1 Important Notions in Measure Theory

第一章 測度論中的重要概念

The modern approach to study Probability Theory is built on Measure Theory. In order to follow this lecture, it is essential to have a minimum understanding and familiarity with Measure Theory. For this reason, we will start by reviewing some notions in Measure Theory: we will recall the important definitions and theorems along with a few proofs. The proofs provided here are classical ones and the techniques involved will also be seen repeatedly in Probability Theory; those which are omitted require more technical details from Measure Theory. Interested readers are invited to take a careful look at Rudin's book "Real and Complex Analysis".

現代機率論是建構在測度論之上，因此要學習這門課程，勢必需要對測度論有足夠的熟練程度。也因此，我們先從測度論的複習開始：我們會回顧測度論中重要的定義以及定理，並且給出部份證明；我們回顧的證明通常是經典證明，也就是在機率論中，相同的概念會一再出現的，其餘比較需要測度論的技巧，我們則省略，有興趣的讀者可以參考 Rudin 的《實分析與複分析》(Real and Complex Analysis) 一書。

1.1 Measurable Spaces and Measures

Given a set, we desire to define a measure (測度) which is a function that attributes a mass to (potentially) all the subsets. Additionally, we want a measure to satisfy some specific properties, such as additivity (加法性) or even σ-additivity (σ 加法性). Hence, we will first define measurable spaces and introduce the notion of σ-algebra, which is, vaguely speaking, a collection of subsets on which we can make sense of a measure.

第一節 可測空間與測度

在測度論中，給定一個集合，我們希望可以定義一個測度去「測量」他的子集合（也有可能是所有的子集合）。此外，我們希望這樣定義出來的測度，能夠具有某些特定的性質及規律，例如加法性 (additivity)，或甚至 σ 加法性 (σ-additivity)。也因此，開始討論測度之前，我們要先定義可測空間及σ 代數的概念，換句話說，也就是討論哪些子集合構成的集合，可以讓我們在上面給出一個合適的測度定義。

1.1.1 Measurable Sets and σ-algebras

First, let us start with the notion of σ-*algebra*.

Definition 1.1.1 : Let E be a set. A subset $\mathcal{A} \subseteq \mathcal{P}(E)$ is called a σ-*algebra* (σ 代數) if the following properties are satisfied.

(a) $E \in \mathcal{A}$.

(b) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

(c) If for all $n \in \mathbb{N}$, $A_n \in \mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

The elements of $\mathcal{A}$ are called *measurable sets* (可測集合) , or $\mathcal{A}$-measurable sets, if one wants to emphasize the underlying σ-algebra. The couple $(E, \mathcal{A})$ is called a *measurable space* (可測空間) .

Remark 1.1.2 : In the above definition, σ means that in the condition (c), the union can be any *countable* union. If we replace this condition by any *finite* union, then the corresponding definition defines what we call an *algebra* (代數) , in the sense of *algebra of sets*, or *set algebra*.

第一小節 可測集合及 σ 代數

讓我們從 σ 代數的概念開始。

定義 1.1.1 ： 令 E 為一集合，若 $\mathcal{A} \subseteq \mathcal{P}(E)$ 滿足

(a) $E \in \mathcal{A}$；

(b) 若 $A \in \mathcal{A}$，則 $A^c \in \mathcal{A}$；

(c) 若對於所有 $n \in \mathbb{N}$，$A_n \in \mathcal{A}$，則 $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$，

則我們稱之為 σ 代數 (σ algebra)。我們把 $\mathcal{A}$ 中的元素稱為可測集合 (measurable sets)；或是當我們想要強調 σ 代數的時候，可以把他稱做 $\mathcal{A}$ 可測集合。此外，我們也稱 $(E, \mathcal{A})$ 為可測空間 (measurable space)。

註解 1.1.2 ： 在上述定義中，σ 意指在條件 (c) 中，聯集是任意可數聯集；若我們將此條件中的「可數」改為「有限」，則我們得到代數 (algebra) 的定義。這裡的代數指的是集合代數。

In Section 1.1.2 below, we will define the notion of measure, which is closely related to the σ-algebra of the considered space. Moreover, later in Probability Theory, we will also see the importance and applications of σ-algebra. For example, in Chapter 5, we will define the notion of *conditional probability* and *conditional expectation.*

Example 1.1.3 : Given a set E. We have two extremal σ-algebras: the finest (最精緻的) is $\mathcal{A} = \mathcal{P}(E)$ and the coarsest (最粗糙) is $\mathcal{A} = \{\varnothing, E\}$. The latter is also called the trivial (平凡) σ-algebra.

