Chapter 6

Exercise 6.1 : Let $(V, \|\cdot\|)$ be a Banach space, and $(a_n)_{n\geqslant 1}$ be a sequence in V. We define an auxiliary sequence $(m_n)_{n\geqslant 1}$ as below,

$$\forall n \in \mathbb{N}, \quad m_n = \frac{a_1 + \cdots + a_n}{n}.$$

We consider the following two properties.

(i) The sequence $(a_n)_{n\geqslant 1}$ converges.

(ii) The sequence $(m_n)_{n\geqslant 1}$ converges.

Answer the following questions.

(1) Show that if (i) holds with limit ℓ, then (ii) also holds with the same limit.

(2) Find an example for which (ii) holds, but (i) does not hold.

(3) We are given a sequence $(w_n)_{n\geqslant 1}$ of non-negative real numbers such that $\sum w_n$ diverges. Define

$$\forall n \in \mathbb{N}, \quad m'_n = \frac{w_1 a_1 + \cdots + w_n a_n}{w_1 + \cdots + w_n}.$$

If we replace the property (ii) by the following property (ii'): the sequence $(m'_n)_{n\geqslant 1}$ converges, does (i) still implies (ii')?

Exercise 6.2 : Let $(V, \|\cdot\|)$ be a non-empty finite-dimensional normed vector space. Let $(a_n)_{n\geqslant 1}$ be a sequence in V and $(r_n)_{n\geqslant 1}$ be a sequence in $\mathbb{R}_+^*$ satisfying

$$\forall n \in \mathbb{N}, \quad \overline{B}(a_{n+1}, r_{n+1}) \subseteq \overline{B}(a_n, r_n).$$

(1) Show that

$$\forall n \in \mathbb{N}, \quad \|a_{n+1} - a_n\| \leqslant r_n - r_{n+1}.$$

(2) Show that the sequence $(a_n)_{n\geqslant 1}$ is convergent.

(3) Show that $\bigcap_{n\geqslant 1} \overline{B}(a_n, r_n)$ is a closed ball. Find its center and radius.

Exercise 6.3 : Let $a \in \mathbb{C}$ and

$$A = \frac{1}{2}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

(1) Show that for $n \in \mathbb{N}$, we have

$$A^n = \frac{1}{2}I + \frac{a}{2}J, \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(2) Find the behavior of the series with general term A^n.

Exercise 6.4 : Let $(a_n)_{n\geqslant 1}$ be a decreasing sequence of non-negative real numbers. Show that if $\sum u_n$ converges, then we have $u_n = o(\frac{1}{n})$.

Exercise 6.5 : Let $\sum_n u_n$ and $\sum_n v_n$ be two convergent series with non-negative terms. Determine the behavior of the series $\sum_n \sqrt{u_n v_n}$ and $\sum_n \max(u_n, v_n)$.

Exercise 6.6 : Let $\sum_n u_n$ be a series with non-negative terms.

(1) Suppose that $\sum_n u_n$ converges. Show that for $\alpha > 1$, the series $\sum_n u_n^\alpha$ converges.

(2) Suppose that $\sum_n u_n$ diverges. Show that for $\alpha \in (0,1)$, the series $\sum_n u_n^\alpha$ diverges.

Exercise 6.7 : Let $(u_n)_{n\geqslant 1}$ be a series with non-negative terms. For $n \geqslant 1$, let $v_n = \frac{u_n}{1+u_n}$.

(1) Show that the function $x \mapsto \frac{x}{1+x}$ is increasing on $[0, +\infty)$.

(2) Show that the series $\sum_n u_n$ and $\sum_n v_n$ have the same behavior.

Exercise 6.8 : Let $(u_n)_{n\geqslant 1}$ be a non-negative decreasing sequence. Show that the series $\sum_n u_n$ and $\sum_n 2^n u_{2^n}$ have the same behavior.

