Chapter 6

Exercise 6.1 : Let $(V, \|\cdot\|)$ be a Banach space, and $(a_n)_{n\geqslant 1}$ be a sequence in V. We define an auxiliary sequence $(m_n)_{n\geqslant 1}$ as below,

$$\forall n \in \mathbb{N}, \quad m_n = \frac{a_1 + \cdots + a_n}{n}.$$

We consider the following two properties.

(i) The sequence $(a_n)_{n\geqslant 1}$ converges.

(ii) The sequence $(m_n)_{n\geqslant 1}$ converges.

Answer the following questions.

(1) Show that if (i) holds with limit ℓ, then (ii) also holds with the same limit.

(2) Find an example for which (ii) holds, but (i) does not hold.

(3) We are given a sequence $(w_n)_{n\geqslant 1}$ of non-negative real numbers such that $\sum w_n$ diverges. Define

$$\forall n \in \mathbb{N}, \quad m'_n = \frac{w_1 a_1 + \cdots + w_n a_n}{w_1 + \cdots + w_n}.$$

If we replace the property (ii) by the following property (ii'): the sequence $(m'_n)_{n\geqslant 1}$ converges, does (i) still implies (ii')?

Exercise 6.2 : Let $(V, \|\cdot\|)$ be a non-empty finite-dimensional normed vector space. Let $(a_n)_{n\geqslant 1}$ be a sequence in V and $(r_n)_{n\geqslant 1}$ be a sequence in $\mathbb{R}_+^*$ satisfying

$$\forall n \in \mathbb{N}, \quad \overline{B}(a_{n+1}, r_{n+1}) \subseteq \overline{B}(a_n, r_n).$$

(1) Show that

$$\forall n \in \mathbb{N}, \quad \|a_{n+1} - a_n\| \leqslant r_n - r_{n+1}.$$

(2) Show that the sequence $(a_n)_{n\geqslant 1}$ is convergent.

(3) Show that $\bigcap_{n\geqslant 1} \overline{B}(a_n, r_n)$ is a closed ball. Find its center and radius.

第六章

習題 6.1 ： 令 $(V, \|\cdot\|)$ 為 Banach 空間，且 $(a_n)_{n\geqslant 1}$ 為在 V 中的序列。我們定義輔助序列 $(m_n)_{n\geqslant 1}$ 如下：

$$\forall n \in \mathbb{N}, \quad m_n = \frac{a_1 + \cdots + a_n}{n}.$$

我們考慮下面兩個性質。

(i) 序列 $(a_n)_{n\geqslant 1}$ 收斂。

(ii) 序列 $(m_n)_{n\geqslant 1}$ 收斂。

回答下面的問題。

(1) 證明如果 (i) 成立且極限為 ℓ，那麼 (ii) 也會成立且有相同的極限。

(2) 找出一個例子使得 (ii) 成立，但是 (i) 不會成立。

(3) 給定非負實數序列 $(w_n)_{n\geqslant 1}$ 使得 $\sum w_n$ 會發散。定義

$$\forall n \in \mathbb{N}, \quad m'_n = \frac{w_1 a_1 + \cdots + w_n a_n}{w_1 + \cdots + w_n}.$$

如果我們性質 (ii) 改成下面這個性質 (ii') 序列 $(m'_n)_{n\geqslant 1}$ 會收斂，那麼 (i) 還是會蘊含 (ii') 嗎？

習題 6.2 ： 令 $(V, \|\cdot\|)$ 為非空有限維度賦範向量空間。令 $(a_n)_{n\geqslant 1}$ 為在 V 中的序列，且 $(r_n)_{n\geqslant 1}$ 為在 $\mathbb{R}_+^*$ 中的序列，滿足

$$\forall n \in \mathbb{N}, \quad \overline{B}(a_{n+1}, r_{n+1}) \subseteq \overline{B}(a_n, r_n).$$

(1) 證明

$$\forall n \in \mathbb{N}, \quad \|a_{n+1} - a_n\| \leqslant r_n - r_{n+1}.$$

(2) 證明序列 $(a_n)_{n\geqslant 1}$ 會收斂。

(3) 證明 $\bigcap_{n\geqslant 1} \overline{B}(a_n, r_n)$ 是顆閉球。找出他的中心和半徑。

Exercise 6.3 : Let $a \in \mathbb{C}$ and

$$A = \frac{1}{2}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

(1) Show that for $n \in \mathbb{N}$, we have

$$A^n = \frac{1}{2}I + \frac{a}{2}J, \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(2) Find the behavior of the series with general term A^n.

Exercise 6.4 : Let $(a_n)_{n\geqslant 1}$ be a decreasing sequence of non-negative real numbers. Show that if $\sum u_n$ converges, then we have $u_n = o(\frac{1}{n})$.

Exercise 6.5 : Let $\sum_n u_n$ and $\sum_n v_n$ be two convergent series with non-negative terms. Determine the behavior of the series $\sum_n \sqrt{u_n v_n}$ and $\sum_n \max(u_n, v_n)$.

Exercise 6.6 : Let $\sum_n u_n$ be a series with non-negative terms.

(1) Suppose that $\sum_n u_n$ converges. Show that for $\alpha > 1$, the series $\sum_n u_n^\alpha$ converges.

(2) Suppose that $\sum_n u_n$ diverges. Show that for $\alpha \in (0,1)$, the series $\sum_n u_n^\alpha$ diverges.

Exercise 6.7 : Let $(u_n)_{n\geqslant 1}$ be a series with non-negative terms. For $n \geqslant 1$, let $v_n = \frac{u_n}{1+u_n}$.

(1) Show that the function $x \mapsto \frac{x}{1+x}$ is increasing on $[0, +\infty)$.

(2) Show that the series $\sum_n u_n$ and $\sum_n v_n$ have the same behavior.

Exercise 6.8 : Let $(u_n)_{n\geqslant 1}$ be a non-negative decreasing sequence. Show that the series $\sum_n u_n$ and $\sum_n 2^n u_{2^n}$ have the same behavior.