In order to construct other examples of σ-algebras, we introduce the notion of *generated σ-algebra* (生成 σ 代數) .

Definition 1.1.4 : Let $\mathcal{C}$ be a subset of $\mathcal{P}(E)$. Then the smallest σ-algebra containing $\mathcal{C}$ exists, denoted by $\sigma(\mathcal{C})$ and is given by,

$$\sigma(\mathcal{C}) = \bigcap_{\substack{\mathcal{A} \text{ is a } \sigma\text{-algebra} \\ \text{s.t. } \mathcal{C} \subseteq \mathcal{A}}} \mathcal{A}.$$

In the definition, the intersection is well defined because $\mathcal{A} = \mathcal{P}(E)$ is a σ-algebra containing $\mathcal{C}$.

Definition 1.1.5 : Let E be a set, and $\mathcal{O}$ be a set of subsets of E satisfying

(a) $E \in \mathcal{O}$ and $\varnothing \in \mathcal{O}$.

(b) For any finitely many open sets $U_1, \ldots, U_n \in \mathcal{O}$, we have $U_1 \cap \cdots \cap U_n \in \mathcal{O}$.

(c) For any family of open sets $(U_i)_{i \in I}$ with $U_i \in \mathcal{O}$, we have $\bigcup_{i \in I} U_i \in \mathcal{O}$.

Then, we call $(E, \mathcal{O})$ a *topological space* (拓撲空間) and the elements in $\mathcal{O}$ open sets (開集).

Definition 1.1.6 : Let $(E, \mathcal{O})$ be a topological space.

(1) We denote by $\sigma(\mathcal{O})$ the σ-algebra generated by $\mathcal{O}$, the open sets of the topological space.

(2) It is also called *Borel σ-algebra* (伯雷爾 σ 代數) of E, which can be denoted by $\mathcal{B}(E)$, if there is a canonical choice for $\mathcal{O}$.

(3) We call *Borel sets* (伯雷爾集合) the elements of $\mathcal{B}(E)$.

In other words, Borel σ-algebra is the smallest σ-algebra containing all the open sets of E.

在下面第 1.1.2 小節中，我們會給出測度的定義，是和可測空間上的 σ 代數息息相關的；稍後在機率論的部份，我們也會看到 σ 代數的重要性及其應用，例如在第五章中，我們要定義的條件機率及條件期望值。

範例 1.1.3 ： 給定任一集合 E，我們有兩個最極端的 σ 代數：最精緻的 (finest) $\mathcal{A} = \mathcal{P}(E)$ 以及最粗糙 (coarsest) $\mathcal{A} = \{\varnothing, E\}$，後者也稱作平凡 (trivial) 的 σ 代數。

為了能夠構造更多 σ 代數，我們介紹生成 σ 代數 (generated σ-algebra) 的概念。

定義 1.1.4 ： 設 $\mathcal{C}$ 為 $\mathcal{P}(E)$ 的子集合，那麼存在一個最小的 σ 代數，使得他包含 $\mathcal{C}$ ，記作 $\sigma(\mathcal{C})$：

$$\sigma(\mathcal{C}) = \bigcap_{\substack{\mathcal{A} \text{ 為 } \sigma \text{ 代數} \\ \text{使得 } \mathcal{C} \subseteq \mathcal{A}}} \mathcal{A}.$$

此定義中的交集是定義良好的，因為 $\mathcal{A} = \mathcal{P}(E)$ 是個包含 $\mathcal{C}$ 的 σ 代數。

定義 1.1.5 ： 令 E 為集合，且 $\mathcal{O}$ 為 E 的子集合所構成的集合，滿足：

(a) $E \in \mathcal{O}$ 以及 $\varnothing \in \mathcal{O}$。

(b) 對於任意有限多個開集 $U_1, \ldots, U_n \in \mathcal{O}$，我們有 $U_1 \cap \cdots \cap U_n \in \mathcal{O}$。

(c) 對於任意開集組 $(U_i)_{i \in I}$，其中 $U_i \in \mathcal{O}$，我們有 $\bigcup_{i \in I} U_i \in \mathcal{O}$。

Then, we call $(E, \mathcal{O})$ a 拓撲空間 (topological space) and the elements in $\mathcal{O}$ 開集 (open sets).

