Chapter 7: Complements on Riemann Integrals

Exercise 7.1 : Let $f : [1, +\infty) \to \mathbb{R}$ be a non-negative continuous function. Suppose that it is integrable on $[1, +\infty)$.

(1) If the limit $\lim_{x\to+\infty} f(x)$ exists, show that it is 0.

(2) Give a counterexample to justify that $\lim_{x\to+\infty} f(x)$ may fail to exist.

(3) Give a counterexample to justify that f may fail to be bounded on $[1, +\infty)$.

(4) Suppose that f is uniformly continuous on $[1, +\infty)$, show that $\lim_{x\to+\infty} f(x) = 0$.

(5) Suppose that f is Lipschitz continuous on $[1, +\infty)$, show that $\liminf_{n\to+\infty} \sqrt{n} f(n) = 0$.

Exercise 7.2 : Let $\alpha, \beta \in \mathbb{R}$. Depending on the values of α and β, find the behavior of the integral

$$\int_I \frac{\mathrm{d}x}{x^\alpha |\ln x|^\beta},$$

where we take

(1) $I = (0, \frac{1}{2}]$; (2) $I = [\frac{1}{2}, 1)$; (3) $I = (1, 2]$; (4) $I = [2, +\infty)$.

Exercise 7.3 : Let $M \in \mathbb{R}$ and $f : [M, +\infty) \to \mathbb{R}_+$ be a piecewise continuous function.

(1) Suppose that f is integrable on $[M, +\infty)$. Show that

$$\int_n^{2n} f(t)\,\mathrm{d}t \xrightarrow[n\to\infty]{} 0.$$

(2) Suppose that f is integrable and decreasing on $[M, +\infty)$. Show that $f(x) = o(\frac{1}{x})$ when $x \to \infty$.

Exercise 7.4 : Find the behavior of the following integrals,

(1) $\displaystyle\int_0^1 \frac{\sinh(\sqrt{t})\ln t}{\sqrt{t} - \sin t}$,

(2) $\displaystyle\int_1^\infty \frac{\ln(t^2 - t)}{(1+t)^2}$,

(3) $\displaystyle\int_0^\infty \frac{\mathrm{d}t}{e^t - 1}$,

(4) $\displaystyle\int_0^\infty \frac{te^{-\sqrt{t}}}{1+t^2}\,\mathrm{d}t$,

(5) $\displaystyle\int_0^\infty \frac{\ln t}{1+t^2}\,\mathrm{d}t$,

(6) $\displaystyle\int_0^\infty \frac{\sqrt{|\ln t|}}{(t-1)\sqrt{t}}\,\mathrm{d}t$.

(7) $\displaystyle\int_0^1 \frac{\mathrm{d}t}{1-\sqrt{t}}$,

(8) $\displaystyle\int_0^\infty \left(1 + t\ln\left(\frac{t}{t+1}\right)\right)\mathrm{d}t$.

第七章 ：黎曼積分的補充

習題 7.1 ： 令 $f : [1, +\infty) \to \mathbb{R}$ 為非負連續函數。假設他在 $[1, +\infty)$ 上是可積的。

(1) 如果極限 $\lim_{x\to+\infty} f(x)$ 存在，證明這個極限為 0。

(2) 請給出反例來解釋為什麼 $\lim_{x\to+\infty} f(x)$ 可能會不存在。

(3) 請給出反例來解釋為什麼 f 可能在 $[1, +\infty)$ 上不會有界。

(4) 假設 f 在 $[1, +\infty)$ 上是均勻連續的，證明 $\lim_{x\to+\infty} f(x) = 0$。

(5) 假設 f 在 $[1, +\infty)$ 上是 Lipschitz 連續的，證明 $\liminf_{n\to+\infty} \sqrt{n} f(n) = 0$。

習題 7.2 ： 令 $\alpha, \beta \in \mathbb{R}$。根據 α 和 β 所取的值，找出下列積分的行為：

$$\int_I \frac{\mathrm{d}x}{x^\alpha |\ln x|^\beta},$$

其中我們選擇

(1) $I = (0, \frac{1}{2}]$; (2) $I = [\frac{1}{2}, 1)$; (3) $I = (1, 2]$; (4) $I = [2, +\infty)$.

習題 7.3 ： 令 $M \in \mathbb{R}$ 以及 $f : [M, +\infty) \to \mathbb{R}_+$ 為片段連續函數。

(1) 假設 f 在 $[M, +\infty)$ 上可積。證明

$$\int_n^{2n} f(t)\,\mathrm{d}t \xrightarrow[n\to\infty]{} 0.$$

(2) 假設 f 在 $[M, +\infty)$ 上可積且遞減。證明 $f(x) = o(\frac{1}{x})$ 當 $x \to \infty$。

習題 7.4 ： 請找出下列積分的行為：

(1) $\displaystyle\int_0^1 \frac{\sinh(\sqrt{t})\ln t}{\sqrt{t} - \sin t}$,

(2) $\displaystyle\int_1^\infty \frac{\ln(t^2 - t)}{(1+t)^2}$,

(3) $\displaystyle\int_0^\infty \frac{\mathrm{d}t}{e^t - 1}$,

(4) $\displaystyle\int_0^\infty \frac{te^{-\sqrt{t}}}{1+t^2}\,\mathrm{d}t$,

(5) $\displaystyle\int_0^\infty \frac{\ln t}{1+t^2}\,\mathrm{d}t$,

(6) $\displaystyle\int_0^\infty \frac{\sqrt{|\ln t|}}{(t-1)\sqrt{t}}\,\mathrm{d}t$.

