Appendix 1: Classical primitives: computations and methods

In what follows, for a non-negative function $g : [a, +\infty) \to \mathbb{R}_+$, we write

$$\int_a^\infty g(t)\,\mathrm{d}t := \lim_{x\to\infty} \int_a^x g(t)\,\mathrm{d}t$$

if the limit on the right exists.

Exercise A1.1 : Find a primitive for each of the following functions,

$$f_1(x) = \frac{1}{x^4 - x^2 - 2}, \quad f_2(x) = \frac{x+1}{(x^2+1)^2}, \quad f_3(x) = \frac{x^2}{x^6 - 1}, \quad f_4(x) = \frac{1}{x(x^2+1)^2}.$$

Exercise A1.2 (Wallis' integral) **:** For every integer $n \geqslant 0$, let us define

$$I_n = \int_0^{\pi/2} \sin^n x\,\mathrm{d}x.$$

(1) Find the values of I_0 and I_1.

(2) Let $n \geqslant 2$. Use an integration by parts to prove that $nI_n = (n-1)I_{n-2}$.

(3) Deduce a formula for I_n depending on the parity of n.

(4) Show that $I_n I_{n-1} = \frac{\pi}{2n}$ for all $n \geqslant 1$.

(5) Deduce that $I_n = \sqrt{\frac{\pi}{2n}}(1 + o(1))$. Hint: see below[1].

Exercise A1.3 : For $n \in \mathbb{N}_0$, we define the following integrals,

$$I_n = \int_0^{\pi/4} \tan^n x\,\mathrm{d}x, \quad J_n = \int_0^{\pi/4} \frac{\mathrm{d}x}{\cos^n x}, \quad K_n = \int_1^e \log^n x\,\mathrm{d}x.$$

For each of the sequences $(I_n)_{n\geqslant 0}$, $(J_n)_{n\geqslant 0}$ and $(K_n)_{n\geqslant 0}$, find a recurrence relation among its terms, and enough number of initial conditions so that an expression for the general terms can be determined. Note that you do not need to find an exact form of the n-th term of each sequence.

Exercise A1.4 : Find a primitive for each of the following functions,

$$h_1(x) = \frac{\cos x}{\sin^2 x + 2\tan^2 x}, \qquad h_2(x) = \frac{\sin x}{\cos^3 x + \sin^3 x},$$

$$h_3(x) = \frac{1}{\sin x + \cos x + 2}, \qquad h_4(x) = \frac{1}{\cosh x\sqrt{\cosh(2x)}}.$$

[1] Do not forget to first check that the sequence $(\sqrt{n}I_n)_{n\geqslant 1}$ converges.

Exercise A1.5 :

(1) Find a primitive of $\frac{1}{3+\sin x}$.

(2) Compute the value of the integral $I = \int_0^{2\pi} \frac{\mathrm{d}x}{3+\sin x}$.

Be careful, the function involved in a change of variables needs to be $\mathcal{C}^1$, and a primitive also needs to be $\mathcal{C}^1$ on its domain of definition.

Exercise A1.6 (Gaussian integral) **:** We want to compute the value of the following integral,

$$I = \int_0^{\infty} e^{-t^2}\,\mathrm{d}t := \lim_{n\to\infty} J_n, \quad \text{where, } J_n = \int_0^{\sqrt{n}} e^{-t^2}\,\mathrm{d}t, \forall n \geqslant 1.$$

(1) Show that for every $x \geqslant 0$, we have $1 - x \leqslant e^{-x} \leqslant \frac{1}{1+x}$.

(2) Check that

$$\forall n \geqslant 1, \quad J_n = \sqrt{n}\int_0^1 e^{-ns^2}\,\mathrm{d}s.$$

(3) Recall the definition of $(I_n)_{n\geqslant 0}$ in Exercise A1.2. For every $n \geqslant 1$, show the two following inequalities,
   (a) $J_n \geqslant \sqrt{n} I_{2n+1}$,
   (b) $J_n \leqslant \sqrt{n} I_{2n-2}$.

(4) Conclude that $I = \frac{\sqrt{\pi}}{2}$.

Exercise A1.7 : Find a primitive for each of the following functions,

$$g_1(x) = \frac{1}{\sqrt[3]{1+x^3}}, \qquad g_2(x) = \frac{1}{x+\sqrt{x^2+2x}},$$

$$g_3(x) = \sqrt{-x^2+4x+10}, \qquad g_4(x) = \sqrt{\frac{x-2}{x+1}}.$$