Exercise 6.9 : Use the comparison theorems (Proposition 6.2.2 and Theorem 6.2.3) to determine the behavior of the series $\sum u_n$ where the general term u_n is given by different expressions. You may also need the Stirling's formula from Exercise 6.12. Let us fix constants $a \geqslant 0, b, c \in \mathbb{R}$.

(1) $u_n = 3^{-\sqrt{n}}$,

(2) $u_n = a^n n!$,

(3) $u_n = ne^{-\sqrt{n}}$,

(4) $u_n = n^{-1-\frac{1}{n}}$,

(5) $u_n = \frac{(n!)^3}{(3n)!}$,

(6) $u_n = \frac{\ln n}{\ln(e^n - 2)}$.

(7) $u_n = \frac{a^n}{n!}$,

(8) $u_n = \frac{a^n}{\binom{2n}{n}}$.

(9) $u_n = 1 - \cos\frac{\pi}{n}$.

(10) $u_n = \left(\frac{n}{n+1}\right)^{n^2}$.

(11) $u_n = e^{1/n} - a - \frac{b}{n}$.

(12) $u_n = \cos(\frac{1}{n}) - a - \frac{b}{n}$.

Exercise 6.10 : Let $(u_n)_{n\geqslant 0}$ be a real-valued sequence defined by $u_0 > 0$ and

$$u_{n+1} = u_n + u_n^2, \quad \forall n \in \mathbb{N}.$$

(1) Show that $u_n \xrightarrow[n\to\infty]{} +\infty$.

(2) Show that

$$\frac{\ln u_{n+1}}{2^{n+1}} - \frac{\ln u_n}{2^n} = o\Big(\frac{1}{2^n}\Big).$$

(3) Deduce that there exists $K > 1$ such that $u_n \sim K^{2^n}$.

Exercise 6.11 : Let $(u_n)_{n\geqslant 0}$ be a real-valued sequence defined by $u_0 = c \in \mathbb{R}$ and

$$u_{n+1} = u_n + e^{-u_n}, \quad \forall n \in \mathbb{N}.$$

(1) Show that $u_n \xrightarrow[n\to\infty]{} +\infty$.

(2) Find an asymptotic expression of u_n up to the order $o\left(\frac{\ln n}{n}\right)$.

Exercise 6.12 : Define the sequence

$$\forall n \geqslant 1, \quad S_n = \sum_{k=1}^{n} \ln k.$$

(1) Show that for every $k \geqslant 2$, we have

$$\int_{k-1}^{k} \ln t \,\mathrm{d}t \leqslant \ln k \leqslant \int_{k}^{k+1} \ln t \,\mathrm{d}t.$$

Deduce that $S_n = n \ln n - n + o(n)$.

(2) By considering the sequence $(A_n)_{n\geqslant 1}$, defined by

$$\forall n \geqslant 1, \quad A_n = S_n - n \ln n + n,$$

show that $A_n - A_{n-1} \sim \frac{1}{2n}$ and deduce that $A_n \sim \frac{1}{2} \ln n$.

(3) Let $D_n := S_n - n \ln n + n - \frac{1}{2} \ln n$ for $n \geqslant 1$. Show that $D_n - D_{n-1} \sim -\frac{1}{12n^2}$.

(4) Show that D_n converges to some D_∞ when $n \to \infty$. Deduce that there exists some constant $C > 0$ such that

$$n! \sim C\left(\frac{n}{e}\right)^n \sqrt{n}.$$

(5) Using the expression of I_{2n} from Exercise A1.2 to show that $C = \sqrt{2\pi}$.

(6) Show that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + o\left(\frac{1}{n}\right)\right).$$

Exercise 6.13 : Our goal is to compute the value of the series $\sum_{n\geqslant 1} \frac{1}{n^2}$. Let $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$ for $n \in \mathbb{N}$ be its n-th partial sum.

