Appendix 1: Classical primitives: computations and methods

In what follows, for a non-negative function $g : [a, +\infty) \to \mathbb{R}_+$, we write

$$\int_a^\infty g(t)\,\mathrm{d}t := \lim_{x\to\infty} \int_a^x g(t)\,\mathrm{d}t$$

if the limit on the right exists.

Exercise A1.1 : Find a primitive for each of the following functions,

$$f_1(x) = \frac{1}{x^4 - x^2 - 2}, \quad f_2(x) = \frac{x+1}{(x^2+1)^2}, \quad f_3(x) = \frac{x^2}{x^6 - 1}, \quad f_4(x) = \frac{1}{x(x^2+1)^2}.$$

Exercise A1.2 (Wallis' integral) **:** For every integer $n \geqslant 0$, let us define

$$I_n = \int_0^{\pi/2} \sin^n x\,\mathrm{d}x.$$

(1) Find the values of I_0 and I_1.

(2) Let $n \geqslant 2$. Use an integration by parts to prove that $nI_n = (n-1)I_{n-2}$.

(3) Deduce a formula for I_n depending on the parity of n.

(4) Show that $I_n I_{n-1} = \frac{\pi}{2n}$ for all $n \geqslant 1$.

(5) Deduce that $I_n = \sqrt{\frac{\pi}{2n}}(1 + o(1))$. Hint: see below[1].

Exercise A1.3 : For $n \in \mathbb{N}_0$, we define the following integrals,

$$I_n = \int_0^{\pi/4} \tan^n x\,\mathrm{d}x, \quad J_n = \int_0^{\pi/4} \frac{\mathrm{d}x}{\cos^n x}, \quad K_n = \int_1^e \log^n x\,\mathrm{d}x.$$

For each of the sequences $(I_n)_{n\geqslant 0}$, $(J_n)_{n\geqslant 0}$ and $(K_n)_{n\geqslant 0}$, find a recurrence relation among its terms, and enough number of initial conditions so that an expression for the general terms can be determined. Note that you do not need to find an exact form of the n-th term of each sequence.

Exercise A1.4 : Find a primitive for each of the following functions,

$$h_1(x) = \frac{\cos x}{\sin^2 x + 2\tan^2 x}, \qquad h_2(x) = \frac{\sin x}{\cos^3 x + \sin^3 x},$$

$$h_3(x) = \frac{1}{\sin x + \cos x + 2}, \qquad h_4(x) = \frac{1}{\cosh x\sqrt{\cosh(2x)}}.$$

[1] Do not forget to first check that the sequence $(\sqrt{n}I_n)_{n\geqslant 1}$ converges.

附錄一：經典原函數：計算與方法

接下來，對於非負函數 $g : [a, +\infty) \to \mathbb{R}_+$，如果下面式子右方的極限存在，則我們記

$$\int_a^\infty g(t)\,\mathrm{d}t := \lim_{x\to\infty} \int_a^x g(t)\,\mathrm{d}t.$$

習題 A1.1 ： 對下面每個函數，請找出一個原函數：

$$f_1(x) = \frac{1}{x^4 - x^2 - 2}, \quad f_2(x) = \frac{x+1}{(x^2+1)^2}, \quad f_3(x) = \frac{x^2}{x^6 - 1}, \quad f_4(x) = \frac{1}{x(x^2+1)^2}.$$

習題 A1.2 【Wallis 積分】 **：** 對於所有整數 $n \geqslant 0$，我們定義

$$I_n = \int_0^{\pi/2} \sin^n x\,\mathrm{d}x.$$

(1) 求 I_0 和 I_1 的值。

(2) 令 $n \geqslant 2$。使用分部積分來證明 $nI_n = (n-1)I_{n-2}$。

(3) 根據 n 的奇偶性，求 I_n 的式子。

(4) 證明 $I_n I_{n-1} = \frac{\pi}{2n}$ 對於所有 $n \geqslant 1$。

(5) 推得 $I_n = \sqrt{\frac{\pi}{2n}}(1 + o(1))$。提示：如下[1]。

習題 A1.3 ： 對於 $n \in \mathbb{N}_0$，我們定義下面的積分：

$$I_n = \int_0^{\pi/4} \tan^n x\,\mathrm{d}x, \quad J_n = \int_0^{\pi/4} \frac{\mathrm{d}x}{\cos^n x}, \quad K_n = \int_1^e \log^n x\,\mathrm{d}x.$$

