Chapter 7: Complements on Riemann Integrals

Exercise 7.1 : Let $f : [1, +\infty) \to \mathbb{R}$ be a non-negative continuous function. Suppose that it is integrable on $[1, +\infty)$.

(1) If the limit $\lim_{x\to+\infty} f(x)$ exists, show that it is 0.

(2) Give a counterexample to justify that $\lim_{x\to+\infty} f(x)$ may fail to exist.

(3) Give a counterexample to justify that f may fail to be bounded on $[1, +\infty)$.

(4) Suppose that f is uniformly continuous on $[1, +\infty)$, show that $\lim_{x\to+\infty} f(x) = 0$.

(5) Suppose that f is Lipschitz continuous on $[1, +\infty)$, show that $\liminf_{n\to+\infty} \sqrt{n} f(n) = 0$.

Exercise 7.2 : Let $\alpha, \beta \in \mathbb{R}$. Depending on the values of α and β, find the behavior of the integral

$$\int_I \frac{\mathrm{d}x}{x^\alpha |\ln x|^\beta},$$

where we take

(1) $I = (0, \frac{1}{2}]$; (2) $I = [\frac{1}{2}, 1)$; (3) $I = (1, 2]$; (4) $I = [2, +\infty)$.

Exercise 7.3 : Let $M \in \mathbb{R}$ and $f : [M, +\infty) \to \mathbb{R}_+$ be a piecewise continuous function.

(1) Suppose that f is integrable on $[M, +\infty)$. Show that

$$\int_n^{2n} f(t)\,\mathrm{d}t \xrightarrow[n\to\infty]{} 0.$$

(2) Suppose that f is integrable and decreasing on $[M, +\infty)$. Show that $f(x) = o(\frac{1}{x})$ when $x \to \infty$.

Exercise 7.4 : Find the behavior of the following integrals,

(1) $\displaystyle\int_0^1 \frac{\sinh(\sqrt{t}) \ln t}{\sqrt{t} - \sin t},$

(2) $\displaystyle\int_1^\infty \frac{\ln(t^2 - t)}{(1+t)^2},$

(3) $\displaystyle\int_0^\infty \frac{\mathrm{d}t}{e^t - 1},$

(4) $\displaystyle\int_0^\infty \frac{te^{-\sqrt{t}}}{1+t^2}\,\mathrm{d}t,$

(5) $\displaystyle\int_0^\infty \frac{\ln t}{1+t^2}\,\mathrm{d}t,$

(6) $\displaystyle\int_0^\infty \frac{\sqrt{|\ln t|}}{(t-1)\sqrt{t}}\,\mathrm{d}t.$

(7) $\displaystyle\int_0^1 \frac{\mathrm{d}t}{1 - \sqrt{t}},$

(8) $\displaystyle\int_0^\infty \Big(1 + t\ln\Big(\frac{t}{t+1}\Big)\Big)\,\mathrm{d}t.$

Exercise 7.5 : Determine the behavior of each of the following integrals, and their values in the case that they are well defined,

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos u}{\sqrt{\sin u}}\,\mathrm{d}u$, (2) $\displaystyle\int_1^{\infty} \ln\Big(1+\frac{1}{t^2}\Big)\,\mathrm{d}t$.

Exercise 7.6 : Let $f \in \mathcal{C}^1(\mathbb{R}_+,\mathbb{R})$ such that both

$$\int_0^{\infty} f(t)\,\mathrm{d}t \quad \text{and} \quad \int_0^{\infty} f'(t)^2\,\mathrm{d}t$$

are convergent.

(1) Let F be the primitive of f such that $F(0)=0$. Show that F is of class $\mathcal{C}^2$.

(2) For $x \geqslant 0$, show that

$$F(x+1) = F(x) + f(x) + R(x), \quad \text{where} \quad R(x) = \int_x^{x+1} (x+1-t) f'(t)\,\mathrm{d}t.$$

(3) Show that $R(x) \to 0$ when $x \to +\infty$.

(4) Deduce that $f(x) \to 0$ when $x \to +\infty$.

Exercise 7.7 : For every $n \in \mathbb{N}$, define

$$I_n = \int_0^{\infty} \frac{\mathrm{d}t}{(1+t^2)^n}.$$

(1) Show that for every $n \geqslant 1$, the integral I_n is well defined.

