# Chapter 5: Theory of Riemann–Stieltjes Integrals

**Exercise 5.1 :** Let $f : [a, b] \to \mathbb{R}$ be a function defined on $[a, b]$. Show that $V_f([a, b]) \geqslant |f(b) - f(a)|$.

**Exercise 5.2 :** Define the function $f : [0, 1] \to \mathbb{R}$ as follows,

$$\forall x \in [0,1], \quad f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Is $f$ a function of bounded variation?

**Exercise 5.3 :** Let $f \in \mathbb{R}[X]$ be a polynomial function and $I = [a, b]$ be a compact segment. Show that the total variation $V_f([a, b])$ is well defined, that is $f$ is of bounded variation on $[a, b]$.

**Exercise 5.4 :** Let $(f_n)_{n \geqslant 1}$ be a sequence of functions of bounded variation on $[a, b]$. Let $f$ be a function on $[a, b]$. We are given the two following conditions,

(a) there exists $M > 0$ such that $V_{f_n}([a, b]) \leqslant M$ for all $n \geqslant 1$;

(b) for every $x \in [a, b]$, we have the convergence $f_n(x) \xrightarrow[n \to \infty]{} f(x)$.

Please answer the following questions.

(1) If both (a) and (b) hold, show that $f$ is of bounded variation and that $V_f([a, b]) \leqslant M$.

(2) If only the condition (b) is satisfied, can you find an example for which the limiting function $f$ is not of bounded variation?

**Exercise 5.5 :** Suppose that $f : [a, b] \to \mathbb{R}$ is a function and there exists $M > 0$ such that $f$ is of bounded variation on every interval $[a + \varepsilon, b]$ for any $\varepsilon > 0$, with $V_f([a + \varepsilon, b]) \leqslant M$.

(1) Show that $f$ if of bounded variation on $[a, b]$.

(2) Do we have $V_f([a, b]) \leqslant M$?

**Exercise 5.6 :** Justify whether each of the following statements is true or false. If it is true, please prove it briefly; otherwise, find a counterexample. ($a < b$ are real numbers.)

(a) A continuous function on $[a, b]$ is of bounded variation.

(b) A function that is continuous on $[a, b]$ and differentiable on $(a, b)$ is of bounded variation.

(c) A function that is continuous and differentiable on $[a, b]$ is of bounded variation.

(d) A function of class $\mathcal{C}^1$ on $[a, b]$ is of bounded variation.

**Exercise 5.7 :** Let $\alpha \in \mathbb{R}$. Define the function $f : [0,1] \to \mathbb{R}$ as follows,

$$\forall x \in [0,1], \quad f(x) = \begin{cases} x^\alpha \sin\left(\frac{1}{x}\right), & \text{if } x \in (0,1], \\ 0, & \text{otherwise.} \end{cases}$$

(1) For which values of $\alpha$ is the function $f$ continuous on $[0,1]$?

(2) For which values of $\alpha$ is the function $f$ uniformly continuous on $[0,1]$?

(3) For which values of $\alpha$ is the derivative $f'(0)$ well defined?

(4) For which values of $\alpha$ is the derivative $f'$ continuous on $[0,1]$?

(5) Show that for $\alpha > 2$, the function $f$ is of bounded variation.

(6) Show that for $\alpha \leqslant 1$, the function $f$ is not of bounded variation.

(7) (Hard) Show that for $1 < \alpha \leqslant 2$, the function $f$ is of bounded variation.

**Exercise 5.8 :** Let $f : [a,b] \to \mathbb{R}$ be a function defined on $[a,b]$. For $\alpha > 0$, we say that $f$ is $\alpha$-Hölder continuous, or satisfies the uniform Lipschitz condition of order $\alpha$, if there exists $M > 0$ such that

$$|f(x) - f(y)| \leqslant M|x-y|^\alpha, \quad \forall x, y \in [a,b]. \tag{5.1}$$

Let $\alpha > 0$ such that $f$ is $\alpha$-Hölder continuous.

(1) If $\alpha > 1$, show that $f$ is constant on $[a,b]$.

(2) If $\alpha = 1$, show that $f$ is of bounded variation.

(3) If $\alpha < 1$, is $f$ of bounded variation? Hint: see below[1].

(4) Find a function $g : [a,b] \to \mathbb{R}$ which is of bounded variation, but is not $\alpha$-Hölder continuous for any $\alpha > 0$.

[1]The answer is no, and you need to find a counterexample.

**Exercise 5.9 :** A function $f : [a, b] \to \mathbb{R}$ is said to be *absolutely continuous* (絕對連續) on $[a, b]$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $n \in \mathbb{N}$ and any disjoint open intervals $(a_k, b_k) \subseteq [a, b]$, $1 \leqslant k \leqslant n$,

$$\sum_{k=1}^{n} (b_k - a_k) < \delta \quad \Rightarrow \quad \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon.$$

(1) Show that the function $f : [0, 1] \to \mathbb{R}$, $x \mapsto \sqrt{x}$ is absolutely continuous.

We are given two functions $f, g : [a, b] \to \mathbb{R}$ that are both absolutely continuous.

(2) Show that $f$ is uniformly continuous, continuous, and of bounded variation on $[a, b]$.

(3) Show that if $f$ is 1-Hölder continuous on $[a, b]$, see Eq. (5.1) for the definition, then $f$ is absolutely continuous on $[a, b]$.

(4) Show that both $|f|$ and $cf$ are absolutely continuous, for any constant $c \in \mathbb{R}$.

(5) Show that $f + g$ and $f \cdot g$ are both absolutely continuous.

(6) Show that if $g$ is bounded away from zero (i.e., $|g| \geqslant c$ for some $c > 0$), then $\frac{f}{g}$ is absolutely continuous.