Chapter 4

Exercise 4.1 : Compute the differentials of the following functions.

(a) $f_1(x,y) = e^{xy}(x+y)$.

(b) $f_2(x,y) = xyz + xy + yz + zx$.

(c) $f_3(r,t) = (r\cos t, r\sin t)$.

Exercise 4.2 : Let V and W be two normed vector spaces. Show that if $f : V \to W$ is differentiable at $x \in V$, then f is locally Lipschitz continuous at x, that is there exists $K > 0$ and $r > 0$ such that for any $y \in B(x,r)$, we have $\|f(y) - f(x)\|_W \leqslant K \|y - x\|_V$.

Exercise 4.3 : Let $n \geqslant 1$ be an integer.

(1) Show that $\mathrm{GL}_n(\mathbb{R}) := \{M \in \mathcal{M}_n(\mathbb{R}) : \det M \neq 0\}$ is open in $\mathcal{M}_n(\mathbb{R})$.

Consider $\varphi : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), M \mapsto M^{-1}$. Recall from the linear algebra class that for $M \in \mathrm{GL}_n(\mathbb{R})$, its inverse can be obtained by $M^{-1} = (\det M)^{-1}\widetilde{M}$, where $\widetilde{M}$ is the adjugate matrix (伴隨矩陣) of M. Note that the coefficients of the adjugate are given by linear combinations of products of the coefficients from the original matrix. This allows us to see that φ is of class $\mathcal{C}^1$ (and actually, of class $\mathcal{C}^\infty$).

(2) Show that φ is differentiable at I_n and compute its differential $\mathrm{d}\varphi_{I_n}$.

(3) Given $M \in \mathrm{GL}_n(\mathbb{R})$. Show that φ is differentiable at M and compute its differential $\mathrm{d}\varphi_M$.

Exercise 4.4 : Let $\varphi : \mathcal{L}(\mathbb{R}^n) \to \mathcal{L}(\mathbb{R}^n)$ defined by $\varphi(u) = u \circ u$. Show that φ is of class $\mathcal{C}^1$.

Exercise 4.5 : We equip $M = \mathbb{R}_n[X]$ with the norm $\|P\|_\infty = \sup_{t\in[0,1]} |P(t)|$. Consider

$$\begin{array}{rccc} \varphi : & M & \to & \mathbb{R} \\ & P & \mapsto & \displaystyle\int_0^1 (P(t))^3 \,\mathrm{d}t \end{array}.$$

Show that φ is differentiable on M and compute its differential. Is the map $P \mapsto \mathrm{d}\varphi_P$ continuous?

Exercise 4.6 (Non-differentiability) :

(1) Consider the normed space $V = \mathcal{C}([0,1],\mathbb{R})$ with supremum norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Let $f \in V$ be such that there are two or more points t in $[0,1]$ with $|f(t)| = \|f\|_\infty$. Show that the supremum norm function $\|\cdot\|_\infty : V \to \mathbb{R}$ is not differentiable at such an f.

(2) Let $V \subseteq \ell^\infty(\mathbb{R})$ be the subspace of all bounded sequence with limit 0. That is,

$$V = \{(a_n)_{n\geqslant 1} \in \ell^\infty(\mathbb{R}) \mid \lim_{n\to\infty} a_n = 0\}.$$

Show that the norm function $\|\cdot\|_\infty : V \to \mathbb{R}$ is differentiable at $a = (a_n)_{\geqslant 1}$ if and only if there is a unique $n \in \mathbb{N}$ such that $|a_n| = \|a\|_\infty$.

(3) Let us come back to the linear form considered in Exercise 3.29, that is

$$\begin{array}{rccc} \varphi : & \mathcal{C}([0,1],\mathbb{R}) & \to & \mathbb{R} \\ & f & \mapsto & f(1) \end{array}.$$

Show that if we equip $\mathcal{C}([0,1],\mathbb{R})$ with the sup norm $\|\cdot\|_\infty$, then φ is differentiable at any $f \in \mathcal{C}([0,1],\mathbb{R})$. Compute the differential map $D\varphi$. This shows that how a norm can changes the continuity and the differentiability of a map.