(2) For each $n \geqslant 1$, find a relation between I_n and I_{n+1}.

(3) Find the value of I_n for $n \geqslant 1$.

Exercise 7.8 : Let $n \in \mathbb{N}$ and $f : [1,+\infty) \to \mathbb{R}$ be a $\mathcal{C}^{\infty}$ function. Recall the Euler's summation formula that we saw in Corollary 5.2.23,

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + \int_1^n f'(x)\Big(\{x\} - \frac{1}{2}\Big)\,\mathrm{d}x + \frac{1}{2}\big[f(n)+f(1)\big].$$

Recall we had defined the 1-periodic functions $(G_p)_{p\geqslant 1}$ in Problem 6 of the midterm exam.

(1) Check that $G_1(x) = \{x\} - \frac{1}{2}$.

(2) Check that for $k \geqslant 2$ and $x \in \mathbb{R}$,

$$G_k(x) = k\int_1^x G_{k-1}(t)\,\mathrm{d}t + B_k,$$

where $B_k = G_k(0)$.

(3) Show that for any $m \geqslant 1$, we have

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + \frac{1}{(2m+1)!}\int_1^n G_{2m+1}(x) f^{(2m+1)}(x)\,\mathrm{d}x + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!}\left[f^{(2r-1)}(n) - f^{(2r-1)}(1)\right] + \frac{1}{2}\left[f(n) + f(1)\right].$$

(4) Fix $m \geqslant 1$. Suppose that $f^{(2m+1)}$ is integrable on $[1, +\infty)$.

(a) Show that the following integral is well defined,

$$\int_1^\infty f^{(2m+1)}(x) G_{2m+1}(x)\,\mathrm{d}x$$

(b) Deduce that

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + C + E(n),$$

where

$$C = \frac{1}{2} f(1) - \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(1) + \frac{1}{(2m+1)!}\int_1^\infty G_{2m+1}(x) f^{(2m+1)}(x)\,\mathrm{d}x$$

and

$$E(n) = \frac{1}{2} f(n) + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(n) - \frac{1}{(2m+1)!}\int_n^\infty G_{2m+1}(x) f^{(2m+1)}(x)\,\mathrm{d}x.$$

(5) Applications: prove the following asymptotics.

(a) For $s > 1$ and $N \geqslant 1$,

$$\sum_{k=1}^{n} \frac{1}{k^s} = \zeta(s) - \frac{1}{(s-1)n^{s-1}} + \frac{1}{2n^s} - \sum_{r=1}^{N} \frac{B_{2r}}{(2r)!} \cdot \frac{(s+2r-2)_{2r-1}}{n^{s+2r-1}} + \mathcal{O}\Big(\frac{1}{n^{s+2N}}\Big) \quad \text{when} \quad n \to \infty,$$

where $(a)_p$ is the falling factorial symbol defined by

$$\forall a \in \mathbb{R}, p \in \mathbb{N}, \quad (a)_p := a \cdot (a-1) \cdot (a-2) \cdots (a-p+1).$$

(b) For $N \geqslant 1$,

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + \frac{1}{2n} - \sum_{k\geqslant 1}^{N} \frac{B_{2k}}{2kn^{2k}} + \mathcal{O}\Big(\frac{1}{n^{2N+1}}\Big) \quad \text{when} \quad n \to \infty.$$

This is the last expression in Example 6.2.9.

(c) For $N \geqslant 1$,

$$\sum_{k=1}^{n} \ln k = (n + \frac{1}{2}) \ln n - n + \frac{1}{2}\ln(2\pi) + \sum_{r=1}^{N} \frac{B_{2r}}{2r(2r-1)n^{2r-1}} + \mathcal{O}\Big(\frac{1}{n^{2N}}\Big) \quad \text{when} \quad n \to \infty.$$

Moreover, deduce the Stirling's approximation formula,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}-\frac{571}{2488320n^4}+\mathcal{O}\left(\frac{1}{n^5}\right)\right) \quad \text{when} \quad n \to \infty.$$

Exercise 7.9 : We want to study properties of the Gamma function defined in Example 7.1.21, and find a characterization of it.