(7) $\displaystyle\int_0^1 \frac{\mathrm{d}t}{1-\sqrt{t}}$,

(8) $\displaystyle\int_0^\infty \left(1 + t\ln\left(\frac{t}{t+1}\right)\right)\mathrm{d}t$.

Exercise 7.5 : Determine the behavior of each of the following integrals, and their values in the case that they are well defined,

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos u}{\sqrt{\sin u}}\,\mathrm{d}u$, (2) $\displaystyle\int_1^{\infty} \ln\Big(1+\frac{1}{t^2}\Big)\,\mathrm{d}t$.

Exercise 7.6 : Let $f \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$ such that both

$$\int_0^{\infty} f(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^{\infty} f'(t)^2\,\mathrm{d}t$$

are convergent.

(1) Let F be the primitive of f such that $F(0) = 0$. Show that F is of class $\mathcal{C}^2$.

(2) For $x \geqslant 0$, show that

$$F(x+1) = F(x) + f(x) + R(x), \quad \text{where} \quad R(x) = \int_x^{x+1} (x+1-t) f'(t)\,\mathrm{d}t.$$

(3) Show that $R(x) \to 0$ when $x \to +\infty$.

(4) Deduce that $f(x) \to 0$ when $x \to +\infty$.

Exercise 7.7 : For every $n \in \mathbb{N}$, define

$$I_n = \int_0^{\infty} \frac{\mathrm{d}t}{(1+t^2)^n}.$$

(1) Show that for every $n \geqslant 1$, the integral I_n is well defined.

(2) For each $n \geqslant 1$, find a relation between I_n and I_{n+1}.

(3) Find the value of I_n for $n \geqslant 1$.

Exercise 7.8 : Let $n \in \mathbb{N}$ and $f : [1, +\infty) \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. Recall the Euler's summation formula that we saw in Corollary 5.2.23,

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + \int_1^n f'(x)\Big(\{x\} - \frac{1}{2}\Big)\,\mathrm{d}x + \frac{1}{2}[f(n) + f(1)].$$

Recall we had defined the 1-periodic functions $(G_p)_{p\geqslant 1}$ in Problem 6 of the midterm exam.

(1) Check that $G_1(x) = \{x\} - \frac{1}{2}$.

(2) Check that for $k \geqslant 2$ and $x \in \mathbb{R}$,

$$G_k(x) = k \int_1^x G_{k-1}(t)\,\mathrm{d}t + B_k,$$

where $B_k = G_k(0)$.

習題 7.5 ： 請找出下列積分的行為，在當他們有定義時，也求出他們的值：

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos u}{\sqrt{\sin u}}\,\mathrm{d}u$, (2) $\displaystyle\int_1^{\infty} \ln\Big(1+\frac{1}{t^2}\Big)\,\mathrm{d}t$.

習題 7.6 ： 令 $f \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$ 使得

$$\int_0^{\infty} f(t)\,\mathrm{d}t \quad \text{以及} \quad \int_0^{\infty} f'(t)^2\,\mathrm{d}t$$

都會收斂。

(1) 令 F 為 f 的原函數，滿足 $F(0) = 0$。證明 F 是 $\mathcal{C}^2$ 類的。

(2) 對於 $x \geqslant 0$，證明

$$F(x+1) = F(x) + f(x) + R(x), \quad \text{其中} \quad R(x) = \int_x^{x+1} (x+1-t) f'(t)\,\mathrm{d}t.$$

(3) 證明當 $x \to +\infty$ 時，我們有 $R(x) \to 0$。

(4) 推得當 $x \to +\infty$ 時，我們有 $f(x) \to 0$。

習題 7.7 ： 對於每個 $n \in \mathbb{N}$，定義

$$I_n = \int_0^{\infty} \frac{\mathrm{d}t}{(1+t^2)^n}.$$

(1) 證明對於每個 $n \geqslant 1$，積分 I_n 定義良好。

(2) 對於每個 $n \geqslant 1$，求 I_n 和 I_{n+1} 之間的關係式。

(3) 對於 $n \geqslant 1$，求 I_n 的值。

習題 7.8 ： 令 $n \in \mathbb{N}$ 以及 $f : [1, +\infty) \to \mathbb{R}$ 為 $\mathcal{C}^\infty$ 函數。我們回顧系理 5.2.23 中的 Euler 求和公式：

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + \int_1^n f'(x)\Big(\{x\} - \frac{1}{2}\Big)\,\mathrm{d}x + \frac{1}{2}[f(n) + f(1)].$$

同時我們也不要忘記，在期中考的問題 6 中，我們定義週期為 1 的函數 $(G_p)_{p\geqslant 1}$。

(1) 檢查 $G_1(x) = \{x\} - \frac{1}{2}$。

(2) 檢查對於 $k \geqslant 2$ 以及 $x \in \mathbb{R}$，我們有

$$G_k(x) = k \int_1^x G_{k-1}(t)\,\mathrm{d}t + B_k,$$

其中 $B_k = G_k(0)$。

(3) Show that for any $m \geqslant 1$, we have

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + \frac{1}{(2m+1)!}\int_1^n G_{2m+1}(x) f^{(2m+1)}(x)\,\mathrm{d}x + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!}\big[f^{(2r-1)}(n) - f^{(2r-1)}(1)\big] + \frac{1}{2}\big[f(n) + f(1)\big].$$

(4) Fix $m \geqslant 1$. Suppose that $f^{(2m+1)}$ is integrable on $[1, +\infty)$.

