Chapter 8: Sequences and series of functions

Exercise 8.1 : Let $I \subseteq \mathbb{R}$ be an interval and $(f_n)_{n\geqslant 1}$ be a sequence of functions from I to $\mathbb{R}$. Suppose that f_n converges pointwise to a function f.

(1) Suppose that every function f_n is convex, show that f is convex.

(2) Suppose that every function f_n is non-decrasing, show that f is non-decreasing.

(3) Suppose that every function f_n is strictly increasing, is f necessarily strictly increasing?

(4) Suppose that every function f_n is periodic with period T, show that f is periodic with period T.

Exercise 8.2 : Consider the sequence of functions $(f_n)_{n\geqslant 1}$ defined as below,

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, \quad f_n(x) = \sin\left(x + \frac{1}{n}\right).$$

Show that $(f_n)_{n\geqslant 1}$ converges uniformly on $\mathbb{R}$.

Exercise 8.3 : For $n \in \mathbb{N}$, define the function u_n on $\mathbb{R}_+$ as below,

$$\forall x \geqslant 0, \quad u_n(x) = \frac{x}{n^2 + x^2}.$$

(1) Show that the series $\sum_{n\geqslant 1} u_n$ converges pointwise on $\mathbb{R}_+$.

(2) Show that the series $\sum_{n\geqslant 1} u_n$ converges uniformly on $[0, A]$ for any $A > 0$.

(3) Show that for every $n \in \mathbb{N}$, we have

$$\sum_{k=n+1}^{2n} \frac{n}{n^2 + k^2} \geqslant \frac{1}{5}.$$

(4) Deduce that the series $\sum_{n\geqslant 1} u_n$ does not converge uniformly on $\mathbb{R}_+$.

Exercise 8.4 :

(1) Let us consider a sequence of functions $(f_n)_{n\geqslant 1}$ defined on $\mathbb{R}_+$ as follows,

$$\forall n \in \mathbb{N}, \forall x \geqslant 0, \quad f_n(x) = \begin{cases} \left(1 - \frac{x}{n}\right)^n & \text{if } x \in [0, n], \\ 0 & \text{if } x > n. \end{cases}$$

Show that $(f_n)_{n\geqslant 1}$ converges uniformly to $f : x \mapsto e^{-x}$ on $\mathbb{R}_+$.

(2) Consider another sequence of functions $(g_n)_{n\geqslant 1}$ defined on $\mathbb{C}$ as follows,

$$\forall n \in \mathbb{N}, \forall z \in \mathbb{C}, \quad g_n(z) = \left(1 + \frac{z}{n}\right)^n.$$

Show that $(g_n)_{n\geqslant 1}$ converges uniformly to g on every compact subset of $\mathbb{C}$.

Exercise 8.5 : For $n \in \mathbb{N}$, define $u_n : \mathbb{R}_+ \to \mathbb{R}$ as below,

$$\forall x \in \mathbb{R}_+, \quad u_n(x) = \frac{x}{n^2 + x^2}.$$

(1) Show that the series of functions $\sum u_n$ converges pointwise on $\mathbb{R}_+$, but does not converge uniformly on $\mathbb{R}_+$.

(2) Show that the series of functions $\sum (-1)^n u_n$ converges uniformly on $\mathbb{R}_+$ but does not converge normally on $\mathbb{R}_+$.

Exercise 8.6 : Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $(P_n)_{n \geqslant 1}$ be a sequence of polynomials that converges uniformly to f on $\mathbb{R}$.

(1) Show that there exists $N \geqslant 1$ such that

$$\forall n \geqslant N, \forall x \in \mathbb{R}, \quad |P_n(x) - f(x)| \leqslant 1.$$

(2) When $n \geqslant N$, what can we say about the polynomial $P_n - P_N$?

(3) Deduce that f is a polynomial function.

Exercise 8.7 : Let $I = [a, b]$ be a segment and $(f_n)_{n \geqslant 1}$ be a sequence of (not necessarily continuous) functions from I to $\mathbb{R}$. Suppose that

(i) for each $n \geqslant 1$, the function f_n is increasing on I;

(ii) the sequence $(f_n)_{n \geqslant 1}$ converges pointwise to a continuous function $f : I \to \mathbb{R}$.

(1) Show that f is increasing on I.

