Chapter 3: Compact spaces and complete spaces

Exercise 3.1 : Let (M, d) be a compact metric space and $f : M \to \mathbb{R}$ be a function. We recall that M satisfies the Borel–Lebesgue property.

(1) Suppose that f is locally bounded, that is for all $x \in M$, there exists $r_x > 0$ and $A_x > 0$ such that

$$\forall y \in B(x, r_x), \quad |f(y)| \leqslant A_x.$$

Show that f is bounded on M.

(2) Suppose that f is locally Lipschitz continuous, that is for all $x \in M$, there exists $r_x > 0$ and $K_x > 0$ such that

$$\forall y, z \in B(x, r_x), \quad |f(y) - f(z)| \leqslant K_x \cdot d(y, z).$$

Show that f is Lipschitz continuous on M.

Exercise 3.2 : Let (M, d) be a metric space, $A \subseteq M$ be a compact subset, and $B \subseteq M$ be a closed subset with $A \cap B = \varnothing$.

(1) Apply the Borel–Lebesgue property to A to show that there exists an open subset $U \subseteq M$ such that $A \subseteq U$ and $\overline{U} \cap B = \varnothing$. Hint: B^c is open.

(2) Suppose that B is also compact. Deduce from the previous question that there exists open sets U and V such that

$$A \subseteq U, \quad B \subseteq V, \quad \text{and} \quad \overline{U} \cap \overline{V} = \varnothing.$$

Exercise 3.3 : Let (M, d) be a metric space and $(x_n)_{n \geqslant 1}$ be a convergent sequence in M with limit ℓ. Show that the set

$$\Gamma = \{x_n : n \geqslant 1\} \cup \{\ell\}$$

is compact using the Borel–Lebesgue property.

Exercise 3.4 : Let (K_1, d_1) and (K_2, d_2) be two compact metric spaces. Show that the product space $K_1 \times K_2$ equipped with the product distance $d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ satisfies the Borel–Lebesgue property, and deduce that it is compact. Show that any finite product of compact metric spaces is compact.

Exercise 3.5 : Let (M, d) be a metric space and $(K_n)_{n \geqslant 1}$ be a sequence of nonempty compact sets of M. Suppose that $K_{n+1} \subseteq K_n$ for all $n \geqslant 1$. Set $K = \cap_{n \geqslant 1} K_n$.

(1) Show that $K \neq \varnothing$.

(2) Let U be an open set containing K. Show that there exists $n \geqslant 1$ such that $K_n \subseteq U$.

We note that when (M, d) is taken to be the Euclidean space $\mathbb{R}^n$, then (1) is the Cantor's intersection theorem.

Exercise 3.6 : Let V and W be two normed vector spaces, $K \subseteq V$ be a compact subset. Let $f : K \to W$ be an injective continuous function. Show that f is a homeomorphism between K and $L = f(K)$.

Exercise 3.7 : Let I and J be intervals in $\mathbb{R}$, and $f : I \to J$ be a continuous and bijective function. Show that f^{-1} is continuous.

Exercise 3.8 : Show Exercise 3.1 using the Bolzano–Weierstraß property.

Exercise 3.9 : Let K_1, K_2 be two compact sets in a normed vector space. Show that the following set is compact,

$$K_1 + K_2 := \{x_1 + x_2 : x_1 \in K_1, x_2 \in K_2\}.$$

Exercise 3.10 : Let K be a compact set in a metric space (M, d). Given a sequence $x = (x_n)_{n \geqslant 1}$ with values in K. Suppose that x only has one subsequential limit ℓ, that is, its set of subsequential limits, defined in Section 2.4.3, is the singleton set $\{\ell\}$. Show that $x_n \xrightarrow[n\to\infty]{} \ell$.

Exercise 3.11 : Let Ω be an open set in the Euclidean space $\mathbb{R}^n$. Show that there exists an exhaustion of Ω by compact sets, that is, a sequence $(K_n)_{n \geqslant 1}$ satisfying

(i) $K_n \subseteq \Omega$ for all $n \geqslant 1$.

(ii) $K_n \subseteq K_{n+1}$ for all $n \geqslant 1$.

(iii) $\Omega = \cup_{n \geqslant 1} K_n$.

Hint: see below[1].