Exercise 4.7 : Let f be a map from a normed space V to a normed space W.

(1) Fix $x \in V$, and explain the difference between the following two statements:
  (a) $\mathrm{d}f$ is continuous at x.
  (b) $\mathrm{d}f_x$ is continuous.

(2) Prove that for a fixed $x \in V$, if $\mathrm{d}f_x$ exists, then f is continuous and differentiable at x.

(3) Give an example of a mapping f such that $\mathrm{d}f$ is continuous but not differentiable.

Exercise 4.8 : Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets. Consider a differentiable function $f : U \to V$ and suppose that

(i) f is differentiable at a certain $a \in U$;

(ii) f has an inverse function $g : V \to U$;

(iii) g is differentiable at $b = f(a) \in V$.

Show that $m = n$.

Exercise 4.9 : Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. Suppose that for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we have $f(\lambda x) = \lambda f(x)$.

(1) Show that $f(0) = 0$.

(2) Show that f is linear.

Exercise 4.10 : Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Consider an open set $A \subseteq V$ and $f : A \to W$. Recall the definition of the differential in Definition 4.1.1. If there exists a map φ satisfying the weaker condition

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \|f(x + \lambda h) - f(x) - \lambda\varphi(h)\|_W = 0$$

for every $h \in V$, then f is said to be *Gâteaux differentiable* at x, and φ is the Gâteaux derivative of f at x. Prove that if f is differentiable at x, then it is Gâteaux differentiable at x, and the two derivatives are equal.

Exercise 4.11 : Let us consider the two functions below,

$$f(x,y) = \begin{cases} y^2 \ln|x| & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

(1) Show that f and g are not continuous at $(0,0)$.

(2) Show that f and g have directional derivatives at $(0,0)$ in any direction $u = (a,b) \in \mathbb{R}^2 \backslash \{(0,0)\}$.

(3) Are f and g differentiable?

Exercise 4.12 : Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(1) Is the function f continuous on $\mathbb{R}^2$?

(2) Is it of class $\mathcal{C}^1$?

(3) Is it of class $\mathcal{C}^2$?

Exercise 4.13 : Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces, and $A \subseteq V$ be a nonempty open subset. Suppose that the function $f : A \to W$ is continuous and differentiable on A, with $\mathrm{d}f_a \equiv 0$ for all $a \in A$. You are not allowed to apply the result from Theorem 2.7.27, but may use the ideas from its proof.

(1) We assume that A is arcwise connected. Show that f is a constant function.

(2) We assume that A is only connected. Fix $x_0 \in A$ and consider $\Gamma = \{x \in A : f(x) = f(x_0)\}$. Show that Γ is open and closed in A, and deduce that f is a constant function.

Exercise 4.14 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$\forall x, y \in \mathbb{R}^2, \quad |f(x) - f(y)| \leqslant \|x - y\|^{1+\varepsilon}$$

for some fixed $\varepsilon > 0$. Show that f is a constant function.

Exercise 4.15 : Let V be a Banach space. Let $K = \overline{B}(0, r)$, where $r > 0$, be a closed ball contained in an open set A contained in a Banach space V. Let $f : A \to V$. Assume that f is differentiable at each point of K and that $f(K) \subset K$. Assume also that $\sup\{|||df_x||| : x \in K\} < 1$. Show that f has a unique fixed point in K. Hint: mean-value theorem and fixed-point theorem.

Exercise 4.16 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of class $\mathcal{C}^1$. Compute the derivatives (univariate function) or partial derivatives (multivariate function) of the following functions.

(a) $g(x, y) = f(y, x)$.

(b) $g(x) = f(x, x)$.

(c) $g(x, y) = f(y, f(x, x))$.