(2) Let us fix $\varepsilon > 0$. Show that we can find a partition $P = (x_k)_{0 \leqslant k \leqslant m} \in \mathcal{P}([a, b])$ such that

$$\forall k = 0, \dots, m-1, \quad |f(x_{k+1}) - f(x_k)| \leqslant \varepsilon.$$

(3) Show that there exists $N \geqslant 1$ such that

$$\forall n \geqslant N, \forall k = 0, \dots, m, \quad |f(x_k) - f_n(x_k)| \leqslant \varepsilon.$$

(4) Deduce that for all $n \geqslant N$ and $x \in [a, b]$, we have $|f_n(x) - f(x)| \leqslant 2\varepsilon$, and conclude that $(f_n)_{n \geqslant 1}$ converges to f uniformly.

Exercise 8.8 :

(1) Let $I = [a, b] \subseteq \mathbb{R}$ be a segment, $(W, \|\cdot\|)$ be a normed vector space, and $K > 0$. Consider a seuqnece of functions $(f_n)_{n\geqslant 1}$ from I to W that are K-Lipschitz continuous. Show that if $(f_n)_{n\geqslant 1}$ converges pointwise to f, then the convergence is uniform.

(2) Let $I = (a, b) \subseteq \mathbb{R}$ be an interval and $(f_n)_{n\geqslant 1}$ be a sequence of convex functions from I to $\mathbb{R}$ that converges pointwise to f. We want to show that this convergence is uniform on every segment of I.

(a) Let $c, d \in I$ such that $[c, d] \subseteq (a, b)$. Consider $p \in (a, c)$ and $q \in (d, b)$. Show that the following two sequences

$$\left(\frac{f_n(p) - f_n(c)}{p - c}\right)_{n\geqslant 1}, \quad \text{and} \quad \left(\frac{f_n(d) - f_n(q)}{d - q}\right)_{n\geqslant 1}$$

are convergent, so bounded.

(b) Let $K > 0$ be a constant that is an upper bound of the absolute value of the terms of the above two sequences. Show that $(f_n)_{n\geqslant 1}$ is a sequence of K-Lipschitz continuous functions on $[c, d]$.

(c) Conclude that $(f_n)_{n\geqslant 1}$ converges uniformly to f on $[c, d]$.

(d) Is it true that $(f_n)_{n\geqslant 1}$ converges to f uniformly on (a, b) in general?

Exercise 8.9 (Cantor–Lebesgue function) **:** We recall the subsets $(C_n)_{n\geqslant 0}$ defined in Exercise 2.21,

$$C_0 = [0, 1], \quad C_{n+1} = \tfrac{1}{3}C_n \cup (\tfrac{1}{3}C_n + \tfrac{2}{3}), \quad \forall n \geqslant 0,$$

and their intersection $\mathcal{C} := \bigcap_{n\geqslant 0} C_n$ called the Cantor set. For $n \geqslant 1$, we also define $I_n := [0, 1]\backslash C_n$, which is an open subset of $\mathbb{R}$. Let us define a sequence of functions $(f_n)_{n\geqslant 0}$ by induction,

$$\forall x \in [0, 1], \quad f_0(x) = x, \quad \text{and} \quad f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & \text{if } x \in [0, \frac{1}{3}], \\ \frac{1}{2} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}], \\ \frac{1}{2} + \frac{1}{2} f_n(3x - 2) & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

(1) Represent graphically the functions f_0, f_1, and f_2.

(2) Show that $\|f_{n+2} - f_{n+1}\|_\infty = \frac{1}{2}\|f_{n+1} - f_n\|_\infty$ for every $n \geqslant 0$.

(3) Deduce that the sequence of functions $(f_n)_{n\geqslant 0}$ converges pointwise to a limit function f, which is continuous.

(4) For any fixed integers $m \leqslant n$, show that f_n is a constant function on every open subinterval of I_m.

(5) Deduce that $f'(x) = 0$ for every $x \in [0, 1]\backslash\mathcal{C}$.

The limit function f is called the *Cantor–Lebesgue function.* It is a non-zero function that has zero derivative on $[0, 1]$ except for the measure zero set $\mathcal{C}$. Therefore, the function f *does not* satisfy the first fundamental theorem of calculus.