Exercise 6.9 : Use the comparison theorems (Proposition 6.2.2 and Theorem 6.2.3) to determine the behavior of the series $\sum u_n$ where the general term u_n is given by different expressions. You may also need the Stirling's formula from Exercise 6.12. Let us fix constants $a \geqslant 0, b, c \in \mathbb{R}$.

(1) $u_n = 3^{-\sqrt{n}}$,

(2) $u_n = a^n n!$,

(3) $u_n = ne^{-\sqrt{n}}$,

(4) $u_n = n^{-1-\frac{1}{n}}$,

(5) $u_n = \frac{(n!)^3}{(3n)!}$,

(6) $u_n = \frac{\ln n}{\ln(e^n - 2)}$.

(7) $u_n = \frac{a^n}{n!}$,

(8) $u_n = \frac{a^n}{\binom{2n}{n}}$.

(9) $u_n = 1 - \cos\frac{\pi}{n}$.

(10) $u_n = \left(\frac{n}{n+1}\right)^{n^2}$.

(11) $u_n = e^{1/n} - a - \frac{b}{n}$.

(12) $u_n = \cos(\frac{1}{n}) - a - \frac{b}{n}$.

習題 6.3 ： 令 $a \in \mathbb{C}$ 以及

$$A = \frac{1}{2}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

(1) 證明對於 $n \in \mathbb{N}$，我們有

$$A^n = \frac{1}{2}I + \frac{a}{2}J, \quad \text{其中} \quad J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(2) 找出一般項為 A^n 的級數的行為。

習題 6.4 ： 令 $(a_n)_{n\geqslant 1}$ 為遞減非負實數序列。證明如果 $\sum u_n$ 收斂，那麼我們會有 $u_n = o(\frac{1}{n})$。

習題 6.5 ： 令 $\sum_n u_n$ 和 $\sum_n v_n$ 為兩個會收斂、且一般項非負的級數。求級數 $\sum_n \sqrt{u_n v_n}$ 和 $\sum_n \max(u_n, v_n)$ 的行為。

習題 6.6 ： 令 $\sum_n u_n$ 為一般項非負的級數。

(1) 假設 $\sum_n u_n$ 收斂。證明對於 $\alpha > 1$，級數 $\sum_n u_n^\alpha$ 收斂。

(2) 假設 $\sum_n u_n$ 發散。證明對於 $\alpha \in (0,1)$，級數 $\sum_n u_n^\alpha$ 發散。

習題 6.7 ： 令 $(u_n)_{n\geqslant 1}$ 為一般項非負的級數。對於 $n \geqslant 1$，令 $v_n = \frac{u_n}{1+u_n}$。

(1) 證明函數 $x \mapsto \frac{x}{1+x}$ 在 $[0, +\infty)$ 上是遞增的。

(2) 證明級數 $\sum_n u_n$ 和 $\sum_n v_n$ 有相同的行為。

習題 6.8 ： 令 $(u_n)_{n\geqslant 1}$ 為一般項非負的級數。證明級數 $\sum_n u_n$ 和 $\sum_n 2^n u_{2^n}$ 有相同的行為。

習題 6.9 ： 使用比較定理（命題 6.2.2 以及定理 6.2.3）找出下列級數 $\sum u_n$ 的行為，其中一般項 u_n 是被不同的式子所給出的。你也會需要用到習題 6.12 中的 Stirling 公式。我們固定常數 $a \geqslant 0$ 以及 $b, c \in \mathbb{R}$。

(1) $u_n = 3^{-\sqrt{n}}$,

(2) $u_n = a^n n!$,

(3) $u_n = ne^{-\sqrt{n}}$,

(4) $u_n = n^{-1-\frac{1}{n}}$,

(5) $u_n = \frac{(n!)^3}{(3n)!}$,

(6) $u_n = \frac{\ln n}{\ln(e^n - 2)}$.

(7) $u_n = \frac{a^n}{n!}$,

(8) $u_n = \frac{a^n}{\binom{2n}{n}}$.

(9) $u_n = 1 - \cos\frac{\pi}{n}$.

(10) $u_n = \left(\frac{n}{n+1}\right)^{n^2}$.

(11) $u_n = e^{1/n} - a - \frac{b}{n}$.

(12) $u_n = \cos(\frac{1}{n}) - a - \frac{b}{n}$.

Exercise 6.10 : Let $(u_n)_{n\geqslant 0}$ be a real-valued sequence defined by $u_0 > 0$ and

$$u_{n+1} = u_n + u_n^2, \quad \forall n \in \mathbb{N}.$$

(1) Show that $u_n \xrightarrow[n\to\infty]{} +\infty$.

(2) Show that
$$\frac{\ln u_{n+1}}{2^{n+1}} - \frac{\ln u_n}{2^n} = o\Big(\frac{1}{2^n}\Big).$$

(3) Deduce that there exists $K > 1$ such that $u_n \sim K^{2^n}$.

Exercise 6.11 : Let $(u_n)_{n\geqslant 0}$ be a real-valued sequence defined by $u_0 = c \in \mathbb{R}$ and

$$u_{n+1} = u_n + e^{-u_n}, \quad \forall n \in \mathbb{N}.$$

(1) Show that $u_n \xrightarrow[n\to\infty]{} +\infty$.

(2) Find an asymptotic expression of u_n up to the order $o\big(\frac{\ln n}{n}\big)$.

Exercise 6.12 : Define the sequence

$$\forall n \geqslant 1, \quad S_n = \sum_{k=1}^{n} \ln k.$$

(1) Show that for every $k \geqslant 2$, we have
$$\int_{k-1}^{k} \ln t \,\mathrm{d}t \leqslant \ln k \leqslant \int_{k}^{k+1} \ln t \,\mathrm{d}t.$$
Deduce that $S_n = n\ln n - n + o(n)$.

(2) By considering the sequence $(A_n)_{n\geqslant 1}$, defined by
$$\forall n \geqslant 1, \quad A_n = S_n - n\ln n + n,$$
show that $A_n - A_{n-1} \sim \frac{1}{2n}$ and deduce that $A_n \sim \frac{1}{2}\ln n$.