(1) Let $f, g : (0, +\infty) \to \mathbb{R}$ be functions. Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that both the functions $|f|^p$ and $|g|^q$ are integrable on $(0, +\infty)$. Show that the product fg is integrable on $(0, +\infty)$ and
$$\int_0^\infty |fg| \leqslant \left(\int_0^\infty |f|^p\right)^{1/p}\left(\int_0^\infty |g|^q\right)^{1/q}.$$

(2) Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that
$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \leqslant \Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}, \quad \forall x, y > 0.$$

(3) Show that $x \mapsto \ln\Gamma(x)$ is convex on $(0, +\infty)$.

(4) Show that for all $x \in (0, 1)$ and $n \in \mathbb{N}$, we have
$$x\ln(n) \leqslant \ln\Gamma(n + x + 1) - \ln(n!) \leqslant x\ln(n + 1).$$

(5) Deduce that for $0 < x < 1$, we have
$$0 \leqslant \ln\Gamma(x) - \ln\left(\frac{n^x n!}{x(x+1)\dots(x+n)}\right) \leqslant x\ln\left(1 + \frac{1}{n}\right).$$

(6) Deduce that
$$\Gamma(x) = \lim_{n\to\infty}\frac{n^x n!}{x(x+1)\dots(x+n)}$$
for $x \in (0, 1)$, and also for all $x > 0$.

(7) Conclude that for $x > 0$,
$$\Gamma(x) = \frac{1}{x}\prod_{n=1}^{\infty}\frac{(1 + 1/n)^x}{1 + x/n}.$$

(8) Prove the following characterization of the Gamma function. Let $f : (0, +\infty) \to \mathbb{R}$ be a function satisfying the following three properties,
   (i) $\ln f(x)$ is convex;
   (ii) $f(x) = (x - 1)f(x - 1)$ for all $x > 1$;
   (iii) $f(1) = 1$.

   Then, $f(x) = \Gamma(x)$. Hint: see below[1].

[1]Repeat the arguments in (4), (5), and (6), using f instead of Γ.

Exercise 7.10 :

(1) Prove that the following improper integral is convergent,

$$I := \int_0^\pi \ln(\sin x)\,\mathrm{d}x.$$

(2) Show that

$$\int_0^{\pi/2} \log(\sin t)\,\mathrm{d}t = \int_0^{\pi/2} \log(\cos t)\,\mathrm{d}t.$$

(3) Compute the value of I by using the change of variables $x = 2t$ and applying the identity $\sin(2t) = 2\sin t \cos t$.

Exercise 7.11 : Let $f \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ such that $\int_0^\infty f(t)\,\mathrm{d}t$ converges.

(1) Find the limit when $x \to +\infty$ of the following integral,

$$\int_{x/2}^{x} f(t)\,\mathrm{d}t.$$

(2) Suppose that f is non-negative and is decreasing. Show that $f(x) = o(\frac{1}{x})$ when $x \to +\infty$.

(3) When the assumption in (2) is not satisfied, find a counterexample.

Exercise 7.12 (Cauchy principal value) **:** Let $I = [a, b]$ be a segment and $c \in (a, b)$. Let $f : I\backslash\{c\} \to \mathbb{R}$ be a piecewise continuous function. If the following limit exists,

$$\lim_{\varepsilon \to 0+} \Big(\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon}^{b} f(x)\,\mathrm{d}x \Big),$$

then we call the above limit *Cauchy principal value* of the improper integral, denoted by

$$\text{p. v.} \int_a^b f(x)\,\mathrm{d}x := \lim_{\varepsilon \to 0+} \Big(\int_a^{c-\varepsilon} f(x)\,\mathrm{d}x + \int_{c+\varepsilon}^{b} f(x)\,\mathrm{d}x \Big).$$

Find the Cauchy principal of the following improper integrals,

(1) $\displaystyle\int_{-a}^{a} \frac{\mathrm{d}x}{x}$, $a > 0$;

(2) $\displaystyle\int_a^b \frac{\mathrm{d}x}{x - c}$, $a < c < b$.

Exercise 7.13 : Let $f : [1, +\infty) \to \mathbb{R}$ be a continuous function. Show that the following two properties are equivalent. Hint: integration by parts.

(1) When $x \to +\infty$, the following limit exists,

$$\frac{1}{x}\int_1^x f(t)\,\mathrm{d}t.$$

(2) When $x \to +\infty$, the following limit exists,

$$x\int_x^\infty \frac{f(t)}{t^2}\,\mathrm{d}t.$$

Show that when the above two properties are satisfied, the two limits are equal.