Exercise 3.12 : Let V be a normed vector space, and $A, B \subseteq V$ be two subsets. Assume that A is closed. Let $f : A \to B$ be a function, and define its graph as

$$\Gamma_f = \{(x, f(x)) : x \in A\}.$$

(1) If f is continuous, show that its graph Γ_f is closed. Note that we have seen a similar statement in Exercise 2.42.

(2) Suppose that B is compact and Γ_f is closed. Show that f is continuous. Hint: you may use Exercise 3.10.

[1] For $n \geqslant 1$, consider $L_n := \{x \in \mathbb{R}^n : d(x, \Omega^c) \geqslant 1/n\}$ and $K_n = \overline{B}(0, n) \cap L_n$.

Exercise 3.13 : Let $f : (E, d_E) \to (F, d_F)$ be a continuous function between two metric spaces.

(1) Suppose that for every compact set $K \subseteq F$, the preimage $f^{-1}(K)$ is also compact. Show that f is a closed map, that is, for any closed subset $A \subseteq E$, the image $f(A)$ is also closed. Hint: use Exercise 3.3.

(2) Are there continuous maps which are not closed?

(3) Let $n \geqslant 1$ be an integer. Consider the real vector space of polynomials of degree at most n, denoted by
$$\mathbb{R}_n[X] = \{P \in \mathbb{R}[X] : \deg(P) \leqslant n\}.$$
We equip $\mathbb{R}_n[X]$ with one of the norms from Example 2.1.9. (For example, $\|P\|_\infty = \max_{0 \leqslant k \leqslant n} |a_k|$ for any $P = \sum_{0 \leqslant k \leqslant n} a_k X^k \in \mathbb{R}_n[X]$.) Let Γ_n be the set of monic polynomials of degree exactly n whose roots are all real. Show that Γ_n is closed in $\mathbb{R}_n[X]$. Hint: see below[2].

Exercise 3.14 : Let (M, d) be a metric space. For any subsets A, B in M, we define
$$d(A, B) = \inf_{\substack{x \in A \\ y \in B}} d(x, y).$$

(1) Let K_1 and K_2 be two compact subsets of M. Show that there exists $x_1 \in K_1$ and $x_2 \in K_2$ such that $d(x_1, x_2) = d(K_1, K_2)$. Deduce that if $K_1 \cap K_2 = \varnothing$, then $d(K_1, K_2) > 0$.

(2) Let $K \subseteq M$ be compact, and $A \subseteq M$ be closed. Show that if $K \cap A = \varnothing$, then $d(K, A) \neq 0$.

(3) In the previous question, is it enough to assume that both K and A are closed?

From now on, let us assume that (M, d) is the Euclidean space $\mathbb{R}^n$ with $n \geqslant 1$.

(4) Let $A \subseteq M = \mathbb{R}^n$ be an unbounded closed subset and $f : A \to \mathbb{R}$ be a continuous map such that
$$\lim_{\substack{\|x\| \to \infty \\ x \in A}} f(x) = +\infty. \tag{E3.1}$$
Show that there exists $x \in A$ such that $f(x) = \inf_{y \in A} f(y)$. Hint: see below[3].

(5) Let $K \subseteq M = \mathbb{R}^n$ be a compact subset and $A \subseteq M = \mathbb{R}^n$ be a closed subset. Show that there exists $x \in K$ and $y \in A$ such that $d(x, y) = d(K, A)$.

(6) If M is an infinite dimensional normed vector space, show that (5) does not hold. In other words, find an infinite dimensional normed vector space M, a compact subset $K \subseteq M$, a closed subset $A \subseteq M$ such that for any $x \in K$ and $y \in A$, we have $d(x, y) \neq d(K, A)$.

[2] A possible proof starts by showing that if r is a root of $P \in \mathbb{P}_n[X]$, then $|r| \leqslant \max\{1, \|P\|_\infty\}$, before applying this result to check the conditions in (1).

[3] Eq. (E3.1) means that for any large enough $M > 0$, there exists $R > 0$ such that $\|x\| \geqslant R$ implies that $f(x) \geqslant M$.