Exercise 8.10 : Consider the sequenece of functions $(f_n)_{n\geqslant 1}$ defined as below,

$$\forall n \in \mathbb{N}, \forall x \in \left[0, \frac{\pi}{2}\right], \quad f_n(x) = \big(\cos x\big)^n \cdot \sin x.$$

(1) Show that $(f_n)_{n\geqslant 1}$ converges uniformly to the zero function. Hint: see below[1].

(2) For $n \geqslant 1$, define $g_n = (n+1)f_n$.

  (a) Show that for any $\delta \in (0, \frac{\pi}{2})$, the sequence of functions $(g_n)_{n\geqslant 1}$ converges uniformly to the zero function on $[\delta, \frac{\pi}{2}]$.

  (b) Find the limit of the following sequence

$$\left(\int_0^{\pi/2} g_n(t)\,\mathrm{d}t\right)_{n\geqslant 1},$$

  and deduce that $(g_n)_{n\geqslant 1}$ does not converge uniformly on $[0, \frac{\pi}{2}]$.

Exercise 8.11 : Let $\sum_{n\geqslant 1} a_n$ and $\sum_{n\geqslant 1} b_n$ be two absolutely convergent series in $\mathbb{R}$, and $c \in \mathbb{R}$.

(1) Show that the following function f is well defined on $\mathbb{R}$,

$$\forall x \in \mathbb{R}, \quad f(x) = c + \sum_{n\geqslant 1} \big(a_n \cos(nx) + b_n \sin(nx)\big).$$

(2) Show that the function f is continuous on $\mathbb{R}$.

(3) If, in addition, the series $\sum n a_n$ and $\sum n b_n$ converges absolutely, show that

$$\forall x \in \mathbb{R}, \quad f'(x) = \sum_{n\geqslant 1} n\big(b_n \cos(nx) - a_n \sin(nx)\big).$$

(4) Find the value of $\int_0^{2\pi} f$.

Exercise 8.12 : Let $g : [0,1] \to \mathbb{R}$ be a continuous function, and $(f_n)_{n\geqslant 0}$ be a sequence of functions on $[0,1]$ defined as follows,

$$f_0 \equiv 0, \quad \text{and} \quad \forall n \in \mathbb{N}_0, \forall x \in [0,1], \quad f_{n+1}(x) = g(x) + \int_0^x f_n(t)\,\mathrm{d}t.$$

(1) Use an induction to show that for every $n \in \mathbb{N}_0$ and $x \in [0,1]$, we have

$$|f_{n+1}(x) - f_n(x)| \leqslant \frac{x^n}{n!}\,\|g\|_\infty\,.$$

(2) Show that $(f_n)_{n\geqslant 0}$ converges uniformly to a continuous function f satisfying

$$\forall x \in [0,1], \quad f(x) = g(x) + \int_0^x f(t)\,\mathrm{d}t.$$

[1] Look at the behavior around 0 and away from 0 separately.

Exercise 8.13 : Show that the following function is of class $\mathcal{C}^\infty$ on $\mathbb{R}_+^* := (0, +\infty)$,

$$\forall x > 0, \quad f(x) = \sum_{n \geqslant 0} \frac{(-1)^n}{x + n}.$$

Exercise 8.14 : Define the functions $u_n : \mathbb{R}_+ \to \mathbb{R}$ as below,

$$\forall n \in \mathbb{N}, \forall x \geqslant 0, \quad u_n(x) = \frac{e^{-x\sqrt{n}}}{n^{3/2}}.$$

(1) Show that the series of functions $S = \sum_{n \geqslant 1} u_n$ is well defined on $\mathbb{R}_+$.

(2) Check that S is continuous and is of class $\mathcal{C}^\infty$ on $\mathbb{R}_+$.

(3) Show that S does not have a right derivative at 0. Hint: see below[2].

Exercise 8.15 :

(1) Check that the following function is well defined,

$$\forall x > 0, \quad f(x) = \sum_{n \geqslant 1} \frac{1}{1 + n^2 x}.$$

(2) Consider the function

$$u: \begin{array}{ccc} \mathbb{R}_+^* \times \mathbb{R}_+ & \to & \mathbb{R}, \\ (x, t) & \mapsto & \frac{1}{1 + xt^2}. \end{array}$$

(a) Check that for every fixed $x > 0$, the function $t \mapsto u(x, t)$ is integrable on $\mathbb{R}_+$.