(1) Recall briefly why for $\alpha > 1$, we have

$$\sum_{k\geqslant n} \frac{1}{k^\alpha} \sim \frac{1}{(\alpha-1)n^{\alpha-1}}, \quad \text{when } n \to \infty.$$

(2) Let $f : [0, \pi] \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Show that

$$\int_0^\pi f(t) \sin\left(\frac{(2n+1)t}{2}\right) \mathrm{d}t \xrightarrow[n\to\infty]{} 0.$$

(3) Consider the function $A_n : (0, \pi] \to \mathbb{R}$ defined by

$$A_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kt), \quad \forall t \in (0, \pi].$$

Show that

$$A_n(t) = \frac{\sin\left(\frac{(2n+1)t}{2}\right)}{2\sin(\frac{t}{2})}.$$

(4) Find $a, b \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N}, \quad \int_0^\pi (at^2 + bt)\cos(nt)\,\mathrm{d}t = \frac{1}{n^2}.$$

In what follows, let us fix such values for a and b.

(5) Check that

$$\forall n \in \mathbb{N}, \quad \int_0^\pi (at^2 + bt)A_n(t)\,\mathrm{d}t = S_n - \frac{\pi^2}{6},$$

and $S_n \xrightarrow[n\to\infty]{} \frac{\pi^2}{6}$.

(6) Deduce that

$$S_n = \frac{\pi^2}{6} - \frac{1}{n} + o\Big(\frac{1}{n}\Big), \quad \text{when } n \to \infty.$$

Exercise 6.14 : Let $(u_n)_{n\geqslant 0}$ be a sequence defined by $u_0 = 1$ and

$$\forall n \in \mathbb{N}, \quad u_{n+1} = \sin u_n.$$

(1) Check that $(u_n)_{n\geqslant 0}$ converges to 0.

(2) Find the limit of $\frac{u_{n+1}}{u_n}$ and $\frac{u_n+u_{n+1}}{u_n}$ when $n \to \infty$.

(3) Find the limit of $\frac{u_n-u_{n+1}}{u_n^3}$ when $n \to \infty$.

(4) Show that when $n \to \infty$, we have the equivalence

$$\frac{1}{u_{n+1}^2} - \frac{1}{u_n^2} \sim \frac{1}{3}.$$

Deduce an equivalence of u_n when $n \to \infty$.

(5) Show that

$$\frac{1}{u_{n+1}^2} - \frac{1}{u_n^2} - \frac{1}{3} = \frac{u_n^2}{15} + o(u_n^2).$$

Deduce that

$$u_n = \frac{\sqrt{3}}{\sqrt{n}} - \frac{3\sqrt{3}}{10}\frac{\ln n}{n^{3/2}} + o\left(\frac{\ln n}{n^{3/2}}\right).$$

Exercise 6.15 : Let $\sum_{n\geqslant 1} u_n$ be a divergent series with non-negative terms. Write $S_n = \sum_{k=1}^{n} u_k$ for all $n \geqslant 1$. Fix $\alpha > 1$.

(1) Show that for $n \geqslant 2$, we have
$$\frac{u_n}{S_n^\alpha} \leqslant \int_{S_{n-1}}^{S_n} \frac{\mathrm{d}t}{t^\alpha}.$$

(2) Show that the series $\sum_n \frac{u_n}{S_n^\alpha}$ is convergent.

Exercise 6.16 : Let $(u_n)_{n\geqslant 1}$ be a sequence in a Banach space $(W, \|\cdot\|)$ and
$$\lambda := \limsup_{n\to\infty} \|u_n\|^{1/n} \in [0, +\infty].$$

Show the following properties.

(1) If $\lambda < 1$, then the series $\sum u_n$ is absolutely convergent.

(2) If $\lambda > 1$, then the series $\sum u_n$ is divergent.

(3) If $\lambda = 1$, then we cannot conclude.