(3) Let $D_n := S_n - n\ln n + n - \frac{1}{2}\ln n$ for $n \geqslant 1$. Show that $D_n - D_{n-1} \sim -\frac{1}{12n^2}$.

(4) Show that D_n converges to some D_∞ when $n \to \infty$. Deduce that there exists some constant $C > 0$ such that
$$n! \sim C\Big(\frac{n}{e}\Big)^n \sqrt{n}.$$

(5) Using the expression of I_{2n} from Exercise A1.2 to show that $C = \sqrt{2\pi}$.

(6) Show that
$$n! \sim \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n \Big(1 + \frac{1}{12n} + o\Big(\frac{1}{n}\Big)\Big).$$

習題 6.10 ： 令 $(u_n)_{n\geqslant 0}$ 為實數序列，被 $u_0 > 0$ 以及下面式子所定義：

$$u_{n+1} = u_n + u_n^2, \quad \forall n \in \mathbb{N}.$$

(1) 證明 $u_n \xrightarrow[n\to\infty]{} +\infty$。

(2) 證明
$$\frac{\ln u_{n+1}}{2^{n+1}} - \frac{\ln u_n}{2^n} = o\Big(\frac{1}{2^n}\Big).$$

(3) 推得存在 $K > 1$ 使得 $u_n \sim K^{2^n}$。

習題 6.11 ： 令 $(u_n)_{n\geqslant 0}$ 為實數序列，被 $u_0 = c \in \mathbb{R}$ 以及下面式子所定義：

$$u_{n+1} = u_n + e^{-u_n}, \quad \forall n \in \mathbb{N}.$$

(1) 證明 $u_n \xrightarrow[n\to\infty]{} +\infty$。

(2) 找出 u_n 到 $o\big(\frac{\ln n}{n}\big)$ 數量級的漸進表示式。

習題 6.12 ： 定義序列

$$\forall n \geqslant 1, \quad S_n = \sum_{k=1}^{n} \ln k.$$

(1) 證明對於每個 $k \geqslant 2$，我們有
$$\int_{k-1}^{k} \ln t \,\mathrm{d}t \leqslant \ln k \leqslant \int_{k}^{k+1} \ln t \,\mathrm{d}t.$$
推得 $S_n = n\ln n - n + o(n)$。

(2) 藉由考慮定義如下的序列 $(A_n)_{n\geqslant 1}$：
$$\forall n \geqslant 1, \quad A_n = S_n - n\ln n + n,$$
證明 $A_n - A_{n-1} \sim \frac{1}{2n}$ 並推得 $A_n \sim \frac{1}{2}\ln n$。

(3) 令 $D_n := S_n - n\ln n + n - \frac{1}{2}\ln n$ 對於 $n \geqslant 1$。證明 $D_n - D_{n-1} \sim -\frac{1}{12n^2}$。

(4) 證明當 $n \to \infty$ 時，D_n 收斂到某個 D_∞。由此推得存在常數 $C > 0$ 滿足
$$n! \sim C\Big(\frac{n}{e}\Big)^n \sqrt{n}.$$

(5) 使用 I_{2n} 在習題 A1.2 中的式子來證明 $C = \sqrt{2\pi}$。

(6) 證明
$$n! \sim \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n \Big(1 + \frac{1}{12n} + o\Big(\frac{1}{n}\Big)\Big).$$

Exercise 6.13 : Our goal is to compute the value of the series $\sum_{n\geqslant 1}\frac{1}{n^2}$. Let $S_n=\sum_{k=1}^{n}\frac{1}{k^2}$ for $n\in\mathbb{N}$ be its n-th partial sum.

(1) Recall briefly why for $\alpha>1$, we have

$$\sum_{k\geqslant n}\frac{1}{k^\alpha}\sim\frac{1}{(\alpha-1)n^{\alpha-1}},\quad\text{when } n\to\infty.$$

(2) Let $f:[0,\pi]\to\mathbb{R}$ be a $\mathcal{C}^1$ function. Show that

$$\int_0^\pi f(t)\sin\left(\frac{(2n+1)t}{2}\right)\mathrm{d}t\xrightarrow[n\to\infty]{}0.$$

(3) Consider the function $A_n:(0,\pi]\to\mathbb{R}$ defined by

$$A_n(t)=\frac{1}{2}+\sum_{k=1}^{n}\cos(kt),\quad\forall t\in(0,\pi].$$

Show that

$$A_n(t)=\frac{\sin\left(\frac{(2n+1)t}{2}\right)}{2\sin(\frac{t}{2})}.$$

(4) Find $a,b\in\mathbb{R}$ such that

$$\forall n\in\mathbb{N},\quad\int_0^\pi(at^2+bt)\cos(nt)\,\mathrm{d}t=\frac{1}{n^2}.$$

In what follows, let us fix such values for a and b.

(5) Check that

$$\forall n\in\mathbb{N},\quad\int_0^\pi(at^2+bt)A_n(t)\,\mathrm{d}t=S_n-\frac{\pi^2}{6},$$

and $S_n\xrightarrow[n\to\infty]{}\frac{\pi^2}{6}$.