(b) For every $x > 0$, compute the following integral

$$\int_0^{+\infty} u(x, t) \, \mathrm{d}t = \frac{\pi}{2\sqrt{x}}.$$

(c) Check that for every $x > 0$, we have

$$\left| f(x) - \int_0^{+\infty} u(x, t) \, \mathrm{d}t \right| \leqslant u(x, 0) = 1.$$

(d) Deduce that when $x \to 0+$, we have

$$f(x) = \frac{\pi}{2\sqrt{x}} + \mathcal{O}(1).$$

[2] Show that the limit $\frac{S(x) - S(0)}{x}$ does not exist when $x \to 0+$.

Exercise 8.16 : Let $\sum a_n z^n$ be a power series with radius of convergence $R \in [0, +\infty)$.

(1) Write R' for the radius of convergence of the power series $\sum a_n z^{2n}$. Determine the relation between R' and R.

(2) Write R'' for the radius of convergence of the power series $\sum a_{2n} z^n$. Determine the relation between R' and R.

Exercise 8.17 : Find the radius of convergence of the power series $\sum a_n z^n$ for different choices of $(a_n)_{n \geqslant 1}$,

(1) $a_n = \cosh(n)$,

(2) $a_n = \sinh(n)$,

(3) $a_n = \frac{\cosh(n)}{n}$,

(4) $a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n$,

(5) $a_n = e^{\sqrt{n}}$,

(6) $a_n = n^\alpha$, $\alpha \in \mathbb{R}$,

(7) $a_n = \sum_{k=1}^n \frac{1}{k}$,

(8) $a_n = n^{(-1)^n}$,

(9) $a_n = \binom{2n}{n}$.

Exercise 8.18 :

(1) Let $\sum a_n z^n$ be a power series with radius of convergence $R > 0$. Show that the radius of convergence of $\sum \frac{a_n}{n!} z^n$ is $+\infty$.

(2) Suppose that the power series $\sum \frac{a_n}{n!} z^n$ has radius of convergence $R < +\infty$. What can we say about the radius of convergence of $\sum a_n z^n$?

Exercise 8.19 : Let $\sum a_n z^n$ and $\sum b_n z^n$ be two power series with radius of convergence R_1 and R_2. Consider the power series $\sum a_n b_n z^n$ and denote its radius of convergence by R.

(1) Show that $R \geqslant R_1 R_2$.

(2) Find an example for which we have $R > R_1 R_2$.

Exercise 8.20 : Let $(a_n)_{n \geqslant 1}$ be a sequence of nonzero complex numbers such that

$$\frac{|a_{n+2}|}{|a_n|} \xrightarrow[n \to \infty]{} 2.$$

Show that the radius of convergence of the power series $\sum a_n z^n$ is $\frac{1}{\sqrt{2}}$.

Exercise 8.21 : Let $\sum a_n z^n$ be a power series with radius of convergence R. Let $S_n = \sum_{k=0}^n a_k$ be the partial sums of $\sum a_n$. Denote the radius of convergence of $\sum S_n z^n$ by r.

(1) Show that $r \leqslant R$.

(2) Show that $\min\{1, R\} \leqslant r$. Hint: see below[3].

[3] The power series $\sum S_n z^n$ can be seen as the Cauchy product between $\sum a_n z^n$ and a specific power series that you need to choose.

Exercise 8.22 : Find the radius of convergence and explicit the sum of each of the following power series,

(1) $\sum_{n\geqslant 0} n^2 z^n$,

(2) $\sum_{n\geqslant 0} \frac{z^n}{2n+1}$,

(3) $\sum_{n\geqslant 1} \frac{z^n}{n(n+2)}$,

(4) $\sum_{n\geqslant 0} \frac{z^n}{n!}\cos(n\theta)$, $\theta \in \mathbb{R}$,

(5) $\sum_{n\geqslant 0} n^{(-1)^n} z^n$,

(6) $\sum_{n\geqslant 1} \left(1 + \cdots + \frac{1}{n}\right) z^n$.