Exercise 6.17 : Let $x \in \mathbb{C}$ and $a \in \mathbb{R}$. Find the behavior of the following series by the ratio test (Theorem 6.3.1) *and* the root test (Theorem 6.3.6).

(1) $\sum \frac{x^n}{n!}$.

(2) $\sum \frac{x^n}{\binom{2n}{n}}$.

(3) $\sum \frac{n!}{n^{an}}$.

(4) $\sum \frac{n^a (\ln n)^n}{n!}$.

(5) $\sum \frac{(n!)^a}{(2n)!}$.

(6) $\sum \frac{a^n n!}{n^n}$.

Exercise 6.18 : Let $(u_n)_{n\geqslant 1}$ be a sequence with strictly positive terms such that
$$\frac{u_{n+1}}{u_n} = 1 + \frac{\alpha}{n} + \mathcal{O}\Big(\frac{1}{n^2}\Big), \quad \text{for some } \alpha \in \mathbb{R}.$$

Fix $\beta \in \mathbb{R}$ and let
$$\forall n \in \mathbb{N}, \quad v_n = \ln((n+1)^\beta u_{n+1}) - \ln(n^\beta u_n).$$

(1) For which value(s) of β does the series $\sum v_n$ converge?

(2) For each of these values, show that there exists $A > 0$ such that $u_n \sim A n^\alpha$.

Exercise 6.19 :

(1) Show that the series $\sum_n \frac{(-1)^n}{\sqrt{n}}$ converges.

(2) Show that for $n \to \infty$, we have

$$\frac{(-1)^n}{\sqrt{n} + (-1)^n} = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + o\Big(\frac{1}{n}\Big).$$

(3) What is the behavior of the following series?

$$\sum_n \frac{(-1)^n}{\sqrt{n} + (-1)^n}.$$

Exercise 6.20 : Use the comparison theorem (Theorem 6.2.3) and alternating series (Theorem 6.4.2) to determine the behavior of the series $\sum u_n$ where the general term u_n is given by different expressions. Let $\alpha > \beta > 0$.

(1) $u_n = \ln\Big(1 + \dfrac{(-1)^n}{2n+1}\Big)$, (2) $u_n = \dfrac{(-1)^n}{\sqrt{n^\alpha + (-1)^n}}$, (3) $u_n = \dfrac{(-1)^n}{\sqrt{n^\alpha + (-1)^n n^\beta}}$.

Exercise 6.21 :

(1) Justify why the alternating series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$ converges. Does it also converge absolutely?

(2) For every $n \in \mathbb{N}_0$, consider the partial sum

$$S_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}.$$

Show that for $n \in \mathbb{N}$, we have

$$S_{2n-1} < \cos(1) < S_{2n} = S_{2n-1} + \frac{1}{(4n)!}.$$

(3) Deduce that $\cos(1)$ is an irrational number.

(4) Similarly, use the fact that $e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ to show that e is an irrational number.

Exercise 6.22 : Let $(u_n)_{n\geqslant 0}$ be a decreasing sequence with strictly positive general terms with limit 0. Consider the alternating series $\sum_{n\geqslant 0}(-1)^n u_n$ which converges by Theorem 6.4.2. Recall that the remainders are defined by

$$\forall n \in \mathbb{N}_0, \quad R_n = \sum_{k=n+1}^{\infty} (-1)^k u_k.$$

Suppose that the following conditions hold,

$$\forall n \in \mathbb{N}_0, \quad u_{n+2} - 2u_{n+1} + u_n \geqslant 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1.$$

(1) Show that for every $n \geqslant 0$, we have $|R_n| + |R_{n+1}| = u_{n+1}$.

(2) Show that the sequence $(|R_n|)_{n\geqslant 0}$ is decreasing.

(3) Deduce that the following equivalence,

$$R_n \sim \frac{(-1)^{n+1} u_n}{2}, \quad \text{when } n \to \infty.$$

(4) Apply this result to the alternating series $\sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n}$ to find an asymptotic formula for its partial sums. Hint: the limit of this series is known, see Example 6.4.4.