(d) $g(x) = f(x, f(x, x))$.

Exercise 4.17 : Let $f : \mathbb{R}^2 \backslash \{(0, 0)\} \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{x^2}{(x^2 + y^2)^{3/4}}, \quad \forall (x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}.$$

(1) Justify that f is of class $\mathcal{C}^\infty$.

(2) Which value can we define at $(0, 0)$ to extend f continuously on $\mathbb{R}^2$? Denote the extended function by $\widetilde{f}$.

(3) Show that the partial derivative $\frac{\partial \widetilde{f}}{\partial x}(0, 0)$ does not exist. Deduce that $\widetilde{f}$ is not differentiable at $(0, 0)$.

Exercise 4.18 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^1$ function. We say that f is homogeneous of degree $r \in \mathbb{R}$ if

$$\forall x, y \in \mathbb{R}^2, \quad \forall t > 0, \quad f(tx, ty) = t^r f(x, y).$$

(1) Show that if f is homogeneous of degree r, then its partial derivatives are homogeneous of degree $r - 1$.

(2) Show that f is homogeneous of degree r if and only if

$$\forall x, y \in \mathbb{R}^2, \quad x\frac{\partial f}{\partial x}(x, y) + y\frac{\partial f}{\partial y}(x, y) = rf(x, y).$$

(3) Suppose that f is of class $\mathcal{C}^2$. Show that

$$x^2\frac{\partial^2 f}{\partial x^2}(x, y) + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = r(r - 1)f(x, y).$$

Exercise 4.19 : Let A be an open subset of $\mathbb{R}^n$, $f : A \to \mathbb{R}$ be a function of class $\mathcal{C}^p$ with $p \geqslant 1$, and $a \in A$. Show the following Taylor formulas by mimicking the proof of Theorem 4.3.2.

(1) Let $h \in \mathbb{R}^n$ such that $[x, x+h] \subseteq A$. Then,

$$f(x+h) = f(x) + \sum_{m=1}^{p-1} \frac{f_h^{(m)}(x)}{m!} + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} f_h^{(p)}(x+th)\, \mathrm{d}t.$$

(2) Show that when $h \to 0$, we have

$$f(x+h) = f(x) + \sum_{m=1}^{p} \frac{f_h^{(m)}(x)}{m!} + o(|h|^p).$$

Exercise 4.20 : Let $A \subseteq \mathbb{R}^m$ be an open subset. Suppose that the function $f : A \to \mathbb{R}^n$ is differentiable and the differential map $x \mapsto \mathrm{d}f_x$ is continuous at $a \in A$. Show that for every $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\|x-a\| < \eta, \|y-a\| < \eta \quad \Rightarrow \quad \|f(y) - f(x) - \mathrm{d}f_a(y-x)\| \leqslant \varepsilon \|y-x\|.$$

Exercise 4.21 (Fundamental theorem of algebra) **:** Let $P \in \mathbb{K}[X]$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Suppose that P is not a constant polynomial, that is its degree is greater or equal to 1.

(1) Show that $\lim_{|z|\to\infty} |P(z)| = \infty$.

(2) Deduce that $|P(z)|$ attains a minimum in $\mathbb{C}$. Let us denote by z_0 the point where the minimum of $|P(z)|$ is attained.

(3) Show that $P(z_0) = 0$ by contradiction. More precisely, suppose that $P(z_0) \neq 0$, and show that there exists z, sufficiently close to z_0, such that the absolute value $|P(z)|$ is strictly less than $|P(z_0)|$. Hint: Taylor expansion around z_0.

Exercise 4.22 : Find the critical points of the following functions, and explain whether they are local minima, local maxima, saddle points.

(a) $f(x,y) = y^2 - x^2 + \frac{x^4}{2}$.

(b) $f(x,y) = x^3 + y^3 - 3xy$.

(c) $f(x,y) = x^4 + y^4 - 4(x-y)^2$.