Exercise 8.23 :

(1) Justify why we may rewrite the following function as a power series,

$$\forall z \in D(0,1), \quad \frac{1}{1-z} = \sum_{n\geqslant 0} z^n.$$

(2) (a) Use (1) and apply a theorem carefully to justify that we have the following power series,

$$\forall x \in (-1,1), \quad \ln(1+x) = \sum_{n\geqslant 0} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} x^n.$$

(b) Use (2a) and apply a theorem carefully to justify that we have

$$\sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} = \ln 2.$$

(3) (a) Use (1) to deduce that

$$\forall x \in (-1,1), \quad \frac{1}{1+x^2} = \sum_{n\geqslant 0} (-1)^n x^{2n}.$$

(b) Then, show that

$$\forall x \in (-1,1), \quad \arctan(x) = \sum_{n\geqslant 0} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

(4) Use (3b) to show that

$$\sum_{n\geqslant 0} \frac{(-1)^n}{2n+1} = \lim_{x\to 1-} \arctan(x) = \frac{\pi}{4}.$$

(5) Mimic the previous questions to show that

$$\sum_{n\geqslant 0} \frac{(-1)^n}{3n+1} = \frac{1}{3}\left(\ln 2 + \frac{\pi}{\sqrt{3}}\right).$$

Exercise 8.24 : Let $f(x) = \sum a_n x^n$ be a real-valued power series with radius of convergence $R > 0$. We say that f is an even function if $f(x) = f(-x)$ for all $x \in (-R, R)$. Show that f is an even function if and only if $a_{2k+1} = 0$ for all $k \in \mathbb{N}_0$.

Exercise 8.25 : Let f be a real-valued power series with radius of convergence $R > 0$. Suppose that there exists $r \in (0, R)$ such that $f(x) = 0$ for $x \in (-r, r)$. Show that $f(x) = 0$ for all $x \in (-R, R)$.

Exercise 8.26 : Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence $R = +\infty$. Suppose that f is bounded on $\mathbb{C}$. Show that f is a constant function on $\mathbb{C}$. Hint: see below[4].

Exercise 8.27 : Expand the following functions into power series around $x = 0$. Do not forget to write down the radius of convergence of each of the power series.

(1) $\ln(a + x)$, $a > 0$,

(2) $\ln(1 + 2x^2)$,

(3) $\frac{1}{a-x}$, $a \neq 0$,

(4) $\sin(x)$.

Exercise 8.28 : Let $(c_n)_{n\geqslant 0}$ be a real sequence defined by $c_0 = 1$ and the following recurrence formula,

$$\forall n \in \mathbb{N}_0, \quad c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}.$$

(1) Suppose that the power series $\sum c_n z^n$ has radius of convergence $R > 0$ and denote the series by $f(z)$. Show that

$$\forall z \in D(0, R), \quad z f(z)^2 = f(z) - 1, \quad \text{and} \quad f(z) = \frac{1}{2z}(1 - \sqrt{1 - 4z}).$$

(2) Show that the function $z \mapsto \frac{1}{2z}(1-\sqrt{1-4z})$ can be extended to 0 by continuity, and can be written as a power series around 0. Find the corresponding power series and its radius of convergence.

(3) Deduce that

$$\forall n \in \mathbb{N}_0, \quad c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Exercise 8.29 : We want to study the following function around 0,

$$f : x \mapsto \int_0^{+\infty} e^{-t^2} \sin(tx) \,\mathrm{d}t.$$

(1) (a) For $k \in \mathbb{N}_0$, show that

$$\int_0^{+\infty} t^{2k+1} e^{-t^2} \,\mathrm{d}t = \frac{k!}{2}.$$

(b) Find an expansion in power series of f around 0 by integrating term by term. Do not forget to justify why you can proceed this way.

(2) Find a differential equation satisfied by f. Apply the method in Example 8.3.35 to determine an expansion in power series of f around 0.

[4]Use the Cauchy's formula in Theorem 8.3.26.

Exercise 8.30 : Let (K, d) be a compact metric space and $\mathcal{F} \subseteq \mathcal{C}(K, \mathbb{R})$ is a subset. Show that if $\mathcal{F}$ is equicontinuous, then $\overline{\mathcal{F}}$ is also equicontinuous.

Exercise 8.31 : Let (K, d) be a compact metric space and $\mathcal{C}(K, \mathbb{R})$ be equipped with the norm $\|\cdot\|_\infty$. Consider a subset $\mathcal{F} \subseteq \mathcal{C}(K, \mathbb{R})$.