Exercise 3.15 : Let $(V, \|\cdot\|)$ be a normed vector space and $K \subseteq V$ be a compact subset. Consider a function $f : K \to K$ satisfying

$$\forall x, y \in K, \quad \|f(x) - f(y)\| \geqslant \|x - y\|.$$

Fix $a_0, b_0 \in K$, and define two iterative sequences as follow

$$\forall n \geqslant 0, \quad a_{n+1} = f(a_n) \quad \text{and} \quad b_{n+1} = f(b_n).$$

(1) Show that for all $\varepsilon > 0$ and integer $p \geqslant 1$, there exists $k \geqslant p$ such that $\|a_k - a_0\| < \varepsilon$ and $\|b_k - b_0\| < \varepsilon$.

(2) Deduce from the previous question that $f(A)$ is dense in A.

(3) Consider $u_n = \|a_n - b_n\|$ for $n \geqslant 0$. Show that $(u_n)_{n\geqslant 0}$ is eventually constant.

(4) Deduce that f is an isometry, so injective.

(5) Show that f is surjective.

Exercise 3.16 : Let (M, d) be a compact metric space and $f : M \to M$ be a function satisfying

$$\forall x, y \in M, x \neq y, \quad d(f(x), f(y)) < d(x, y).$$

(1) Show that f has a unique fixed point, that we denote by α in what follows. Hint: see below[4].

(2) Let $x_0 \in M$. Define iteratively the sequence $x_{n+1} = f(x_n)$ for $n \geqslant 0$. Show that $x_n \xrightarrow[n\to\infty]{} \alpha$.

(3) If (M, d) is only a complete metric space, are these results still valid?

Exercise 3.17 : Let $V = \mathcal{C}([0, 2\pi], \mathbb{C})$ be equipped with the 2-norm $\|\cdot\|_2$. For $n \in \mathbb{N}$, set $f_n(x) = e^{\mathrm{i}\, nx}$.

(1) Find the value of $\|f_n - f_m\|_2$ for $n, m \in \mathbb{N}$.

(2) Deduce that the bounded closed ball $\overline{B}(0, 1)$ is not compact.

Exercise 3.18 : Let V be a finite dimensional normed vector space and $K \subseteq V$ be a compact subset. Let $r > 0$ and $K_r := \cup_{x\in K} \overline{B}(x, r)$. Show that K_r is a compact subset of V. What happens if V is an infinite dimensional normed vector space?

Exercise 3.19 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Show that the following conditions are equivalent.

(i) For all $M > 0$, there exists $R > 0$ such that $\|x\| > R$ implies that $|f(x)| > M$.

(ii) For any bounded subset $B \subseteq \mathbb{R}$, the preimage $f^{-1}(B)$ is bounded in $\mathbb{R}^n$.

(iii) For any compact subset $K \subseteq \mathbb{R}$, the preimage $f^{-1}(K)$ is compact in $\mathbb{R}^n$.

[4] Look at the map $x \mapsto d(x, f(x))$.

Exercise 3.20 (Characterization of complete metric spaces) **:** Let (X, d) be a metric space. Show that the following statements are equivalent.

(i) The metric space (X, d) is complete.

(ii) Each sequence $(x_n)_{n\geqslant 1}$ in X having the property $\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty$ is convergent.

(iii) Each Cauchy sequence $(x_n)_{n\geqslant 1}$ in X has a convergent subsequence.

Exercise 3.21 : Show that a metric space (M, d) is compact if and only if it is precompact and complete.

Exercise 3.22 : Given a sequence of metric spaces $(M_1, d_1), \ldots, (M_n, d_n)$ and consider the product metric space (M, d) given by $M = M_1 \times \cdots \times M_n$ and the product distance defined in Definition 2.6.1. Show that the following properties are equivalent.

(i) (M, d) is complete.

(ii) (M_i, d_i) is complete for all $1 \leqslant i \leqslant n$.

Exercise 3.23 : Let (M, d) and (M', d') be two metric spaces, and $A \subseteq M$ be a dense subset.

(1) Consider a continuous function $f : (A, d) \to (M', d')$ and suppose that

$$\forall x \in M \backslash A, \quad \lim_{\substack{y \to x \\ y \in A}} f(y) \text{ there exists.}$$

Show that there exists a unique continuous function $g : M \to M'$ such that $g_{|A} \equiv f$. The function g is called the *continuation* of f on M.

(2) Suppose that (M', d') is complete and consider a uniformly continuous function $f : (A, d) \to (M', d')$. Show that there exists a unique uniformly continuous function $g : M \to M'$ such that $g_{|A} \equiv f$. The function g is called the *uniform continuation* of f on M.