定義 1.1.6 ： 假設 $(E, \mathcal{O})$ 為拓撲空間 (topological space)。

(1) 我們把由拓撲空間開集 $\mathcal{O}$ 生成的 σ 代數記作 $\sigma(\mathcal{O})$。

(2) 這也被稱作伯雷爾 σ 代數 (Borel σ-algebra) ，當我們有標準的方法選擇 $\mathcal{O}$ 時，也可以把他記作 $\mathcal{B}(E)$。

(3) 我們把 $\mathcal{B}(E)$ 之中的元素稱作伯雷爾集合 (Borel sets) 。

換句話說，伯雷爾 σ 代數是包含所有 E 的開集中，最小的 σ 代數；

Next, whenever we talk about topological spaces, the σ-algebra we consider will always be its Borel σ-algebra. For example, when we want to equip $\mathbb{R}$ or $\mathbb{R}^d$ with a σ-algebra to turn them into measurable spaces, we take the open sets to be given by any equivalent norm on $\mathbb{R}$ or $\mathbb{R}^d$, and the corresponding Borel σ-algebra. Actually, in the class of Probability Theory, the most of the spaces we will discuss fall into this setting.

Question 1.1.7: Prove that the σ-algebra generated by each of the three following sets of intervals is equal to $\mathcal{B}(\mathbb{R})$,

(1) $\mathcal{I} = \{(a,b) : a < b, a, b \in \mathbb{R}\}$,

(2) $\mathcal{I} = \{(-\infty, a) : a \in \mathbb{R}\}$,

(3) $\mathcal{I} = \{(-\infty, a) : a \in \mathbb{Q}\}$.

The next important notion to introduce is *prodcut σ-algebra.*

Definition 1.1.8 : Given two measurable spaces $(E_1, \mathcal{A}_1)$ and $(E_2, \mathcal{A}_2)$, we can define the *product σ-algebra* (積 σ 代數) on their product space $E_1 \times E_2$,

$$\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma(A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2).$$

Question 1.1.9: Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

1.1.2 Measures

Given a measurable space $(E, \mathcal{A})$, we want to define a *measure* on it.

Definition 1.1.10 : A function $\mu : \mathcal{A} \longrightarrow [0, \infty]$ is called a *positive measure* (正測度) or simply *measure* (測度) if the following axioms (公理) are satisfied,

- $\mu(\varnothing) = 0$,
- (σ-additivity) For any disjoint sequence $(A_n)_{n \geqslant 1}$ on $\mathcal{A}$, we have

$$\mu\Big(\bigcup_{n\geqslant 1} A_n\Big) = \sum_{n \geqslant 1} \mu(A_n). \tag{1.1}$$

Moreover, we call $(E, \mathcal{A}, \mu)$ a *measure space* (測度空間) .

Proposition 1.1.11 : *Let $(E, \mathcal{A}, \mu)$ be a measure space. We have the following properties.*

(1) If $A, B \in \mathcal{A}$, $A \subseteq B$, $\mu(A) \leqslant \mu(B)$ and $\mu(A) < \infty$, then

$$\mu(B \backslash A) = \mu(B) - \mu(A).$$

接下來，每當我們提到拓撲空間時，我們要考慮的 σ 代數都會是其伯雷爾 σ 代數；例如當我們想要在實空間 $\mathbb{R}$ 或 $\mathbb{R}^d$ 之上賦予 σ 代數、把他們變成可測空間時，我們考慮由任何 $\mathbb{R}$ 或 $\mathbb{R}^d$ 上等價範數所定義的開集，再取相對應的伯雷爾 σ 代數。在機率論中，大部分的時候，我們會討論的都是這樣的情況。

問題 1.1.7：證明下列三種不同類型區間構成的集合所生成的 σ 代數皆為 $\mathcal{B}(\mathbb{R})$：

(1) $\mathcal{I} = \{(a,b) : a < b, a, b \in \mathbb{R}\}$,

(2) $\mathcal{I} = \{(-\infty, a) : a \in \mathbb{R}\}$,

(3) $\mathcal{I} = \{(-\infty, a) : a \in \mathbb{Q}\}$.

下一個要介紹的重要概念是積 σ 代數。

定義 1.1.8 ：給定兩個可測空間 $(E_1, \mathcal{A}_1)$ 及 $(E_2, \mathcal{A}_2)$，我們可以在 $E_1 \times E_2$ 之上定義積 σ 代數 (product σ-algebra) ：

$$\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma(A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2).$$

問題 1.1.9：證明 $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$。

第二小節 測度

給定一可測空間 $(E, \mathcal{A})$，我們要給出在此空間上測度的定義。

定義 1.1.10 ：令 $\mu : \mathcal{A} \longrightarrow [0, \infty]$，若 μ 滿足下列公理 (axioms)，我們稱之為正測度 (positive measure) 或測度 (measure) ：

- $\mu(\varnothing) = 0$；
- 【σ 加法性】對於任何 $\mathcal{A}$ 之上的互斥序列 $(A_n)_{n \geqslant 1}$，我們有

$$\mu\Big(\bigcup_{n\geqslant 1} A_n\Big) = \sum_{n \geqslant 1} \mu(A_n). \tag{1.1}$$