(6) Deduce that

$$S_n=\frac{\pi^2}{6}-\frac{1}{n}+o\Big(\frac{1}{n}\Big),\quad\text{when } n\to\infty.$$

習題 6.13 ： 我們的目的是計算級數 $\sum_{n\geqslant 1}\frac{1}{n^2}$ 的值。對於 $n\in\mathbb{N}$，令 $S_n=\sum_{k=1}^{n}\frac{1}{k^2}$ 為他的第 n 個部份和。

(1) 請簡短證明對於 $\alpha>1$，我們有

$$\sum_{k\geqslant n}\frac{1}{k^\alpha}\sim\frac{1}{(\alpha-1)n^{\alpha-1}},\quad\text{當 } n\to\infty.$$

(2) 令 $f:[0,\pi]\to\mathbb{R}$ 為 $\mathcal{C}^1$ 函數。證明

$$\int_0^\pi f(t)\sin\left(\frac{(2n+1)t}{2}\right)\mathrm{d}t\xrightarrow[n\to\infty]{}0.$$

(3) 考慮函數 $A_n:(0,\pi]\to\mathbb{R}$ 定義做

$$A_n(t)=\frac{1}{2}+\sum_{k=1}^{n}\cos(kt),\quad\forall t\in(0,\pi].$$

證明

$$A_n(t)=\frac{\sin\left(\frac{(2n+1)t}{2}\right)}{2\sin(\frac{t}{2})}.$$

(4) 求 $a,b\in\mathbb{R}$ 滿足

$$\forall n\in\mathbb{N},\quad\int_0^\pi(at^2+bt)\cos(nt)\,\mathrm{d}t=\frac{1}{n^2}.$$

接下來，我們把 a 和 b 固定為這樣的值。

(5) 檢查

$$\forall n\in\mathbb{N},\quad\int_0^\pi(at^2+bt)A_n(t)\,\mathrm{d}t=S_n-\frac{\pi^2}{6},$$

以及 $S_n\xrightarrow[n\to\infty]{}\frac{\pi^2}{6}$。

(6) 推得

$$S_n=\frac{\pi^2}{6}-\frac{1}{n}+o\Big(\frac{1}{n}\Big),\quad\text{當 } n\to\infty.$$

Exercise 6.14 : Let $(u_n)_{n\geqslant 0}$ be a sequence defined by $u_0 = 1$ and

$$\forall n \in \mathbb{N}, \quad u_{n+1} = \sin u_n.$$

(1) Check that $(u_n)_{n\geqslant 0}$ converges to 0.

(2) Find the limit of $\frac{u_{n+1}}{u_n}$ and $\frac{u_n+u_{n+1}}{u_n}$ when $n \to \infty$.

(3) Find the limit of $\frac{u_n-u_{n+1}}{u_n^3}$ when $n \to \infty$.

(4) Show that when $n \to \infty$, we have the equivalence

$$\frac{1}{u_{n+1}^2} - \frac{1}{u_n^2} \sim \frac{1}{3}.$$

Deduce an equivalence of u_n when $n \to \infty$.

(5) Show that

$$\frac{1}{u_{n+1}^2} - \frac{1}{u_n^2} - \frac{1}{3} = \frac{u_n^2}{15} + o(u_n^2).$$

Deduce that

$$u_n = \frac{\sqrt{3}}{\sqrt{n}} - \frac{3\sqrt{3}}{10}\frac{\ln n}{n^{3/2}} + o\left(\frac{\ln n}{n^{3/2}}\right).$$

Exercise 6.15 : Let $\sum_{n\geqslant 1} u_n$ be a divergent series with non-negative terms. Write $S_n = \sum_{k=1}^n u_k$ for all $n \geqslant 1$. Fix $\alpha > 1$.

(1) Show that for $n \geqslant 2$, we have

$$\frac{u_n}{S_n^\alpha} \leqslant \int_{S_{n-1}}^{S_n} \frac{\mathrm{d}t}{t^\alpha}.$$

(2) Show that the series $\sum_n \frac{u_n}{S_n^\alpha}$ is convergent.

Exercise 6.16 : Let $(u_n)_{n\geqslant 1}$ be a sequence in a Banach space $(W, \|\cdot\|)$ and

$$\lambda := \limsup_{n\to\infty} \|u_n\|^{1/n} \in [0, +\infty].$$

Show the following properties.

(1) If $\lambda < 1$, then the series $\sum u_n$ is absolutely convergent.

(2) If $\lambda > 1$, then the series $\sum u_n$ is divergent.

(3) If $\lambda = 1$, then we cannot conclude.

習題 6.14 ： 令 $(u_n)_{n\geqslant 0}$ 為序列，被 $u_0 = 1$ 以及下面式子所定義：

$$\forall n \in \mathbb{N}, \quad u_{n+1} = \sin u_n.$$

(1) 檢查 $(u_n)_{n\geqslant 0}$ 收斂到 0。

(2) 求 $\frac{u_{n+1}}{u_n}$ 和 $\frac{u_n+u_{n+1}}{u_n}$ 在 $n \to \infty$ 時的極限。

(3) 求 $\frac{u_n-u_{n+1}}{u_n^3}$ 在 $n \to \infty$ 時的極限。

(4) 證明當 $n \to \infty$ 時，我們有等價關係

$$\frac{1}{u_{n+1}^2} - \frac{1}{u_n^2} \sim \frac{1}{3}.$$

由此推得 u_n 在 $n \to \infty$ 時的一個等價關係。

(5) 證明

$$\frac{1}{u_{n+1}^2} - \frac{1}{u_n^2} - \frac{1}{3} = \frac{u_n^2}{15} + o(u_n^2).$$

由此推得

$$u_n = \frac{\sqrt{3}}{\sqrt{n}} - \frac{3\sqrt{3}}{10}\frac{\ln n}{n^{3/2}} + o\left(\frac{\ln n}{n^{3/2}}\right).$$

習題 6.15 ： 令 $\sum_{n\geqslant 1} u_n$ 為一般項非負的發散級數。對於所有 $n \geqslant 1$，我們記 $S_n = \sum_{k=1}^n u_k$。固定 $\alpha > 1$。

(1) 證明對於 $n \geqslant 2$，我們有

$$\frac{u_n}{S_n^\alpha} \leqslant \int_{S_{n-1}}^{S_n} \frac{\mathrm{d}t}{t^\alpha}.$$

(2) 證明級數 $\sum_n \frac{u_n}{S_n^\alpha}$ 會收斂。

習題 6.16 ： 令 $(u_n)_{n\geqslant 1}$ 為在 Banach 空間 $(W, \|\cdot\|)$ 中的數列，以及

$$\lambda := \limsup_{n\to\infty} \|u_n\|^{1/n} \in [0, +\infty].$$

證明下列性質。

(1) 如果 $\lambda < 1$，那麼級數 $\sum u_n$ 會絕對收斂。

(2) 如果 $\lambda > 1$，那麼級數 $\sum u_n$ 會發散。

(3) 如果 $\lambda = 1$，那麼我們無法總結。

Exercise 6.17 : Let $x \in \mathbb{C}$ and $a \in \mathbb{R}$. Find the behavior of the following series by the ratio test (Theorem 6.3.1) *and* the root test (Theorem 6.3.6).

