.e.,

$$\exists A > 0, \forall n \geqslant 1, \quad \delta(M_n) < A.$$

Define the product space $M = \prod_{n \geqslant 1} M_n$ and the following function on $M \times M$,

$$d(x, y) = \sum_{n \geqslant 1} 2^{-n} d_n(x_n, y_n), \quad \forall x, y \in M.$$

Show that d is a distance on M.

Exercise 2.47 : Let $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$.

(1) Show that for any $z_0 \in \mathbb{U}$, the subset $\mathbb{U}\backslash\{z_0\}$ is connected. (Hint: show that it is arcwise connected.)

(2) Construct a continuous bijective map from $[0, 1)$ to $\mathbb{U}$.

(3) Show that that there is no continuous bijective map from $\mathbb{U}$ to $[0, 1]$.

(4) Deduce that $\mathbb{U}$ and $[0, 1]$ are not homeomorphic.

Exercise 2.48 : Let (M, d) be a metric space, A and B be subsets of M. Suppose that B is connected and A satisfies

$$B \cap \operatorname{int}(A) \neq \varnothing \quad \text{and} \quad B \cap \operatorname{int}(M\backslash A) \neq \varnothing.$$

Show that $B \cap \partial A \neq \varnothing$.

Exercise 2.49 (Question 2.7.10) **:** Let $(C_i)_{i\in I}$ be a countable family of connected subsets, i.e., $I = \{1, \ldots, p\}$ for some $p \geqslant 1$ or $I = \mathbb{N}$. Suppose that for every $i \in I$, $i \neq 1$, we have $C_{i-1} \cap C_i \neq \varnothing$. Show that $C = \cup_{i\in I} C_i$ is connected by rewriting the proof of Proposition 2.7.8.

Exercise 2.50 : Let (M, d) be a metric space. Show that the the following statements are equivalent.

(i) (M, d) is connected.

(ii) Every proper nonempty subset of M has nonempty boundary in M.

(iii) Any real-valued continuous function defined on M has intermediate value property. That is, for any continuous function $f : (M, d) \to (\mathbb{R}, |\cdot|)$, if $x \in \mathbb{R}$ is such that $f(a) < x < f(b)$ for some $a, b \in M$, then $x \in f(M)$.

Exercise 2.51 : Let A and B be connected metric spaces. Let $X \subsetneq A$ and $Y \subsetneq B$ be proper subsets. Show that $C := (A \times B)\backslash(X \times Y)$ is connected in $A \times B$. Why do we need to take X and Y to be proper subsets?

Exercise 2.52 : Let $A \subseteq \mathbb{R}^2$ be defined as follows,

$$A := \{(0, 0)\} \cup \{(x, \sin(1/x)) : x \in (0, 1]\}.$$

We also define

$$A' := \{(x, \sin(1/x)) : x \in (0, 1]\}.$$

(1) Show that A' is arcwise connected and deduce that it is connected.

(2) Show that A is not arcwise connected.

(3) Find $\overline{A'}$ and deduce that A is connected.

Exercise 2.53 : In this exercise, we will distinguish the notion between two "equinumerous sets" and "homeomorphic sets".

(1) Given two disjoint sets A and B, show that $\mathcal{P}(A) \times \mathcal{P}(B) \sim \mathcal{P}(A \cup B)$.

(2) Use the fact that $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$ to deduce that $\mathbb{R}^2 \sim \mathbb{R}$.

(3) Show that that there is no continuous bijective map from $\mathbb{R}^2$ to $\mathbb{R}$.

Exercise 2.54 : Let (M, d) be a connected metric space and $A \subseteq M$ be a closed subset of M. Suppose that the boundary ∂A is connected. Let $f : A \to D = \{0, 1\}$ be a continuous function.

(1) Show that $f_{|\partial A}$ is constant.

Without loss of generality, we may assume that $f_{|\partial A} \equiv 0$. Define the function $g : M \to D$ as below,

$$g(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \in A^c. \end{cases}$$

(1) Show that $g^{-1}(\varnothing)$, $g^{-1}(D)$, $g^{-1}(\{0\})$, and $g^{-1}(\{1\})$ are closed subsets of M, and deduce that g is continuous.

(2) Deduce that A is connected.

(3) If we remove the assumption that $A \subseteq M$ is a closed subset, is it still true that A is connected?

Exercise 2.55 : Let $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ and $f : \mathbb{U} \to \mathbb{R}$ be a continuous function. Show that there exists $z \in \mathbb{U}$ such that $f(z) = f(-z)$.