(a) Show that the following integral is well defined,

$$\int_1^{\infty} f^{(2m+1)}(x) G_{2m+1}(x)\,\mathrm{d}x$$

(b) Deduce that

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + C + E(n),$$

where

$$C = \frac{1}{2}f(1) - \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(1) + \frac{1}{(2m+1)!}\int_1^{\infty} G_{2m+1}(x) f^{(2m+1)}(x)\,\mathrm{d}x$$

and

$$E(n) = \frac{1}{2}f(n) + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(n) - \frac{1}{(2m+1)!}\int_n^{\infty} G_{2m+1}(x) f^{(2m+1)}(x)\,\mathrm{d}x.$$

(5) Applications: prove the following asymptotics.

(a) For $s > 1$ and $N \geqslant 1$,

$$\sum_{k=1}^{n} \frac{1}{k^s} = \zeta(s) - \frac{1}{(s-1)n^{s-1}} + \frac{1}{2n^s} - \sum_{r=1}^{N} \frac{B_{2r}}{(2r)!} \cdot \frac{(s+2r-2)_{2r-1}}{n^{s+2r-1}} + \mathcal{O}\Big(\frac{1}{n^{s+2N}}\Big) \quad \text{when} \quad n \to \infty,$$

where $(a)_p$ is the falling factorial symbol defined by

$$\forall a \in \mathbb{R}, p \in \mathbb{N}, \quad (a)_p := a \cdot (a-1) \cdot (a-2) \cdots (a-p+1).$$

(b) For $N \geqslant 1$,

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + \frac{1}{2n} - \sum_{k \geqslant 1}^{N} \frac{B_{2k}}{2kn^{2k}} + \mathcal{O}\Big(\frac{1}{n^{2N+1}}\Big) \quad \text{when} \quad n \to \infty.$$

This is the last expression in Example 6.2.9.

(c) For $N \geqslant 1$,

$$\sum_{k=1}^{n} \ln k = (n+\frac{1}{2}) \ln n - n + \frac{1}{2}\ln(2\pi) + \sum_{r=1}^{N} \frac{B_{2r}}{2r(2r-1)n^{2r-1}} + \mathcal{O}\Big(\frac{1}{n^{2N}}\Big) \quad \text{when} \quad n \to \infty.$$

(3) 證明對於任意 $m \geqslant 1$，我們有

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + \frac{1}{(2m+1)!}\int_1^n G_{2m+1}(x) f^{(2m+1)}(x)\,\mathrm{d}x + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!}\big[f^{(2r-1)}(n) - f^{(2r-1)}(1)\big] + \frac{1}{2}\big[f(n) + f(1)\big].$$

(4) 固定 $m \geqslant 1$。假設 $f^{(2m+1)}$ 在 $[1, +\infty)$ 上是可積的。

(a) 證明下列積分定義良好：

$$\int_1^{\infty} f^{(2m+1)}(x) G_{2m+1}(x)\,\mathrm{d}x$$

(b) 推得

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + C + E(n),$$

其中

$$C = \frac{1}{2}f(1) - \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(1) + \frac{1}{(2m+1)!}\int_1^{\infty} G_{2m+1}(x) f^{(2m+1)}(x)\,\mathrm{d}x$$

以及

$$E(n) = \frac{1}{2}f(n) + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(n) - \frac{1}{(2m+1)!}\int_n^{\infty} G_{2m+1}(x) f^{(2m+1)}(x)\,\mathrm{d}x.$$

(5) 應用：證明下列漸進式。

(a) 對於 $s > 1$ 以及 $N \geqslant 1$，我們有

$$\sum_{k=1}^{n} \frac{1}{k^s} = \zeta(s) - \frac{1}{(s-1)n^{s-1}} + \frac{1}{2n^s} - \sum_{r=1}^{N} \frac{B_{2r}}{(2r)!} \cdot \frac{(s+2r-2)_{2r-1}}{n^{s+2r-1}} + \mathcal{O}\Big(\frac{1}{n^{s+2N}}\Big) \quad 當 \quad n \to \infty,$$

其中 $(a)_p$ 是遞降階層記號，定義做：

$$\forall a \in \mathbb{R}, p \in \mathbb{N}, \quad (a)_p := a \cdot (a-1) \cdot (a-2) \cdots (a-p+1).$$

(b) 對於 $N \geqslant 1$，我們有

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + \frac{1}{2n} - \sum_{k \geqslant 1}^{N} \frac{B_{2k}}{2kn^{2k}} + \mathcal{O}\Big(\frac{1}{n^{2N+1}}\Big) \quad 當 \quad n \to \infty.$$

這是範例 6.2.9 中最後所提到的式子。

(c) 對於 $N \geqslant 1$，我們有

$$\sum_{k=1}^{n} \ln k = (n+\frac{1}{2}) \ln n - n + \frac{1}{2}\ln(2\pi) + \sum_{r=1}^{N} \frac{B_{2r}}{2r(2r-1)n^{2r-1}} + \mathcal{O}\Big(\frac{1}{n^{2N}}\Big) \quad 當 \quad n \to \infty.$$

Moreover, deduce the Stirling's approximation formula,

$$n! = \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n \Big(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}-\frac{571}{2488320n^4}+\mathcal{O}\Big(\frac{1}{n^5}\Big)\Big) \quad \text{when} \quad n\to\infty.$$

Exercise 7.9 : We want to study properties of the Gamma function defined in Example 7.1.21, and find a characterization of it.