Exercise 4.23 : Prove that the function $f(x,y) = xy + \sqrt{9 - x^2 - y^2}$ attains maximum and minimum on the set

$$S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 9\}.$$

At which points (x,y) does f achieve its maximum and minimum?

Exercise 4.24 (Rolle's theorem) **:** Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. We assume that f is constant on the unit sphere $S(0,1)$. Show that there exists $x_0 \in B(0,1)$ such that $\mathrm{d}f_{x_0} = 0$.

Exercise 4.25 : Let V and W be two normed vector spaces, and $f : V \to W$ be a map of class $\mathcal{C}^1$. Let $y_0 \in W$ be such that $\mathrm{d}f$ is invertible at each point of $f^{-1}(y_0)$. Prove that $f^{-1}(y_0)$ is a discrete set. That is, for any $x \in f^{-1}(y_0)$, the singleton $\{x\}$ is an open subset of $f^{-1}(y_0)$.

Exercise 4.26 : Assume that the polynomial

$$P(x) = x^3 + a_2x^2 + a_1x + a_0$$

has three different real roots for $(a_2, a_1, a_0) = (p_0, q_0, r_0)$. Show that there exists $\varepsilon > 0$ such that $P(x)$ has three different real roots $\lambda_1 < \lambda_2 < \lambda_3$ whenever $(a_2, a_1, a_0) \in B((p_0, q_0, r_0), \varepsilon)$, where λ_j, $1 \leqslant j \leqslant 3$, are $\mathcal{C}^1$ functions of a_2, a_1, a_0.

Exercise 4.27 : Let $\varphi : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R}), M \mapsto M^2$. Show that there exists $\varepsilon > 0$ such that for $A \in \mathcal{M}_n(\mathbb{R})$ with $|||A - I||| < \varepsilon$, then we may define a square root $\sqrt{A}$ of A. Show that $A \mapsto \sqrt{A}$ is of class $\mathcal{C}^\infty$ on $B(I, \varepsilon)$.

Exercise 4.28 : It is the second part of Exercise 4.4. Recall that $\varphi : \mathcal{L}(\mathbb{R}^n) \to \mathcal{L}(\mathbb{R}^n), u \mapsto u \circ u$.

(1) Show that φ is of class $\mathcal{C}^1$.

(2) Show that $\mathrm{d}\varphi$ at $\mathrm{Id}_{\mathbb{R}^n}$ is a map in $\mathcal{L}_c(\mathcal{L}_c(\mathbb{R}^n))$ and writes, $u \mapsto 2u$. Deduce that $\mathrm{d}\varphi_{\mathrm{Id}_{\mathbb{R}^n}}$ is invertible.

(3) Show that there exists an open set $U \subseteq \mathcal{L}(\mathbb{R}^n)$ containing $\mathrm{Id}_{\mathbb{R}^n}$ such that for any $u \in U$, there exists $v \in \mathcal{L}(\mathbb{R}^n)$ such that $u = v \circ v$. In other words, all linear operators near $\mathrm{Id}_{\mathbb{R}^n}$ have a "square root". Hint: Inversion theorem.

Exercise 4.29 : Let $(V, \|\cdot\|)$ be a Banach space. Show that there exists $\varepsilon > 0$ such that whenever $f \in \mathcal{L}_c(V)$ satisfying $|||f - \mathrm{Id}||| < \varepsilon$, we may find $g \in \mathcal{L}_c(V)$ such that $f = \exp(g)$, that we may also write as $g = \ln(f)$. Hint: use Remark 3.2.21 and the local inversion theorem.

Exercise 4.30 : Let $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2 \backslash \{(0,0)\}$ be defined by $f(x,y) = (x^2 - y^2, 2xy)$. Show that f is a local $\mathcal{C}^1$-diffeomorphism, but not a global $\mathcal{C}^1$-diffeomorphism.