(1) $\sum \frac{x^n}{n!}$.

(2) $\sum \frac{x^n}{\binom{2n}{n}}$.

(3) $\sum \frac{n!}{n^{an}}$.

(4) $\sum \frac{n^a(\ln n)^n}{n!}$.

(5) $\sum \frac{(n!)^a}{(2n)!}$.

(6) $\sum \frac{a^n n!}{n^n}$.

Exercise 6.18 : Let $(u_n)_{n\geqslant 1}$ be a sequence with strictly positive terms such that

$$\frac{u_{n+1}}{u_n} = 1 + \frac{\alpha}{n} + \mathcal{O}\Big(\frac{1}{n^2}\Big), \quad \text{for some } \alpha \in \mathbb{R}.$$

Fix $\beta \in \mathbb{R}$ and let

$$\forall n \in \mathbb{N}, \quad v_n = \ln((n+1)^\beta u_{n+1}) - \ln(n^\beta u_n).$$

(1) For which value(s) of β does the series $\sum v_n$ converge?

(2) For each of these values, show that there exists $A > 0$ such that $u_n \sim An^\alpha$.

Exercise 6.19 :

(1) Show that the series $\sum_n \frac{(-1)^n}{\sqrt{n}}$ converges.

(2) Show that for $n \to \infty$, we have

$$\frac{(-1)^n}{\sqrt{n}+(-1)^n} = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + o\Big(\frac{1}{n}\Big).$$

(3) What is the behavior of the following series?

$$\sum_n \frac{(-1)^n}{\sqrt{n}+(-1)^n}.$$

Exercise 6.20 : Use the comparison theorem (Theorem 6.2.3) and alternating series (Theorem 6.4.2) to determine the behavior of the series $\sum u_n$ where the general term u_n is given by different expressions. Let $\alpha > \beta > 0$.

(1) $u_n = \ln\Big(1 + \frac{(-1)^n}{2n+1}\Big)$,

(2) $u_n = \frac{(-1)^n}{\sqrt{n^\alpha + (-1)^n}}$,

(3) $u_n = \frac{(-1)^n}{\sqrt{n^\alpha + (-1)^n n^\beta}}$.

習題 6.17 ： 令 $x \in \mathbb{C}$ 以及 $a \in \mathbb{R}$。使用商檢測法（定理 6.3.1）以及根檢測法（定理 6.3.6）來求下列級數的行為。

(1) $\sum \frac{x^n}{n!}$.

(2) $\sum \frac{x^n}{\binom{2n}{n}}$.

(3) $\sum \frac{n!}{n^{an}}$.

(4) $\sum \frac{n^a(\ln n)^n}{n!}$.

(5) $\sum \frac{(n!)^a}{(2n)!}$.

(6) $\sum \frac{a^n n!}{n^n}$.

習題 6.18 ： 令 $(u_n)_{n\geqslant 1}$ 為每項皆嚴格為正的序列，滿足

$$\frac{u_{n+1}}{u_n} = 1 + \frac{\alpha}{n} + \mathcal{O}\Big(\frac{1}{n^2}\Big), \quad \text{對於選定的 } \alpha \in \mathbb{R}.$$

固定 $\beta \in \mathbb{R}$ 並令

$$\forall n \in \mathbb{N}, \quad v_n = \ln((n+1)^\beta u_{n+1}) - \ln(n^\beta u_n).$$

(1) 對哪些 β 的值會讓級數 $\sum v_n$ 收斂呢？

(2) 對每個這樣的值，證明存在 $A > 0$ 使得 $u_n \sim An^\alpha$。

習題 6.19 ：

(1) 證明級數 $\sum_n \frac{(-1)^n}{\sqrt{n}}$ 會收斂。

(2) 證明對於 $n \to \infty$，我們有

$$\frac{(-1)^n}{\sqrt{n}+(-1)^n} = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + o\Big(\frac{1}{n}\Big).$$

(3) 下面級數的行為是什麼？

$$\sum_n \frac{(-1)^n}{\sqrt{n}+(-1)^n}.$$

習題 6.20 ： 使用比較定理（定理 6.2.3）以及交錯級數（定理 6.4.2）來求級數 $\sum u_n$ 的性質，其中一般項 u_n 是由下列不同的表示式所給出的。令 $\alpha > \beta > 0$。

(1) $u_n = \ln\Big(1 + \frac{(-1)^n}{2n+1}\Big)$,

(2) $u_n = \frac{(-1)^n}{\sqrt{n^\alpha + (-1)^n}}$,

(3) $u_n = \frac{(-1)^n}{\sqrt{n^\alpha + (-1)^n n^\beta}}$.

Exercise 6.21 :

(1) Justify why the alternating series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$ converges. Does it also converge absolutely?

(2) For every $n \in \mathbb{N}_0$, consider the partial sum

$$S_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}.$$

Show that for $n \in \mathbb{N}$, we have

$$S_{2n-1} < \cos(1) < S_{2n} = S_{2n-1} + \frac{1}{(4n)!}.$$

(3) Deduce that $\cos(1)$ is an irrational number.

(4) Similarly, use the fact that $e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ to show that e is an irrational number.

Exercise 6.22 : Let $(u_n)_{n\geqslant 0}$ be a decreasing sequence with strictly positive general terms with limit 0. Consider the alternating series $\sum_{n\geqslant 0}(-1)^n u_n$ which converges by Theorem 6.4.2. Recall that the remainders are defined by

$$\forall n \in \mathbb{N}_0, \quad R_n = \sum_{k=n+1}^{\infty} (-1)^k u_k.$$

Suppose that the following conditions hold,

$$\forall n \in \mathbb{N}_0, \quad u_{n+2} - 2u_{n+1} + u_n \geqslant 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.$$

(1) Show that for every $n \geqslant 0$, we have $|R_n| + |R_{n+1}| = u_{n+1}$.