(1) Let $f, g : (0,+\infty) \to \mathbb{R}$ be functions. Let $p, q > 1$ be such that $\frac{1}{p}+\frac{1}{q} = 1$. Suppose that both the functions $|f|^p$ and $|g|^q$ are integrable on $(0,+\infty)$. Show that the product fg is integrable on $(0,+\infty)$ and

$$\int_0^\infty |fg| \leqslant \Big(\int_0^\infty |f|^p\Big)^{1/p}\Big(\int_0^\infty |g|^q\Big)^{1/q}.$$

(2) Let $p, q > 1$ be such that $\frac{1}{p}+\frac{1}{q} = 1$. Show that

$$\Gamma\Big(\frac{x}{p}+\frac{y}{q}\Big) \leqslant \Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}, \quad \forall x, y > 0.$$

(3) Show that $x \mapsto \ln\Gamma(x)$ is convex on $(0,+\infty)$.

(4) Show that for all $x \in (0,1)$ and $n \in \mathbb{N}$, we have

$$x\ln(n) \leqslant \ln\Gamma(n+x+1) - \ln(n!) \leqslant x\ln(n+1).$$

(5) Deduce that for $0 < x < 1$, we have

$$0 \leqslant \ln\Gamma(x) - \ln\Big(\frac{n^x n!}{x(x+1)\dots(x+n)}\Big) \leqslant x\ln\Big(1+\frac{1}{n}\Big).$$

(6) Deduce that

$$\Gamma(x) = \lim_{n\to\infty}\frac{n^x n!}{x(x+1)\dots(x+n)}$$

for $x \in (0,1)$, and also for all $x > 0$.

(7) Conclude that for $x > 0$,

$$\Gamma(x) = \frac{1}{x}\prod_{n=1}^\infty \frac{(1+1/n)^x}{1+x/n}.$$

(8) Prove the following characterization of the Gamma function. Let $f : (0,+\infty) \to \mathbb{R}$ be a function satisfying the following three properties,

(i) $\ln f(x)$ is convex;

(ii) $f(x) = (x-1)f(x-1)$ for all $x > 1$;

(iii) $f(1) = 1$.

Then, $f(x) = \Gamma(x)$. Hint: see below[1].

[1]Repeat the arguments in (4), (5), and (6), using f instead of Γ.

並推得 Stirling 近似公式：

$$n! = \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n \Big(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}-\frac{571}{2488320n^4}+\mathcal{O}\Big(\frac{1}{n^5}\Big)\Big) \quad 當 \quad n\to\infty.$$

習題 7.9 ： 我們想要討論範例 7.1.21 中定義的 Gamma 函數 Γ 的性質，並找出方法來刻劃他。

(1) 令 $f, g : (0,+\infty) \to \mathbb{R}$ 為函數。令 $p, q > 1$ 滿足 $\frac{1}{p}+\frac{1}{q} = 1$。假設函數 $|f|^p$ 和 $|g|^q$ 兩者在 $(0,+\infty)$ 上都是可積的。證明乘積 fg 在 $(0,+\infty)$ 上是可積的，而且我們有

$$\int_0^\infty |fg| \leqslant \Big(\int_0^\infty |f|^p\Big)^{1/p}\Big(\int_0^\infty |g|^q\Big)^{1/q}.$$

(2) 令 $p, q > 1$ 滿足 $\frac{1}{p}+\frac{1}{q} = 1$。證明

$$\Gamma\Big(\frac{x}{p}+\frac{y}{q}\Big) \leqslant \Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}, \quad \forall x, y > 0.$$

(3) 證明 $x \mapsto \ln\Gamma(x)$ 在 $(0,+\infty)$ 上是個凸函數。

(4) 證明對於所有 $x \in (0,1)$ 以及 $n \in \mathbb{N}$，我們有

$$x\ln(n) \leqslant \ln\Gamma(n+x+1) - \ln(n!) \leqslant x\ln(n+1).$$

(5) 推得對於 $0 < x < 1$，我們有

$$0 \leqslant \ln\Gamma(x) - \ln\Big(\frac{n^x n!}{x(x+1)\dots(x+n)}\Big) \leqslant x\ln\Big(1+\frac{1}{n}\Big).$$

(6) 推得

$$\Gamma(x) = \lim_{n\to\infty}\frac{n^x n!}{x(x+1)\dots(x+n)}$$

對於 $x \in (0,1)$ 成立，也會對於所有 $x > 0$ 成立。

(7) 總結對於 $x > 0$，我們有

$$\Gamma(x) = \frac{1}{x}\prod_{n=1}^\infty \frac{(1+1/n)^x}{1+x/n}.$$

(8) 證明下面這個對於 Γ 函數的刻劃。令 $f : (0,+\infty) \to \mathbb{R}$ 為滿足下列三個性質的函數：

(i) $\ln f(x)$ 是凸函數；

(ii) $f(x) = (x-1)f(x-1)$ 對於所有 $x > 1$；

(iii) $f(1) = 1$。

那麼，我們有 $f(x) = \Gamma(x)$。提示：如下[1]。

[1]把 (4)、(5) 和 (6) 敘述裡面的 Γ 換成 f。

Exercise 7.10 :

(1) Prove that the following improper integral is convergent,

$$I := \int_0^{\pi} \ln(\sin x)\,\mathrm{d}x.$$

(2) Show that

$$\int_0^{\pi/2} \log(\sin t)\,\mathrm{d}t = \int_0^{\pi/2} \log(\cos t)\,\mathrm{d}t.$$

(3) Compute the value of I by using the change of variables $x = 2t$ and applying the identity $\sin(2t) = 2\sin t \cos t$.