Exercise 3.24 : Let (M, d) be a complete metric space and $p \geqslant 1$ be an integer. Consider a map $f : M \to M$ such that f^p is a contraction.

(1) Show that f has a unique fixed point, denoted by x.

(2) For any $x_0 \in M$, define $x_{n+1} = f(x_n)$ for $n \geqslant 0$, and show that $x_n \xrightarrow[n\to\infty]{} x$.

Exercise 3.25 : Let $\varphi : [0,1] \to [0,1]$ be a continuous function which is not identically 1 and $\alpha \in \mathbb{R}$. We denote by $\mathcal{C}^1([0,1],\mathbb{R})$ the space of continuous and differentiable functions from $[0,1]$ to $\mathbb{R}$ such that the derivative is also continuous. We want to show that there exists a unique solution $f \in \mathcal{C}^1([0,1],\mathbb{R})$ to the differential equation,

$$f(0) = \alpha, \quad f'(x) = f(\varphi(x)), \quad \forall x \in [0,1].$$

Let $M = \mathcal{C}([0.1],\mathbb{R})$ be equipped with $\|\cdot\|_\infty$, which is a Banach space as we will see later in Exercise 3.30. Define $T : M \to M$ as below,

$$\forall x \in [0,1], \quad Tf(x) = \alpha + \int_0^x f(\varphi(t))\,\mathrm{d}t.$$

Show that $T^2 = T \circ T$ is a contraction, and conclude using Exercise 3.23.

Exercise 3.26 : Let V be a normed vector space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Consider a linear form $f \in \mathcal{L}(V,\mathbb{K})$ which is not identically zero, then its kernel

$$\mathrm{Ker}(f) := \{x \in V : f(x) = 0\} \tag{E3.2}$$

is called an hyperplane (超平面) of V.

(1) Show that $\ker f$ is either closed or dense in V.

(2) Show that f is continuous if and only if $\ker f$ is closed in V.

Exercise 3.27 : Let $\ell^\infty(\mathbb{R})$ be the normed space of bounded sequences of real numbers, equipped with the infinite norm $\|\cdot\|_\infty$. Consider the subspace $V \subseteq \ell^\infty(\mathbb{R})$ consisting of the convergent sequences. Let us define the map

$$\begin{array}{rccc} \varphi : & V & \to & \mathbb{R} \\ & (a_n)_{n\geqslant 1} & \mapsto & \lim\limits_{n\to\infty} a_n. \end{array}$$

(1) Check that V is a subvector space, and that φ is a linear form, that is $\varphi \in \mathcal{L}(V,\mathbb{R})$.

(2) Show that φ is continuous, and that $|||\varphi||| \leqslant 1$.

(3) Find a sequence $a = (a_n)_{n\geqslant 1}$ such that $|\varphi(a)| = \|a\|_\infty$ and deduce that $|||\varphi||| = 1$.

Exercise 3.28 : Let $\mathcal{C}([0,1],\mathbb{R})$ be the space of real continuous functions on $[0,1]$. Consider the subspace

$$V = \{f \in \mathcal{C}([0,1],\mathbb{R}) : f(0) = 0\}.$$

Let $g \in \mathcal{C}([0,1],\mathbb{R})$ be the function $g : x \mapsto 1 - x$. Consider the endomorphism

$$\begin{array}{rccc} F : & V & \to & V \\ & f & \mapsto & fg. \end{array}$$

(1) Show that F is linear and continuous.

(2) Show that $|||F||| = 1$.

Exercise 3.29 : Consider the linear form

$$\varphi : \begin{array}{ccc} \mathcal{C}([0,1],\mathbb{R}) & \to & \mathbb{R} \\ f & \mapsto & f(1) \end{array},$$

where we equip $\mathcal{C}([0,1],\mathbb{R})$ with $\|\cdot\|_1$.

(1) For every integer $n \geqslant 1$, consider the function $f_n : t \mapsto t^n$. Compute $\varphi(f_n)$ and $\|f_n\|_1$.

(2) Show that φ is not continuous.

Exercise 3.30 : Show that the space of sequences $\ell^1(\mathbb{R})$ and $\ell^2(\mathbb{R})$, defined in Example 2.1.6, are Banach spaces. Is $\ell^\infty(\mathbb{R})$ a Banach space?