Exercise 6.23 :

(1) Show that the Cauchy product of the following two divergent series

$$(-1-2-2-2-2-\cdots)(-1+2-2+2-2+\cdots)$$

is absolutely convergent.

(2) Consider the alternating series $\sum_{n\geqslant 0} \frac{(-1)^n}{\sqrt{n+1}}$. Show that the Cauchy product of this series with itself is divergent.

Exercise 6.24 : This exercise is a generalization of Theorem 6.6.3. Let $\sum_{n\geqslant 0} a_n$ be an absolutely convergent series and $\sum_{n\geqslant 0} b_n$ be a convergent series with terms in a complete normed algebra $(\mathcal{A}, \cdot, |\cdot|)$. Their Cauchy product is the series $\sum_{n\geqslant 0} c_n$ given by

$$\forall n \in \mathbb{N}_0, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Show that the series $\sum c_n$ is convergent, and its sum equals

$$\sum_{n\geqslant 0} c_n = \left(\sum_{p\geqslant 0} a_p\right)\left(\sum_{q\geqslant 0} b_q\right).$$

Exercise 6.25 : Consider the double sequence $(u_{m,n})_{m,n\geqslant 1}$ defined by

$$\forall m, n \geqslant 1, \quad u_{m,n} = \Big(1 + \frac{1}{m}\Big)^n.$$

(1) Find the iterated limits $\lim_{n\to\infty} \lim_{m\to\infty} u_{m,n}$ and $\lim_{m\to\infty} \lim_{n\to\infty} u_{m,n}$.

(2) Does the limit of the sequence $(u_{m,n})_{m,n\geqslant 1}$ exist? How about the limit of $(u_{n,n})_{n\geqslant 1}$?

Exercise 6.26 : For a real number $k > 1$, let us define the Riemann zeta function

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

(1) Explain briefly why the series $\zeta(k)$ is well defined for $k > 1$.

(2) Show the following identity,

$$\sum_{k=2}^{\infty} (\zeta(k) - 1) = 1.$$

(3) Show the following identity,

$$\sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k} = 1 - \gamma, \quad \text{where} \quad \gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln n\right].$$

Exercise 6.27 : Let $(u_{m,n})_{m,n\geqslant 1}$ be a double sequence in a Banach space $(W, \|\cdot\|)$. The associated double series $\sum_{m,n} u_{m,n}$ is the double sequence $(s_{m,n})_{m,n\geqslant 1}$ given by

$$\forall m, n \geqslant 1, \quad s_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{i,j}.$$

If the limit $\lim_{m,n\to\infty} s_{m,n}$ is well defined, then we say that the double series $\sum_{m,n} u_{m,n}$ converges; if the limit $\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \|u_{i,j}\|$ is well-defined, then we say that the double series $\sum_{m,n} u_{m,n}$ converges absolutely.

(1) Show that the iterated series $\sum_m (\sum_n \|u_{m,n}\|)$ is convergent if and only if the double series $\sum_{m,n} u_{m,n}$ is absolutely convergent.

(2) If the double series $\sum_{m,n} u_{m,n}$ converges absolutely, show that it also converges.

(3) Suppose that the double series $\sum_{m,n} u_{m,n}$ is absolutely convergent. We define

$$\forall n \geqslant 2, \quad c_n = \sum_{i=1}^{n-1} u_{i,(n-1)} = u_{1,(n-1)} + u_{2,(n-2)} + \cdots + u_{(n-1),1}.$$

Show that the series $\sum_n c_n$ is also absolutely convergent and

$$\sum_{n\geqslant 2} c_n = \sum_{m,n\geqslant 1} u_{m,n}.$$

(4) Let $(u_{m,n})_{m,n\geqslant 1}$ be the double sequence given by

$$\forall m,n\geqslant 1,\quad u_{m,n}=\begin{cases}+1 & \text{if } m-n=1,\\ -1 & \text{if } m-n=-1,\\ 0 & \text{otherwise.}\end{cases}$$

(a) Show that both iterated series $\sum_n(\sum_m u_{m,n})$ and $\sum_m(\sum_n u_{m,n})$ converge, but their limits are not equal.