Exercise 7.11 : Let $f \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ such that $\int_0^{\infty} f(t)\,\mathrm{d}t$ converges.

(1) Find the limit when $x \to +\infty$ of the following integral,

$$\int_{x/2}^{x} f(t)\,\mathrm{d}t.$$

(2) Suppose that f is non-negative and is decreasing. Show that $f(x) = o(\frac{1}{x})$ when $x \to +\infty$.

(3) When the assumption in (2) is not satisfied, find a counterexample.

Exercise 7.12 (Cauchy principal value) **:** Let $I = [a, b]$ be a segment and $c \in (a, b)$. Let $f : I \backslash \{c\} \to \mathbb{R}$ be a piecewise continuous function. If the following limit exists,

$$\lim_{\varepsilon \to 0+} \Big(\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon}^{b} f(x)\,\mathrm{d}x \Big),$$

then we call the above limit *Cauchy principal value* of the improper integral, denoted by

$$\mathrm{p.\,v.} \int_a^b f(x)\,\mathrm{d}x := \lim_{\varepsilon \to 0+} \Big(\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon}^{b} f(x)\,\mathrm{d}x \Big).$$

Find the Cauchy principal of the following improper integrals,

(1) $\displaystyle\int_{-a}^{a} \frac{\mathrm{d}x}{x}$, $a > 0$;

(2) $\displaystyle\int_{a}^{b} \frac{\mathrm{d}x}{x - c}$, $a < c < b$.

習題 7.10 ：

(1) 證明下面這個瑕積分會收斂：

$$I := \int_0^{\pi} \ln(\sin x)\,\mathrm{d}x.$$

(2) 證明

$$\int_0^{\pi/2} \log(\sin t)\,\mathrm{d}t = \int_0^{\pi/2} \log(\cos t)\,\mathrm{d}t.$$

(3) 使用變數變換 $x = 2t$ 以及關係式 $\sin(2t) = 2\sin t \cos t$ 來計算 I 的值。

習題 7.11 ： 令 $f \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ 使得 $\int_0^{\infty} f(t)\,\mathrm{d}t$ 會收斂。

(1) 找出下列積分在 $x \to +\infty$ 的極限：

$$\int_{x/2}^{x} f(t)\,\mathrm{d}t.$$

(2) 假設 f 是非負且遞減的。證明當 $x \to +\infty$ 時，我們有 $f(x) = o(\frac{1}{x})$。

(3) 當 (2) 中的假設沒有滿足時，找一個反例。

習題 7.12 【柯西主值】**：** 令 $I = [a, b]$ 為區間以及 $c \in (a, b)$。令 $f : I \backslash \{c\} \to \mathbb{R}$ 為片段連續函數。如果下列極限存在：

$$\lim_{\varepsilon \to 0+} \Big(\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon}^{b} f(x)\,\mathrm{d}x \Big),$$

則我們把他稱作是瑕積分的柯西主值記作

$$\mathrm{p.\,v.} \int_a^b f(x)\,\mathrm{d}x := \lim_{\varepsilon \to 0+} \Big(\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon}^{b} f(x)\,\mathrm{d}x \Big).$$

求下列瑕積分的柯西主值：

(1) $\displaystyle\int_{-a}^{a} \frac{\mathrm{d}x}{x}$, $a > 0$;

(2) $\displaystyle\int_{a}^{b} \frac{\mathrm{d}x}{x - c}$, $a < c < b$.

Exercise 7.13 : Let $f : [1, +\infty) \to \mathbb{R}$ be a continuous function. Show that the following two properties are equivalent. Hint: integration by parts.

(1) When $x \to +\infty$, the following limit exists,
$$\frac{1}{x}\int_1^x f(t)\,\mathrm{d}t.$$

(2) When $x \to +\infty$, the following limit exists,
$$x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t.$$

Show that when the above two properties are satisfied, the two limits are equal.

Exercise 7.14 : Suppose that $f : [a, \infty) \to \mathbb{R}$ is continuous and decreases to 0 as $x \to +\infty$.

(1) Show that if $\int_a^\infty |f(x)\cos x|\,\mathrm{d}x$ converges, then both of the integrals $\int_a^\infty f(x)\sin x\,\mathrm{d}x$ and $\int_a^\infty f(x)\cos x\,\mathrm{d}x$ are absolutely convergent.

(2) Show that if $\int_a^\infty |f(x)\cos x|\,\mathrm{d}x$ diverges, then both of the integrals $\int_a^\infty f(x)\sin x\,\mathrm{d}x$ and $\int_a^\infty f(x)\cos x\,\mathrm{d}x$ are conditionally convergent.

Exercise 7.15 : Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $\ell := \lim_{x\to\infty} f(x)$ exists. Let $a < b$. We want to study the following integral
$$I := \int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x.$$

Let $g : (0, \infty) \to \mathbb{R}$ be defined by
$$\forall x \in (0, \infty), \quad g(x) = \frac{f(ax) - f(bx)}{x}.$$

(1) (a) Assume that f is differentiable at $0+$. Find an equivalence of g at $0+$ and deduce that g is integrable on $(0, 1]$.

(b) Find a function f such that g is not integrable on $(0, 1]$.

(c) Let $\varepsilon > 0$. Show that if $f(x) = \ell + o(x^{-\varepsilon})$ when $x \to +\infty$, then g is integrable on $[1, +\infty)$.