Exercise 3.31 : Let (M, N) be a complete normed vector space. Show that $\mathcal{C}([0,1], M)$, the space of continuous functions from $[0,1]$ to M, equipped with the norm

$$\forall f \in \mathcal{C}([0,1], M), \quad \|f\|_\infty = \sup_{x\in[0,1]} N(f(x)) < \infty$$

is a Banach space. In particular, the space $\mathcal{C}([0,1],\mathbb{R})$ equipped with $\|\cdot\|_\infty$ is Banach. Hint: see below[5].

Exercise 3.32 : Let E be an Euclidean space and $u \in \mathcal{L}(E)$. Suppose that u is symmetric, that is

$$\forall x, y \in E, \quad \langle u(x), y\rangle = \langle x, u(y)\rangle.$$

Let S be the centered unit sphere of E and $\varphi : S \to \mathbb{R}$ be a map defined by $\varphi(x) = \langle x, u(x)\rangle$.

(1) Justify φ attains its maximum on S. We will write $x_0 \in S$ where this maximum is attained.

(2) Let y be a unit vector that is orthogonal to x. We define the following two functions on $\mathbb{R}$. For $t \in \mathbb{R}$, let

$$x(t) = (\cos t)x_0 + (\sin t)y \quad \text{and} \quad f(t) = \langle u(x(t)), x(t)\rangle.$$

Show that f attains its maximum at 0, and deduce that y is orthogonal to $u(x_0)$.

(3) Show that x_0 is an eigenvalue of u.

Exercise 3.33 : Let $(V, \|\cdot\|)$ be a normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\hat{V}$ be the completion of V as in Proposition 3.3.6. Define the addition and the scalar product on $\hat{V}$ by

$$(x_n)_{n\geqslant 1} + (y_n)_{n\geqslant 1} := (x_n + y_n)_{n\geqslant 1}, \quad \text{and} \quad a \cdot (x_n)_{n\geqslant 1} := (ax_n)_{n\geqslant 1}$$

for all $(x_n)_{n\geqslant 1}, (y_n)_{n\geqslant 1} \in \hat{V}$ and $a \in \mathbb{K}$. Show that these two operations makes $\hat{V}$ into a vector space, and thus a Banach space.

[5]You may follow the steps suggested in Remark 3.2.19.

Exercise 3.34 : In this exercise, we give another construction of the completion of a metric space. Let (X, d) be a nonempty metric space, and fix a point $x_0 \in X$.

(1) Let $\mathcal{B}(X, \mathbb{R})$ be the set of all the bounded real-valued functions on X, equipped with the norm $\|\cdot\|_\infty$. Show that $(\mathcal{B}(X, \mathbb{R}), \|\cdot\|_\infty)$ is complete.

Hint: It is very similar to Exercise 3.30. If $(f_n)_{n \geqslant 1}$ is a Cauchy sequence in $\mathcal{B}(X, \mathbb{R})$, then $(f_n(x))_{n \geqslant 1}$ is a Cauchy sequence in $\mathbb{R}$ for all $x \in X$. The limit of $(f_n)_{n \geqslant 1}$ will be $g(x) := \lim_{n \to \infty} f_n(x)$.

(2) For every $x \in X$, define the function $f_x : X \to \mathbb{R}$ by

$$f_x(y) = d(y, x) - d(y, x_0).$$

Show that f_x is bounded and thus $f_x \in \mathcal{B}(X, \mathbb{R})$.

(3) Show that the map $F : X \to \mathcal{B}(X, \mathbb{R})$, $x \mapsto f_x$, is an isometry.

(4) Deduce that $(\overline{F(X)}, \|\cdot\|_\infty)$ is a completion of (X, d).

Exercise 3.35 : The extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is defined to be the union of $\mathbb{C}$ with an extra point ∞.

(1) Show that the function d defined on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ by

$$\begin{aligned} d(z_1, z_2) &= \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}} && \text{for all } z_1, z_2 \in \mathbb{C}, \\ d(z_1, \infty) &= \frac{2}{\sqrt{1 + |z_1|^2}} && \text{for all } z_1 \in \mathbb{C}, \end{aligned}$$

is a metric on $\widehat{\mathbb{C}}$.

(2) Show that $(\widehat{\mathbb{C}}, d)$ is a complete and compact metric space.