對 $(I_n)_{n\geqslant 0}$、$(J_n)_{n\geqslant 0}$ 以及 $(K_n)_{n\geqslant 0}$ 中的每個序列，找出不同項之間的遞迴關係式，以及足夠的初始條件，使得我們可以解出一般項。注意到，你不需要寫出每個序列第 n 項的確切形式。

習題 A1.4 ： 對下面每個函數，請找出一個原函數：

$$h_1(x) = \frac{\cos x}{\sin^2 x + 2\tan^2 x}, \qquad h_2(x) = \frac{\sin x}{\cos^3 x + \sin^3 x},$$

$$h_3(x) = \frac{1}{\sin x + \cos x + 2}, \qquad h_4(x) = \frac{1}{\cosh x\sqrt{\cosh(2x)}}.$$

[1] 不要忘記先檢查序列 $(\sqrt{n}I_n)_{n\geqslant 1}$ 會收斂

Exercise A1.5 :

(1) Find a primitive of $\dfrac{1}{3+\sin x}$.

(2) Compute the value of the integral $I = \displaystyle\int_0^{2\pi} \frac{\mathrm{d}x}{3+\sin x}$.

Be careful, the function involved in a change of variables needs to be $\mathcal{C}^1$, and a primitive also needs to be $\mathcal{C}^1$ on its domain of definition.

Exercise A1.6 (Gaussian integral) **:** We want to compute the value of the following integral,

$$I = \int_0^{\infty} e^{-t^2}\,\mathrm{d}t := \lim_{n\to\infty} J_n, \quad \text{where, } J_n = \int_0^{\sqrt{n}} e^{-t^2}\,\mathrm{d}t, \forall n \geqslant 1.$$

(1) Show that for every $x \geqslant 0$, we have $1 - x \leqslant e^{-x} \leqslant \frac{1}{1+x}$.

(2) Check that

$$\forall n \geqslant 1, \quad J_n = \sqrt{n}\int_0^1 e^{-ns^2}\,\mathrm{d}s.$$

(3) Recall the definition of $(I_n)_{n\geqslant 0}$ in Exercise A1.2. For every $n \geqslant 1$, show the two following inequalities,

(a) $J_n \geqslant \sqrt{n} I_{2n+1}$,

(b) $J_n \leqslant \sqrt{n} I_{2n-2}$.

(4) Conclude that $I = \frac{\sqrt{\pi}}{2}$.

Exercise A1.7 : Find a primitive for each of the following functions,

$$g_1(x) = \frac{1}{\sqrt[3]{1+x^3}}, \qquad g_2(x) = \frac{1}{x+\sqrt{x^2+2x}},$$

$$g_3(x) = \sqrt{-x^2+4x+10}, \qquad g_4(x) = \sqrt{\frac{x-2}{x+1}}.$$

習題 A1.5 ：

(1) 求 $\dfrac{1}{3+\sin x}$ 的一個原函數。

(2) 求積分 $I = \displaystyle\int_0^{2\pi} \frac{\mathrm{d}x}{3+\sin x}$ 的值。

這裡要小心的是，變數變換中的函數要是 $\mathcal{C}^1$ 函數，且原函數在他的定義域上面也是個 $\mathcal{C}^1$ 函數。

習題 A1.6 【高斯積分】 **：** 我們想要求下面積分的值：

$$I = \int_0^{\infty} e^{-t^2}\,\mathrm{d}t := \lim_{n\to\infty} J_n, \quad \text{其中}, J_n = \int_0^{\sqrt{n}} e^{-t^2}\,\mathrm{d}t, \forall n \geqslant 1.$$

(1) 證明對於每個 $x \geqslant 0$，我們有 $1 - x \leqslant e^{-x} \leqslant \frac{1}{1+x}$。

(2) 檢查

$$\forall n \geqslant 1, \quad J_n = \sqrt{n}\int_0^1 e^{-ns^2}\,\mathrm{d}s.$$

(3) 我們回顧在習題 A1.2 中定義的 $(I_n)_{n\geqslant 0}$。對於每個 $n \geqslant 1$，證明下面兩個不等式：

(a) $J_n \geqslant \sqrt{n} I_{2n+1}$,

(b) $J_n \leqslant \sqrt{n} I_{2n-2}$.

(4) 總結 $I = \frac{\sqrt{\pi}}{2}$。

習題 A1.7 ： 對下面每個函數，請找出一個原函數：

$$g_1(x) = \frac{1}{\sqrt[3]{1+x^3}}, \qquad g_2(x) = \frac{1}{x+\sqrt{x^2+2x}},$$

$$g_3(x) = \sqrt{-x^2+4x+10}, \qquad g_4(x) = \sqrt{\frac{x-2}{x+1}}.$$