(2) For $y \geqslant x > 0$, let
$$I(x, y) := \int_x^y \frac{f(at) - f(bt)}{t}\,\mathrm{d}t.$$

(a) Make a change of variables to show that for $y \geqslant x > 0$, we have
$$I(x, y) = \int_{ax}^{bx} \frac{f(u)}{u}\,\mathrm{d}u - \int_{ay}^{by} \frac{f(u)}{u}\,\mathrm{d}u.$$

習題 7.13 ： 令 $f : [1, +\infty) \to \mathbb{R}$ 為連續函數。證明下面兩個性質是等價的。提示：分部積分。

(1) 當 $x \to +\infty$ 時，下列極限存在：
$$\frac{1}{x}\int_1^x f(t)\,\mathrm{d}t.$$

(2) 當 $x \to +\infty$ 時，下列極限存在：
$$x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t.$$

證明如果上面兩個性質成立的話，則兩個極限相同。

習題 7.14 ： 假設 $f : [a, \infty) \to \mathbb{R}$ 是連續的，且當 $x \to +\infty$ 時會遞減到 0。

(1) 證明如果 $\int_a^\infty |f(x)\cos x|\,\mathrm{d}x$ 收斂，那麼 $\int_a^\infty f(x)\sin x\,\mathrm{d}x$ 和 $\int_a^\infty f(x)\cos x\,\mathrm{d}x$ 兩個積分都會絕對收斂。

(2) 證明如果 $\int_a^\infty |f(x)\cos x|\,\mathrm{d}x$ 發散，那麼 $\int_a^\infty f(x)\sin x\,\mathrm{d}x$ 和 $\int_a^\infty f(x)\cos x\,\mathrm{d}x$ 兩個積分都會條件收斂。

習題 7.15 ： 令 $f : [0, \infty) \to \mathbb{R}$ 為連續函數，使得 $\ell := \lim_{x\to\infty} f(x)$ 存在。令 $a < b$。我們想要討論下面這個積分：
$$I := \int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x.$$

令 $g : (0, \infty) \to \mathbb{R}$，定義做
$$\forall x \in (0, \infty), \quad g(x) = \frac{f(ax) - f(bx)}{x}.$$

(1) (a) 假設 f 在 $0+$ 可微。找出 g 在 $0+$ 附近的等價式，並推得 g 在 $(0, 1]$ 上是可積的。

(b) 找出一個函數 f 使得 g 在 $(0, 1]$ 上不可積。

(c) 令 $\varepsilon > 0$。證明如果 $f(x) = \ell + o(x^{-\varepsilon})$ 當 $x \to +\infty$，那麼 g 在 $[1, +\infty)$ 上是可積的。

(2) 對於 $y \geqslant x > 0$，令
$$I(x, y) := \int_x^y \frac{f(at) - f(bt)}{t}\,\mathrm{d}t.$$

(a) 使用變數變換來證明對於 $y \geqslant x > 0$，我們有
$$I(x, y) = \int_{ax}^{bx} \frac{f(u)}{u}\,\mathrm{d}u - \int_{ay}^{by} \frac{f(u)}{u}\,\mathrm{d}u.$$

(b) 證明
$$\lim_{x\to 0+}\int_{ax}^{bx} \frac{f(u)}{u}\,\mathrm{d}u = f(0)\ln\Big(\frac{b}{a}\Big) \quad 以及 \quad \lim_{y\to +\infty}\int_{ay}^{by} \frac{f(u)}{u}\,\mathrm{d}u = \ell\ln\Big(\frac{b}{a}\Big).$$

(b) Show that

$$\lim_{x\to 0+}\int_{ax}^{bx}\frac{f(u)}{u}\,\mathrm{d}u = f(0)\ln\left(\frac{b}{a}\right) \quad \text{and} \quad \lim_{y\to+\infty}\int_{ay}^{by}\frac{f(u)}{u}\,\mathrm{d}u = \ell\ln\left(\frac{b}{a}\right).$$

(c) Deduce the value of the limit $\lim_{x\to 0+}\lim_{y\to+\infty} I(x,y)$.

(3) Can you explain why when f is only continuous, we may find a function f such that g is not integrable on $(0,1]$, while the limit in (2c) is well defined?

(4) Show that the function $x\mapsto \frac{1}{x}(e^{-ax}-e^{-bx})$ is integrable on $(0,+\infty)$ and find the value of the following integral,

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x.$$

Exercise 7.16 : Find the behavior of the following integrals,

(1) $\displaystyle\int_1^\infty \frac{\sin t}{\sqrt{t}}\,\mathrm{d}t$,

(2) $\displaystyle\int_1^\infty \ln\left(1+\frac{\sin t}{\sqrt{t}}\right)\mathrm{d}t$,

(3) $\displaystyle\int_0^\infty \cos(t^2+t)\,\mathrm{d}t$,

(4) $\displaystyle\int_1^\infty \frac{\sin t}{t^\alpha}\ln\left(\frac{t+1}{t-1}\right)\mathrm{d}t,\ \alpha\in\mathbb{R}$.

Exercise 7.17 : Consider the function $f:\mathbb{R}_+\to\mathbb{R}$ defined by

$$f(0)=0, \quad \text{and} \quad f(t)=\frac{\sin t - t}{t^2}, \quad \forall t>0.$$

(1) Check that f is continuous on $\mathbb{R}_+$.

(2) Find the limit when $x\to 0+$ of

$$\int_x^{3x}\frac{\sin t}{t^2}\,\mathrm{d}t.$$

(3) Explain why the following integral converges,

$$\int_0^\infty \frac{\sin^3 t}{t^2}\,\mathrm{d}t.$$

(4) Find the value of the integral in the previous question. Hint: see below[2].