(2) Show that the sequence $(|R_n|)_{n\geqslant 0}$ is decreasing.

(3) Deduce that the following equivalence,

$$R_n \sim \frac{(-1)^{n+1}u_n}{2}, \quad \text{when } n \to \infty.$$

(4) Apply this result to the alternating series $\sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n}$ to find an asymptotic formula for its partial sums. Hint: the limit of this series is known, see Example 6.4.4.

Exercise 6.23 :

(1) Show that the Cauchy product of the following two divergent series

$$(-1-2-2-2-2-\cdots)(-1+2-2+2-2+\cdots)$$

is absolutely convergent.

(2) Consider the alternating series $\sum_{n\geqslant 0} \frac{(-1)^n}{\sqrt{n+1}}$. Show that the Cauchy product of this series with itself is divergent.

習題 6.21 ：

(1) 解釋為什麼交錯級數 $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$ 收斂。他會絕對收斂嗎？

(2) 對於每個 $n \in \mathbb{N}_0$，考慮部份和

$$S_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}.$$

證明對於 $n \in \mathbb{N}$，我們有

$$S_{2n-1} < \cos(1) < S_{2n} = S_{2n-1} + \frac{1}{(4n)!}.$$

(3) 推得 $\cos(1)$ 是個無理數。

(4) 同理，使用式子 $e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ 來證明 e 是個無理數。

習題 6.22 ： 令 $(u_n)_{n\geqslant 0}$ 為每項皆嚴格為正，且遞減到 0 的序列。考慮交錯級數 $\sum_{n\geqslant 0}(-1)^n u_n$，根據定理 6.4.2 ，他是個收斂級數。我們回顧餘項的定義：

$$\forall n \in \mathbb{N}_0, \quad R_n = \sum_{k=n+1}^{\infty} (-1)^k u_k.$$

假設下列條件成立：

$$\forall n \in \mathbb{N}_0, \quad u_{n+2} - 2u_{n+1} + u_n \geqslant 0 \quad \text{以及} \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.$$

(1) 證明對於每個 $n \geqslant 0$，我們有 $|R_n| + |R_{n+1}| = u_{n+1}$。

(2) 證明序列 $(|R_n|)_{n\geqslant 0}$ 是遞減的。

(3) 推得下面等價關係：

$$R_n \sim \frac{(-1)^{n+1}u_n}{2}, \quad \text{當 } n \to \infty.$$

(4) 把這個結果用在交錯級數 $\sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n}$ 上，以給出他部份和的漸進式。提示：此級數的極限是已知的，見範例 6.4.4 。

習題 6.23 ：

(1) 證明下面這兩個發散序列的柯西積

$$(-1-2-2-2-2-\cdots)(-1+2-2+2-2+\cdots)$$

會絕對收斂。

(2) 考慮交錯級數 $\sum_{n\geqslant 0} \frac{(-1)^n}{\sqrt{n+1}}$。證明他和自己的柯西積會發散。

Exercise 6.24 : This exercise is a generalization of Theorem 6.6.3. Let $\sum_{n\geqslant 0} a_n$ be an absolutely convergent series and $\sum_{n\geqslant 0} b_n$ be a convergent series with terms in a complete normed algebra $(\mathcal{A}, \cdot, |\cdot|)$. Their Cauchy product is the series $\sum_{n\geqslant 0} c_n$ given by

$$\forall n \in \mathbb{N}_0, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Show that the series $\sum c_n$ is convergent, and its sum equals

$$\sum_{n\geqslant 0} c_n = \Big(\sum_{p\geqslant 0} a_p\Big)\Big(\sum_{q\geqslant 0} b_q\Big).$$

Exercise 6.25 : Consider the double sequence $(u_{m,n})_{m,n\geqslant 1}$ defined by

$$\forall m,n \geqslant 1, \quad u_{m,n} = \Big(1+\frac{1}{m}\Big)^n.$$

(1) Find the iterated limits $\lim_{n\to\infty}\lim_{m\to\infty} u_{m,n}$ and $\lim_{m\to\infty}\lim_{n\to\infty} u_{m,n}$.

(2) Does the limit of the sequence $(u_{m,n})_{m,n\geqslant 1}$ exist? How about the limit of $(u_{n,n})_{n\geqslant 1}$?

Exercise 6.26 : For a real number $k > 1$, let us define the Riemann zeta function

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

(1) Explain briefly why the series $\zeta(k)$ is well defined for $k > 1$.

(2) Show the following identity,
$$\sum_{k=2}^{\infty}(\zeta(k)-1) = 1.$$

(3) Show the following identity,
$$\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k} = 1-\gamma, \quad \text{where} \quad \gamma = \lim_{n\to\infty}\Big[\sum_{k=1}^{n}\frac{1}{k} - \ln n\Big].$$

Exercise 6.27 : Let $(u_{m,n})_{m,n\geqslant 1}$ be a double sequence in a Banach space $(W, \|\cdot\|)$. The associated double series $\sum_{m,n} u_{m,n}$ is the double sequence $(s_{m,n})_{m,n\geqslant 1}$ given by

$$\forall m,n \geqslant 1, \quad s_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} u_{i,j}.$$

If the limit $\lim_{m,n\to\infty} s_{m,n}$ is well defined, then we say that the double series $\sum_{m,n} u_{m,n}$ converges; if the limit $\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\|u_{i,j}\|$ is well-defined, then we say that the double series $\sum_{m,n} u_{m,n}$ converges absolutely.