Exercise 7.14 : Suppose that $f : [a, \infty) \to \mathbb{R}$ is continuous and decreases to 0 as $x \to +\infty$.

(1) Show that if $\int_a^\infty |f(x)\cos x|\,\mathrm{d}x$ converges, then both of the integrals $\int_a^\infty f(x)\sin x\,\mathrm{d}x$ and $\int_a^\infty f(x)\cos x\,\mathrm{d}x$ are absolutely convergent.

(2) Show that if $\int_a^\infty |f(x)\cos x|\,\mathrm{d}x$ diverges, then both of the integrals $\int_a^\infty f(x)\sin x\,\mathrm{d}x$ and $\int_a^\infty f(x)\cos x\,\mathrm{d}x$ are conditionally convergent.

Exercise 7.15 : Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $\ell := \lim_{x\to\infty} f(x)$ exists. Let $a < b$. We want to study the following integral

$$I := \int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x.$$

Let $g : (0, \infty) \to \mathbb{R}$ be defined by

$$\forall x \in (0, \infty), \quad g(x) = \frac{f(ax) - f(bx)}{x}.$$

(1) (a) Assume that f is differentiable at $0+$. Find an equivalence of g at $0+$ and deduce that g is integrable on $(0, 1]$.
  (b) Find a function f such that g is not integrable on $(0, 1]$.
  (c) Let $\varepsilon > 0$. Show that if $f(x) = \ell + o(x^{-\varepsilon})$ when $x \to +\infty$, then g is integrable on $[1, +\infty)$.

(2) For $y \geqslant x > 0$, let

$$I(x, y) := \int_x^y \frac{f(at) - f(bt)}{t}\,\mathrm{d}t.$$

  (a) Make a change of variables to show that for $y \geqslant x > 0$, we have

$$I(x, y) = \int_{ax}^{bx} \frac{f(u)}{u}\,\mathrm{d}u - \int_{ay}^{by} \frac{f(u)}{u}\,\mathrm{d}u.$$

  (b) Show that

$$\lim_{x\to 0+}\int_{ax}^{bx} \frac{f(u)}{u}\,\mathrm{d}u = f(0)\ln\Big(\frac{b}{a}\Big) \quad \text{and} \quad \lim_{y\to +\infty}\int_{ay}^{by} \frac{f(u)}{u}\,\mathrm{d}u = \ell\ln\Big(\frac{b}{a}\Big).$$

(c) Deduce the value of the limit $\lim_{x\to 0+}\lim_{y\to+\infty} I(x,y)$.

(3) Can you explain why when f is only continuous, we may find a function f such that g is not integrable on $(0,1]$, while the limit in (2c) is well defined?

(4) Show that the function $x \mapsto \frac{1}{x}(e^{-ax} - e^{-bx})$ is integrable on $(0,+\infty)$ and find the value of the following integral,

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,\mathrm{d}x.$$

Exercise 7.16 : Find the behavior of the following integrals,

(1) $\displaystyle\int_1^\infty \frac{\sin t}{\sqrt{t}}\,\mathrm{d}t,$

(2) $\displaystyle\int_1^\infty \ln\left(1 + \frac{\sin t}{\sqrt{t}}\right)\mathrm{d}t,$

(3) $\displaystyle\int_0^\infty \cos(t^2 + t)\,\mathrm{d}t,$

(4) $\displaystyle\int_1^\infty \frac{\sin t}{t^\alpha}\ln\left(\frac{t+1}{t-1}\right)\mathrm{d}t,\ \alpha\in\mathbb{R}.$

Exercise 7.17 : Consider the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f(0) = 0, \quad \text{and} \quad f(t) = \frac{\sin t - t}{t^2}, \quad \forall t > 0.$$

(1) Check that f is continuous on $\mathbb{R}_+$.

(2) Find the limit when $x \to 0+$ of

$$\int_x^{3x} \frac{\sin t}{t^2}\,\mathrm{d}t.$$

(3) Explain why the following integral converges,

$$\int_0^\infty \frac{\sin^3 t}{t^2}\,\mathrm{d}t.$$

(4) Find the value of the integral in the previous question. Hint: see below[2].