Exercise 4.31 : Let V be a Banach space and $\varphi : V \to V$ be a function of class $\mathcal{C}^1$. Assume that $\mathrm{d}\varphi_u \in \mathcal{L}_c(V, V)$ is a bicontinuous isomorphism for all $u \in V$, and there exists $c \in (0,1)$ such that

$$\|\varphi(u) - \varphi(v) - (u - v)\|_V \leqslant c\,\|u - v\|_V \quad \forall u, v \in V.$$

Follow the following steps to show that φ is a $\mathcal{C}^1$-diffeomorphism.

(1) Show that φ is injective.

(2) Fix $w \in V$. Set $u_0 = w$ and $u_{n+1} = u_n + (w - \varphi(u_n))$ for all $n \geqslant 0$. Show that $(u_n)_{n \geqslant 0}$ is a Cauchy sequence in V.

(3) Deduce that φ is surjective and conclude that φ is a $\mathcal{C}^1$-diffeomorphism.

Exercise 4.32 : Consider the function

$$f: \begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^3, \\ (x,y,z) & \mapsto & (e^{2y}+e^{2z}, e^{2x}-e^{2z}, x-y). \end{array}$$

Show that the image of f is a proper open subset of $\mathbb{R}^3$.

Exercise 4.33 : Consider the solutions to the equation $x+y+z+\sin(xyz)=0$.

(1) Show that around the point $(0,0,0)$, we may write z as a function of x and y. That is, there exist an open set $X \subseteq \mathbb{R}^2$ containing 0, an open set $Z \subseteq \mathbb{R}$ containing 0, and a function $\varphi : X \to Z$ such that $f(x,y,z)=0$ has a unique solution $z=\varphi(x,y) \in Z$.

(2) Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(3) Deduce that $\varphi(x,y) = -(x+y) + o(\|(x,y)\|)$ when $(x,y) \to 0$.

(4) Show that we have a higher-order expansion $\varphi(x,y) = -(x+y) + xy(x+y) + o(\|(x,y)\|^3)$ when $(x,y) \to 0$.

Exercise 4.34 : Let $f(x)$ be a non-negative continuous function satisfying $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$.

(1) Prove that among all closed intervals $[a,b]$ such that $\int_a^b f(x)\,\mathrm{d}x = \frac{1}{2}$, there is one with the shortest length.

(2) If $[a,b]$ is one of the shortest closed intervals in (a), show that $f(a)=f(b)$.

Exercise 4.35 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y,z) = x^2 - xy^3 - y^2 z + z^3$, and the surface $\mathcal{S}$ be defined as the set of the solutions to $f(x,y,z)=0$.

(1) Show that around the point $(1,1,1)$, the surface $\mathcal{S}$ can be defined by an equation $z=\varphi(x,y)$ where φ is of class $\mathcal{C}^\infty$ around $(1,1)$.

(2) Find the equation of the tangent plane $\mathcal{P}$ at $(1,1,1)$ to the surface $\mathcal{S}$.

(3) Find the partial derivatives of φ up to order 2 around $(1,1)$ and at $(1,1)$.

(4) What is the position of the surface $\mathcal{S}$ with respect to the tangent plane $\mathcal{P}$?

Exercise 4.36 (AM–GM inequality) **:** Let $n \geqslant 2$ and $f : \mathbb{R}^n \to \mathbb{R}$, $(x_1,\ldots,x_n) \mapsto x_1 \ldots x_n$. Consider $\Gamma = \{(x_1,\ldots,x_n) \in \mathbb{R}^n_{\geqslant 0} : x_1 + \cdots + x_n = 1\}$.

(1) Show that f has a global maximum on Γ and find its value.

(2) Deduce the AM–GM inequality, that is, prove the following,

$$\Big(\prod_{i=1}^n x_i\Big)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^n x_i, \quad \forall (x_1,\ldots,x_n) \in \mathbb{R}^n_{\geqslant 0}.$$