習題 6.24 ： 這個習題是定理 6.6.3 的推廣。令 $\sum_{n\geqslant 0} a_n$ 為絕對收斂級數以及 $\sum_{n\geqslant 0} b_n$ 是個收斂級數，並假設他們的項都在完備賦範代數 $(\mathcal{A}, \cdot, |\cdot|)$ 裡面。他們的柯西積是由級數 $\sum_{n\geqslant 0} c_n$ 所給定，定義做

$$\forall n \in \mathbb{N}_0, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

證明級數 $\sum c_n$ 會收斂，且他的和等於

$$\sum_{n\geqslant 0} c_n = \Big(\sum_{p\geqslant 0} a_p\Big)\Big(\sum_{q\geqslant 0} b_q\Big).$$

習題 6.25 ： 考慮雙下標序列 $(u_{m,n})_{m,n\geqslant 1}$，定義做：

$$\forall m,n \geqslant 1, \quad u_{m,n} = \Big(1+\frac{1}{m}\Big)^n.$$

(1) 求迭代極限 $\lim_{n\to\infty}\lim_{m\to\infty} u_{m,n}$ 和 $\lim_{m\to\infty}\lim_{n\to\infty} u_{m,n}$。

(2) 序列 $(u_{m,n})_{m,n\geqslant 1}$ 的極限是否存在？那 $(u_{n,n})_{n\geqslant 1}$ 的極限呢？

習題 6.26 ： 對於實數 $k > 1$，我們可以定義黎曼 ζ 函數如下：

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

(1) 簡單解釋為什麼當 $k > 1$ 時，級數 $\zeta(k)$ 會是定義良好的。

(2) 證明下列關係式：
$$\sum_{k=2}^{\infty}(\zeta(k)-1) = 1.$$

(3) 證明下列關係式：
$$\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k} = 1-\gamma, \quad \text{其中} \quad \gamma = \lim_{n\to\infty}\Big[\sum_{k=1}^{n}\frac{1}{k} - \ln n\Big].$$

習題 6.27 ： 令 $(u_{m,n})_{m,n\geqslant 1}$ 為在 Banach 空間 $(W, \|\cdot\|)$ 中的雙下標序列。他所對應到的雙下標級數 $\sum_{m,n} u_{m,n}$ 會是雙下標序列 $(s_{m,n})_{m,n\geqslant 1}$，定義做

$$\forall m,n \geqslant 1, \quad s_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} u_{i,j}.$$

如果極限 $\lim_{m,n\to\infty} s_{m,n}$ 定義良好，則我們說雙下標級數 $\sum_{m,n} u_{m,n}$ 會收斂；如果極限 $\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\|u_{i,j}\|$ 定義良好，則我們說雙下標級數 $\sum_{m,n} u_{m,n}$ 會絕對收斂。

(1) Show that the iterated series $\sum_m(\sum_n \|u_{m,n}\|)$ is convergent if and only if the double series $\sum_{m,n} u_{m,n}$ is absolutely convergent.

(2) If the double series $\sum_{m,n} u_{m,n}$ converges absolutely, show that it also converges.

(3) Suppose that the double series $\sum_{m,n} u_{m,n}$ is absolutely convergent. We define

$$\forall n \geqslant 2, \quad c_n = \sum_{i=1}^{n-1} u_{i,(n-1)} = u_{1,(n-1)} + u_{2,(n-2)} + \cdots + u_{(n-1),1}.$$

Show that the series $\sum_n c_n$ is also absolutely convergent and

$$\sum_{n\geqslant 2} c_n = \sum_{m,n\geqslant 1} u_{m,n}.$$

(4) Let $(u_{m,n})_{m,n\geqslant 1}$ be the double sequence given by

$$\forall m,n \geqslant 1, \quad u_{m,n} = \begin{cases} +1 & \text{if } m-n=1, \\ -1 & \text{if } m-n=-1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that both iterated series $\sum_n(\sum_m u_{m,n})$ and $\sum_m(\sum_n u_{m,n})$ converge, but their limits are not equal.

(b) Show that the double series $\sum_{m,n} u_{m,n}$ does not converge.

(c) Show that the limit $\lim_{n\to\infty} s_{n,n}$ exists.

(5) Show that if the double series and one of the iterated series associated to $(u_{m,n})_{m,n\geqslant 1}$ converge, then these two limits are equal.

(6) Show that the convergence of the double series does not imply the convergence of the iterated series.

(7) Show that the convergence of the double series does not imply that $\lim_{n\to\infty} u_{m,n} = 0$ for each $m \geqslant 1$.

Exercise 6.28 : Let $u_n = \frac{(-1)^n}{n}$ for $n \in \mathbb{N}$.

(1) Show that $\prod_{n\geqslant 2}(1+u_n)$ converges with limit 1.

(2) Does the series $\sum_{n\geqslant 2} u_n$ converge, absolutely converge, or conditionally converge?

Exercise 6.29 : Let $(u_n)_{n\geqslant 1}$ be a sequence defined by

$$\forall n \in \mathbb{N}, \quad u_{2n-1} = -\frac{1}{\sqrt{n}}, \quad u_{2n} = \frac{1}{\sqrt{n}} + \frac{1}{n}.$$

(1) Show that $\prod_{n\geqslant 2}(1+u_n)$ converges to a nonzero limit.

(2) Does the series $\sum_{n\geqslant 2} u_n$ converge, absolutely converge, or conditionally converge?

(1) 證明迭代級數 $\sum_m(\sum_n \|u_{m,n}\|)$ 會收斂，若且唯若雙下標級數 $\sum_{m,n} u_{m,n}$ 會絕對收斂。

(2) 如果雙下標級數 $\sum_{m,n} u_{m,n}$ 會絕對收斂，證明他也會收斂。

(3) 假設雙下標級數 $\sum_{m,n} u_{m,n}$ 會絕對收斂。我們定義

$$\forall n \geqslant 2, \quad c_n = \sum_{i=1}^{n-1} u_{i,(n-1)} = u_{1,(n-1)} + u_{2,(n-2)} + \cdots + u_{(n-1),1}.$$

證明級數 $\sum_n c_n$ 也會絕對收斂，而且我們有

$$\sum_{n\geqslant 2} c_n = \sum_{m,n\geqslant 1} u_{m,n}.$$

(4) 令 $(u_{m,n})_{m,n\geqslant 1}$ 為雙下標序列，定義做：

$$\forall m,n \geqslant 1, \quad u_{m,n} = \begin{cases} +1 & \text{若 } m-n=1, \\ -1 & \text{若 } m-n=-1, \\ 0 & \text{其他情況.} \end{cases}$$