(1) Suppose that $\mathcal{F}$ is precompact.

   (a) Show that for every $\varepsilon > 0$, there exists $f \in \mathcal{F}$ such that $B(f, \varepsilon) \cap \mathcal{F}$ is infinite.

   (b) Deduce that if $(f_n)_{n \geqslant 1}$ is a sequence in $\mathcal{F}$, then we may extract a Cauchy subsequence from $(f_n)_{n \geqslant 1}$.

(2) Suppose that from any sequence $(f_n)_{n \geqslant 1}$ in $\mathcal{F}$, we may extract a Cauchy subsequence.

   (a) Check that $\overline{\mathcal{F}}$ is complete.

   (b) Let $(g_n)_{n \geqslant 1}$ be a sequence in $\overline{\mathcal{F}}$. For every $n \geqslant 1$, check that we may choose $f_n \in \mathcal{F}$ with $\|f_n - g_n\|_\infty \leqslant 2^{-n}$. Show that $(g_n)_{n \geqslant 1}$ has a convergent subsequence in $\overline{\mathcal{F}}$.

   (c) Deduce that $\mathcal{F}$ is precompact.

Exercise 8.32 : Define the following function on $\mathbb{R}_{>0}$,

$$\forall x > 0, \quad g(x) = \int_0^\infty \frac{e^{-t}}{t + x} \, \mathrm{d}t.$$

(1) Explain why g is well defined on $\mathbb{R}_{>0}$.

(2) Show that $g(x) \sim \frac{1}{x}$ when $x \to +\infty$. Hint: see below[5].

Exercise 8.33 : Use the dominated convergence theorem (Theorem 8.5.5) to show the following convergence,

$$\int_0^{\sqrt{n}} \Big(1 - \frac{t^2}{n}\Big)^n \, \mathrm{d}t \xrightarrow[n \to \infty]{} \int_0^\infty e^{-t^2} \, \mathrm{d}t.$$

Then, use Wallis' integrals (Exercise A1.2) to deduce the value of the integral

$$\int_0^\infty e^{-t^2} \, \mathrm{d}t.$$

Hint: see below[6].

[5] Apply the dominated convergence theorem to the integral defining $xg(x)$.

[6] Consider the sequence of piecewise continuous functions $(f_n)_{n \geqslant 1}$ defined by $f_n : t \mapsto (1 - \frac{t^2}{n})^n \mathbb{1}_{[0, \sqrt{n}]}(t)$.

Exercise 8.34 : We define

$$\forall n \in \mathbb{N}, \quad I_n = \int_0^1 \ln(1+t^n)\,\mathrm{d}t.$$

(1) Show that $\lim_{n\to\infty} I_n = 0$.

(2) Use the change of variables $t = u^{1/n}$ and the dominated convergence theorem to show that

$$nI_n \xrightarrow[n\to\infty]{} \int_0^1 \frac{\ln(1+u)}{u}\,\mathrm{d}u.$$

(3) Deduce that

$$I_n \sim \frac{1}{n}\int_0^1 \frac{\ln(1+u)}{u}\,\mathrm{d}u.$$

Exercise 8.35 : Let

$$\forall x > 0, \quad g(x) = \int_0^{+\infty} \frac{e^{-t} - e^{-xt}}{t}\,\mathrm{d}t.$$

(1) Show that g is well defined on $(0, +\infty)$.

(2) Show that g is of class $\mathcal{C}^1$. Find g' and g.

Exercise 8.36 : Show that for $x > 1$, we have

$$\Gamma(x)\,\zeta(x) = \int_0^{+\infty} \frac{t^{x-1}}{e^t - 1}\,\mathrm{d}t.$$

Exercise 8.37 : Let

$$\forall x \geqslant 0, \quad g(x) = \int_0^{+\infty} \frac{e^{-x^2t^2}}{1+t^2}\,\mathrm{d}t, \quad \text{and} \quad I = \int_0^{+\infty} e^{-u^2}\,\mathrm{d}u.$$

(1) Show that g is well defined on $\mathbb{R}_+$.

(2) Show that g is of class $\mathcal{C}^1$ and is a solution to the differential equation,

$$y' - 2xy = -2I.$$

(3) Deduce that $I = \frac{\sqrt{\pi}}{2}$.