[2]Note that $4\sin^3 t = 3\sin t - \sin(3t)$.

(c) 推得極限 $\lim_{x\to 0+}\lim_{y\to+\infty} I(x,y)$ 的值。

(3) 你有辦法解釋為什麼當 f 只有假設是連續時，我們可以找到函數 f 使得 g 在 $(0,1]$ 上不可積，但是 (2c) 中的極限卻是定義良好的嗎？

(4) 證明函數 $x\mapsto \frac{1}{x}(e^{-ax}-e^{-bx})$ 在 $(0,+\infty)$ 上是可積的，並且求下列積分：

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,\mathrm{d}x.$$

習題 7.16 ： 求下列積分的行為：

(1) $\displaystyle\int_1^\infty \frac{\sin t}{\sqrt{t}}\,\mathrm{d}t$,

(2) $\displaystyle\int_1^\infty \ln\left(1+\frac{\sin t}{\sqrt{t}}\right)\mathrm{d}t$,

(3) $\displaystyle\int_0^\infty \cos(t^2+t)\,\mathrm{d}t$,

(4) $\displaystyle\int_1^\infty \frac{\sin t}{t^\alpha}\ln\left(\frac{t+1}{t-1}\right)\mathrm{d}t,\ \alpha\in\mathbb{R}$.

習題 7.17 ： 考慮函數 $f:\mathbb{R}_+\to\mathbb{R}$ 定義做：

$$f(0)=0, \quad \text{以及} \quad f(t)=\frac{\sin t - t}{t^2}, \quad \forall t>0.$$

(1) 檢查 f 在 $\mathbb{R}_+$ 上是連續的。

(2) 求當 $x\to 0+$ 時下面式子的極限：

$$\int_x^{3x}\frac{\sin t}{t^2}\,\mathrm{d}t.$$

(3) 解釋為什麼下列積分收斂：

$$\int_0^\infty \frac{\sin^3 t}{t^2}\,\mathrm{d}t.$$

(4) 求上題中積分的值。提示：見下方[2]。

[2]注意到 $4\sin^3 t = 3\sin t - \sin(3t)$。

Exercise 7.18 : Consider the function $f : [0, \pi] \to \mathbb{R}$ defined by

$$\forall x \in \mathbb{R}_+, \quad f(x) = \begin{cases} \frac{1}{x} - \frac{1}{2\sin(\frac{x}{2})}, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(1) Show that f is of class $\mathcal{C}^1$ on $[0, \pi]$.

(2) For every $n \in \mathbb{N}_0$, show that the following integral is well defined,

$$I_n := \int_0^\pi \frac{\sin\left(\frac{2n+1}{2}t\right)}{\sin\left(\frac{t}{2}\right)} \,\mathrm{d}t.$$

(3) For every $n \in \mathbb{N}_0$, find the value of $I_{n+1} - I_n$ and the value of I_n.

(4) Let $g : [0, \pi] \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Show that

$$\lim_{\lambda \to +\infty} \int_0^\pi g(t) \sin(\lambda t) \,\mathrm{d}t = 0.$$

(5) Show that the following integral is convergent and find its value,

$$I = \int_0^\infty \frac{\sin t}{t} \,\mathrm{d}t.$$

Exercise 7.19 (Laplace's method, Theorem 7.3.1) **:** In this exercise, we are going to proof the Laplace's method stated in Theorem 7.3.1. First, note that we may assume $g(c) > 0$.

(1) Show that for any $\lambda \geqslant 1$, the function $x \mapsto g(x)e^{\lambda h(x)}$ is integrable on (a, b).

(2) For any $\varepsilon \in (0, g(c)) \cap (0, -h''(c))$, show that there exists $\delta > 0$ such that

$$h(c) + \frac{1}{2}(h''(c) + \varepsilon)(x - c)^2 \geqslant h(x) \geqslant h(c) + \frac{1}{2}(h''(c) - \varepsilon)(x - c)^2,$$

and

$$g(c) + \varepsilon \geqslant g(x) \geqslant g(c) - \varepsilon,$$

for all $x \in [c - \delta, c + \delta] \subseteq (a, b)$.

Next, we consider the decomposition

$$\int_a^b g(x)e^{\lambda h(x)} \,\mathrm{d}x = \int_a^{c-\delta} g(x)e^{\lambda h(x)} \,\mathrm{d}x + \int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)} \,\mathrm{d}x + \int_{c+\delta}^b g(x)e^{\lambda h(x)} \,\mathrm{d}x.$$

(3) We first deal with the term $\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)} \,\mathrm{d}x$. Show that

$$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)} \,\mathrm{d}x \geqslant (g(c) - \varepsilon)e^{\lambda h(c)} \sqrt{\frac{1}{\lambda(-h''(c) + \varepsilon)}} \int_{-\delta\sqrt{\lambda(-h''(c)+\varepsilon)}}^{\delta\sqrt{\lambda(-h''(c)+\varepsilon)}} e^{-\frac{1}{2}t^2} \,\mathrm{d}t$$

習題 7.18 ： 考慮函數 $f : [0, \pi] \to \mathbb{R}$ 定義做：

$$\forall x \in \mathbb{R}_+, \quad f(x) = \begin{cases} \frac{1}{x} - \frac{1}{2\sin(\frac{x}{2})}, & \text{若 } x > 0, \\ 0, & \text{若 } x = 0. \end{cases}$$