(a) 證明兩個迭代級數 $\sum_n(\sum_m u_{m,n})$ 和 $\sum_m(\sum_n u_{m,n})$ 都會收斂，但極限不相同。

(b) 證明雙下標級數 $\sum_{m,n} u_{m,n}$ 不會收斂。

(c) 證明極限 $\lim_{n\to\infty} s_{n,n}$ 存在。

(5) 證明如果對應到序列 $(u_{m,n})_{m,n\geqslant 1}$ 的雙下標級數和其中一個迭代級數都收斂，那麼這兩個極限會相等。

(6) 證明雙下標級數的收斂性不會蘊含迭代級數的收斂性。

(7) 證明雙下標級數的收斂性不會蘊含對於每個 $m \geqslant 1$，我們有 $\lim_{n\to\infty} u_{m,n} = 0$。

習題 6.28 ： 令 $u_n = \frac{(-1)^n}{n}$ 對於 $n \in \mathbb{N}$。

(1) 證明 $\prod_{n\geqslant 2}(1+u_n)$ 收斂且極限為 1。

(2) 級數 $\sum_{n\geqslant 2} u_n$ 會收斂、絕對收斂，或條件收斂嗎？

習題 6.29 ： 令 $(u_n)_{n\geqslant 1}$ 為定義如下的序列：

$$\forall n \in \mathbb{N}, \quad u_{2n-1} = -\frac{1}{\sqrt{n}}, \quad u_{2n} = \frac{1}{\sqrt{n}} + \frac{1}{n}.$$

(1) 證明 $\prod_{n\geqslant 2}(1+u_n)$ 收斂到非零極限。

(2) 級數 $\sum_{n\geqslant 2} u_n$ 會收斂、絕對收斂，或條件收斂嗎？

Exercise 6.30 : Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers with $a_n > -1$ for all $n \in \mathbb{N}$.

(1) Show that if the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is convergent.

(2) Suppose that the series $\sum_{n\geqslant 1} a_n$ is convergent. Show that the infinite product $\prod(1+a_n)$ is convergent if and only if the series $\sum_{n=1}^{\infty} a_n^2$ is convergent. Hint: see below[1].

Exercise 6.31 : Justify the convergence of the following infinite products, and prove their limits.

(1) $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$. (2) $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)=\frac{2}{\pi}$.

Exercise 6.32 : Let $P_n = \prod_{k=2}^{n}\left(1+\frac{(-1)^k}{\sqrt{k}}\right)$ for $n \geqslant 2$. Show that there exists $\lambda \in \mathbb{R}$ such that $P_n \sim \frac{e^{\lambda}}{\sqrt{n}}$ when $n \to \infty$.

Exercise 6.33 : Let $\mathcal{P}$ be the set of all the primes. In this exercise, we will prove $\sum_{p\in\mathcal{P}} \frac{1}{p}$ is divergent.

(1) Show that for $s > 1$, we have
$$-\sum_{p\in\mathcal{P}} \log\left(1-\frac{1}{p^s}\right) = \log\zeta(s).$$

(2) Deduce that there exists $M > 0$ such that for any $s > 1$, we have
$$\left|\sum_{p\in\mathcal{P}} \frac{1}{p^s} - \log\zeta(s)\right| < M.$$

(3) Show that as $s \to 1+$, we have $\zeta(s) \to +\infty$.

(4) Conclude that $\sum_{p\in\mathcal{P}} \frac{1}{p}$ is divergent.

[1]Show that there exist positive constants A and B such that if $|x| < \frac{1}{2}$, then $Ax^2 \leqslant x - \log(1+x) \leqslant Bx^2$.

習題 6.30 ： 令 $(a_n)_{n\geqslant 1}$ 為實數序列滿足 $a_n > -1$ 對於所有 $n \in \mathbb{N}$。

(1) 證明如果級數 $\sum_{n=1}^{\infty} a_n$ 會絕對收斂，那麼無窮積 $\prod_{n=1}^{\infty}(1+a_n)$ 會收斂。

(2) 假設級數 $\sum_{n\geqslant 1} a_n$ 會收斂。證明無窮積 $\prod(1+a_n)$ 會收斂若且唯若級數 $\sum_{n=1}^{\infty} a_n^2$ 會收斂。提示：如下[1]。

習題 6.31 ： 解釋下列無窮積的收斂性，並證明他們的極限。

(1) $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$. (2) $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)=\frac{2}{\pi}$.

習題 6.32 ： 對於 $n \geqslant 2$，令 $P_n = \prod_{k=2}^{n}\left(1+\frac{(-1)^k}{\sqrt{k}}\right)$。證明存在 $\lambda \in \mathbb{R}$ 使得當 $n \to \infty$ 時，我們有 $P_n \sim \frac{e^{\lambda}}{\sqrt{n}}$。

習題 6.33 ： 令 $\mathcal{P}$ 為所有質數所構成的集合。在這個習題中，我們會證明 $\sum_{p\in\mathcal{P}} \frac{1}{p}$ 是發散的。

(1) 證明對於 $s > 1$，我們有
$$-\sum_{p\in\mathcal{P}} \log\left(1-\frac{1}{p^s}\right) = \log\zeta(s).$$

(2) 推得存在 $M > 0$ 使得對於任意 $s > 1$，我們有
$$\left|\sum_{p\in\mathcal{P}} \frac{1}{p^s} - \log\zeta(s)\right| < M.$$

(3) 證明當 $s \to 1+$ 時，我們有 $\zeta(s) \to +\infty$。

(4) 總結 $\sum_{p\in\mathcal{P}} \frac{1}{p}$ 是發散的。

[1]證明存在正數 A 和 B 使得如果 $|x| < \frac{1}{2}$，則我們有 $Ax^2 \leqslant x - \log(1+x) \leqslant Bx^2$。