(b) Show that the double series $\sum_{m,n} u_{m,n}$ does not converge.

(c) Show that the limit $\lim_{n\to\infty} s_{n,n}$ exists.

(5) Show that if the double series and one of the iterated series associated to $(u_{m,n})_{m,n\geqslant 1}$ converge, then these two limits are equal.

(6) Show that the convergence of the double series does not imply the convergence of the iterated series.

(7) Show that the convergence of the double series does not imply that $\lim_{n\to\infty} u_{m,n}=0$ for each $m\geqslant 1$.

Exercise 6.28 : Let $u_n=\frac{(-1)^n}{n}$ for $n\in\mathbb{N}$.

(1) Show that $\prod_{n\geqslant 2}(1+u_n)$ converges with limit 1.

(2) Does the series $\sum_{n\geqslant 2} u_n$ converge, absolutely converge, or conditionally converge?

Exercise 6.29 : Let $(u_n)_{n\geqslant 1}$ be a sequence defined by

$$\forall n\in\mathbb{N},\quad u_{2n-1}=-\frac{1}{\sqrt{n}},\quad u_{2n}=\frac{1}{\sqrt{n}}+\frac{1}{n}.$$

(1) Show that $\prod_{n\geqslant 2}(1+u_n)$ converges to a nonzero limit.

(2) Does the series $\sum_{n\geqslant 2} u_n$ converge, absolutely converge, or conditionally converge?

Exercise 6.30 : Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers with $a_n>-1$ for all $n\in\mathbb{N}$.

(1) Show that if the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is convergent.

(2) Suppose that the series $\sum_{n\geqslant 1} a_n$ is convergent. Show that the infinite product $\prod(1+a_n)$ is convergent if and only if the series $\sum_{n=1}^{\infty} a_n^2$ is convergent. Hint: see below[1].

Exercise 6.31 : Justify the convergence of the following infinite products, and prove their limits.

(1) $\displaystyle\prod_{n=1}^{\infty}\Big(1-\frac{1}{n^2}\Big)=\frac{1}{2}.$

(2) $\displaystyle\prod_{n=1}^{\infty}\Big(1-\frac{1}{4n^2}\Big)=\frac{2}{\pi}.$

[1]Show that there exist positive constants A and B such that if $|x|<\frac{1}{2}$, then $Ax^2\leqslant x-\log(1+x)\leqslant Bx^2$.

Exercise 6.32 : Let $P_n = \prod_{k=2}^{n} \left(1 + \frac{(-1)^k}{\sqrt{k}}\right)$ for $n \geqslant 2$. Show that there exists $\lambda \in \mathbb{R}$ such that $P_n \sim \frac{e^\lambda}{\sqrt{n}}$ when $n \to \infty$.

Exercise 6.33 : Let $\mathcal{P}$ be the set of all the primes. In this exercise, we will prove $\sum_{p \in \mathcal{P}} \frac{1}{p}$ is divergent.

(1) Show that for $s > 1$, we have

$$-\sum_{p \in \mathcal{P}} \log\left(1 - \frac{1}{p^s}\right) = \log \zeta(s).$$

(2) Deduce that there exists $M > 0$ such that for any $s > 1$, we have

$$\left| \sum_{p \in \mathcal{P}} \frac{1}{p^s} - \log \zeta(s) \right| < M.$$

(3) Show that as $s \to 1+$, we have $\zeta(s) \to +\infty$.

(4) Conclude that $\sum_{p \in \mathcal{P}} \frac{1}{p}$ is divergent.