[2]Note that $4\sin^3 t = 3\sin t - \sin(3t)$.

Exercise 7.18 : Consider the function $f : [0, \pi] \to \mathbb{R}$ defined by

$$\forall x \in \mathbb{R}_+, \quad f(x) = \begin{cases} \frac{1}{x} - \frac{1}{2\sin(\frac{x}{2})}, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(1) Show that f is of class $\mathcal{C}^1$ on $[0, \pi]$.

(2) For every $n \in \mathbb{N}_0$, show that the following integral is well defined,

$$I_n := \int_0^\pi \frac{\sin\left(\frac{2n+1}{2}t\right)}{\sin\left(\frac{t}{2}\right)}\,\mathrm{d}t.$$

(3) For every $n \in \mathbb{N}_0$, find the value of $I_{n+1} - I_n$ and the value of I_n.

(4) Let $g : [0, \pi] \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Show that

$$\lim_{\lambda \to +\infty} \int_0^\pi g(t)\sin(\lambda t)\,\mathrm{d}t = 0.$$

(5) Show that the following integral is convergent and find its value,

$$I = \int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t.$$

Exercise 7.19 (Laplace's method, Theorem 7.3.1) **:** In this exercise, we are going to proof the Laplace's method stated in Theorem 7.3.1. First, note that we may assume $g(c) > 0$.

(1) Show that for any $\lambda \geqslant 1$, the function $x \mapsto g(x)e^{\lambda h(x)}$ is integrable on (a, b).

(2) For any $\varepsilon \in (0, g(c)) \cap (0, -h''(c))$, show that there exists $\delta > 0$ such that

$$h(c) + \frac{1}{2}(h''(c) + \varepsilon)(x - c)^2 \geqslant h(x) \geqslant h(c) + \frac{1}{2}(h''(c) - \varepsilon)(x - c)^2,$$

and

$$g(c) + \varepsilon \geqslant g(x) \geqslant g(c) - \varepsilon,$$

for all $x \in [c - \delta, c + \delta] \subseteq (a, b)$.

Next, we consider the decomposition

$$\int_a^b g(x)e^{\lambda h(x)}\,\mathrm{d}x = \int_a^{c-\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x + \int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x + \int_{c+\delta}^b g(x)e^{\lambda h(x)}\,\mathrm{d}x.$$

(3) We first deal with the term $\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x$. Show that

$$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x \geqslant (g(c) - \varepsilon)e^{\lambda h(c)}\sqrt{\frac{1}{\lambda(-h''(c) + \varepsilon)}}\int_{-\delta\sqrt{\lambda(-h''(c)+\varepsilon)}}^{\delta\sqrt{\lambda(-h''(c)+\varepsilon)}} e^{-\frac{1}{2}t^2}\,\mathrm{d}t$$

and

$$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x \leqslant (g(c)+\varepsilon)e^{\lambda h(c)}\sqrt{\frac{1}{\lambda(-h''(c)-\varepsilon)}}\int_{-\delta\sqrt{\lambda(-h''(c)-\varepsilon)}}^{\delta\sqrt{\lambda(-h''(c)-\varepsilon)}} e^{-\frac{1}{2}t^2}\,\mathrm{d}t.$$

Deduce that

$$\int_{c-\delta}^{c+\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x \sim \sqrt{\frac{2\pi}{-\lambda h''(c)}}\cdot g(c)e^{\lambda h(c)}, \quad \text{when } \lambda \to +\infty.$$

(4) Now, we deal with the remaining terms. Show that there exists $\eta > 0$ such that $|x-c| \geqslant \delta$ implies $h(x) \leqslant h(c) - \eta$. Then, deduce that for any $\lambda \geqslant 1$, we have

$$\left|e^{-\lambda h(c)}\int_{a}^{c-\delta} g(x)e^{\lambda h(x)}\,\mathrm{d}x\right| \leqslant M\cdot e^{-\eta\lambda}, \quad \text{and} \quad \left|e^{-\lambda h(c)}\int_{c+\delta}^{b} g(x)e^{\lambda h(x)}\,\mathrm{d}x\right| \leqslant M\cdot e^{-\eta\lambda},$$

where

$$M = e^{-h(c)+\eta}\int_a^b |g(x)|e^{h(x)}\,\mathrm{d}x$$

is a constant.

(5) Conclude the theorem.