(1) 證明 f 在 $[0, \pi]$ 上是 $\mathcal{C}^1$ 類的。

(2) 對於每個 $n \in \mathbb{N}_0$，證明下列積分定義良好：

$$I_n := \int_0^\pi \frac{\sin\left(\frac{2n+1}{2}t\right)}{\sin\left(\frac{t}{2}\right)} \,\mathrm{d}t.$$

(3) 對於每個 $n \in \mathbb{N}_0$，求 $I_{n+1} - I_n$ 的值以及 I_n 的值。

(4) 令 $g : [0, \pi] \to \mathbb{R}$ 為 $\mathcal{C}^1$ 函數。證明

$$\lim_{\lambda \to +\infty} \int_0^\pi g(t) \sin(\lambda t) \,\mathrm{d}t = 0.$$

(5) 證明下列積分會收斂，並求他的值：

$$I = \int_0^\infty \frac{\sin t}{t} \,\mathrm{d}t.$$

習題 7.19 【Laplace 方法，定理 7.3.1 】 **：** 在這個習題中，我們會證明定理 7.3.1 當中敘述的 Laplace 方法。首先，我們注意到我們可以假設 $g(c) > 0$。

(1) 證明對於任意 $\lambda \geqslant 1$，函數 $x \mapsto g(x)e^{\lambda h(x)}$ 在 (a, b) 上是可積的。

(2) 對於任意 $\varepsilon \in (0, g(c)) \cap (0, -h''(c))$，證明存在 $\delta > 0$ 使得

$$h(c) + \frac{1}{2}(h''(c) + \varepsilon)(x - c)^2 \geqslant h(x) \geqslant h(c) + \frac{1}{2}(h''(c) - \varepsilon)(x - c)^2,$$

還有

$$g(c) + \varepsilon \geqslant g(x) \geqslant g(c) - \varepsilon,$$

對於所有 $x \in [c - \delta, c + \delta] \subseteq (a, b)$。

再來，我們考慮下列分解：

$$\int_a^b g(x)e^{\lambda h(x)} \,\mathrm{d}x = \int_a^{c-\delta} g(x)e^{\lambda h(x)} \,\mathrm{d}x + \int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)} \,\mathrm{d}x + \int_{c+\delta}^b g(x)e^{\lambda h(x)} \,\mathrm{d}x.$$

(3) 我們首先處理這一項：$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)} \,\mathrm{d}x$。證明

$$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)} \,\mathrm{d}x \geqslant (g(c) - \varepsilon)e^{\lambda h(c)} \sqrt{\frac{1}{\lambda(-h''(c) + \varepsilon)}} \int_{-\delta\sqrt{\lambda(-h''(c)+\varepsilon)}}^{\delta\sqrt{\lambda(-h''(c)+\varepsilon)}} e^{-\frac{1}{2}t^2} \,\mathrm{d}t$$

and

$$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x \leqslant (g(c)+\varepsilon)e^{\lambda h(c)}\sqrt{\frac{1}{\lambda(-h''(c)-\varepsilon)}}\int_{-\delta\sqrt{\lambda(-h''(c)-\varepsilon)}}^{\delta\sqrt{\lambda(-h''(c)-\varepsilon)}} e^{-\frac{1}{2}t^2}\,\mathrm{d}t.$$

Deduce that

$$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x \sim \sqrt{\frac{2\pi}{-\lambda h''(c)}}\cdot g(c)e^{\lambda h(c)}, \quad \text{when } \lambda \to +\infty.$$

(4) Now, we deal with the remaining terms. Show that there exists $\eta > 0$ such that $|x-c| \geqslant \delta$ implies $h(x) \leqslant h(c) - \eta$. Then, deduce that for any $\lambda \geqslant 1$, we have

$$\left|e^{-\lambda h(c)}\int_a^{c-\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x\right| \leqslant M\cdot e^{-\eta\lambda}, \quad \text{and} \quad \left|e^{-\lambda h(c)}\int_{c+\delta}^{b} g(x)e^{\lambda h(x)}\,\mathrm{d}x\right| \leqslant M\cdot e^{-\eta\lambda},$$

where

$$M = e^{-h(c)+\eta}\int_a^b |g(x)|e^{h(x)}\,\mathrm{d}x$$

is a constant.

(5) Conclude the theorem.

還有

$$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x \leqslant (g(c)+\varepsilon)e^{\lambda h(c)}\sqrt{\frac{1}{\lambda(-h''(c)-\varepsilon)}}\int_{-\delta\sqrt{\lambda(-h''(c)-\varepsilon)}}^{\delta\sqrt{\lambda(-h''(c)-\varepsilon)}} e^{-\frac{1}{2}t^2}\,\mathrm{d}t.$$

推得

$$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x \sim \sqrt{\frac{2\pi}{-\lambda h''(c)}}\cdot g(c)e^{\lambda h(c)}, \quad \text{當 } \lambda \to +\infty.$$

(4) 現在我們來處理剩下的項。證明存在 $\eta > 0$ 使得 $|x-c| \geqslant \delta$ 會蘊含 $h(x) \leqslant h(c) - \eta$。接著，推得對於任意 $\lambda \geqslant 1$，我們有

$$\left|e^{-\lambda h(c)}\int_a^{c-\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x\right| \leqslant M\cdot e^{-\eta\lambda}, \quad \text{以及} \quad \left|e^{-\lambda h(c)}\int_{c+\delta}^{b} g(x)e^{\lambda h(x)}\,\mathrm{d}x\right| \leqslant M\cdot e^{-\eta\lambda},$$

其中

$$M = e^{-h(c)+\eta}\int_a^b |g(x)|e^{h(x)}\,\mathrm{d}x$$

是個常數。

(5) 總結定理。