此外，我們稱 $(E, \mathcal{A}, \mu)$ 為測度空間 (measure space) 。

命題 1.1.11 ：對任意測度空間 $(E, \mathcal{A}, \mu)$，下列性質成立：

(1) 若 $A, B \in \mathcal{A}$，$A \subseteq B$，$\mu(A) \leqslant \mu(B)$ 且 $\mu(A) < \infty$，則有

$$\mu(B \backslash A) = \mu(B) - \mu(A).$$

(2) *If $A, B \in \mathcal{A}$, then*

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B).$$

(3) *(Lower continuity) Let $(A_n)_{n\geqslant 1}$ be a sequence of increasing elements in $\mathcal{A}$ (i.e., $A_n \subseteq A_{n+1}$ for all n). Then, we have*

$$\mu\Big(\bigcup_{n\geqslant 1} A_n\Big) = \lim_{n\to\infty} \uparrow \mu(A_n).$$

(4) *(Upper continuity) Let $(B_n)_{n\geqslant 1}$ be a sequence of decreasing elements in $\mathcal{A}$ (i.e., $B_{n+1} \subseteq B_n$ for all n). If there exists n_0 such that $\mu(B_{n_0}) < \infty$, then*

$$\mu\Big(\bigcap_{n\geqslant 1} B_n\Big) = \lim_{n\to\infty} \downarrow \mu(B_n).$$

(5) *If $(A_n)_{n\geqslant 1}$ is a sequence of elements in $\mathcal{A}$, then*

$$\mu\Big(\bigcup_{n\geqslant 1} A_n\Big) \leqslant \sum_{n\geqslant 1} \mu(A_n).$$

Proof : See Exercise 1.5. □

Definition 1.1.12 : Given a measure μ.

- The quantity $\mu(E)$ is called the *total mass* (總質量) of μ.
- If $\mu(E) = 1$, we say that μ is a *probability measure* (機率測度).
- If $\mu(E) < \infty$, we say that μ is a *finite measure* (有限測度).
- If there exists an increasing measurable subsets (E_n) such that $E = \cup E_n$ and that for all n, we have $\mu(E_n) < \infty$, then we say that μ is a *σ-finite measure* (σ 有限測度).
- Let $x \in E$ and suppose that $\{x\} \in \mathcal{A}$. If $\mu(\{x\}) > 0$, then we say that x is an *atom* (原子) of μ and that μ is an *atomic measure* (原子測度). Otherwise, we say that μ is an *atomless measure* (無原子測度).

Example 1.1.13 :

(1) If μ is a finite measure on $(E, \mathcal{A})$, then $\frac{\mu}{\mu(E)}$ is a probability measure on the same measurable space.

(2) On the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the Lebesgue measure μ is the unique measure such that $\mu([a, b]) = \mu((a, b)) = b - a$ for any $a < b$. Its uniqueness is a consequence of the monotone

(2) 若 $A, B \in \mathcal{A}$，則

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B).$$

(3) 【下連續性】令 $(A_n)_{n\geqslant 1}$ 為取值在 $\mathcal{A}$ 中的遞增序列（也就是說對於所有 n，我們有 $A_n \subseteq A_{n+1}$），則

$$\mu\Big(\bigcup_{n\geqslant 1} A_n\Big) = \lim_{n\to\infty} \uparrow \mu(A_n).$$

(4) 【上連續性】令 $(B_n)_{n\geqslant 1}$ 為取值在 $\mathcal{A}$ 中的遞減序列（也就是說對於所有 n，我們有 $B_{n+1} \subseteq B_n$）。若存在 n_0 使得 $\mu(B_{n_0}) < \infty$，則

$$\mu\Big(\bigcap_{n\geqslant 1} B_n\Big) = \lim_{n\to\infty} \downarrow \mu(B_n).$$

(5) 若 $(A_n)_{n\geqslant 1}$ 為取值在 $\mathcal{A}$ 中的序列，則

$$\mu\Big(\bigcup_{n\geqslant 1} A_n\Big) \leqslant \sum_{n\geqslant 1} \mu(A_n).$$

證明：參照習題 1.5 。 □

定義 1.1.12 ： 給定一測度 μ。

- 我們稱 $\mu(E)$ 為 μ 的總質量 (total mass) 。
- 若 $\mu(E) = 1$，我們說 μ 是個 機率測度 (probability measure) .
- 若 $\mu(E) < \infty$，我們說 μ 是個有限測度 (finite measure) 。
- 若存在遞增可測的子集合 (E_n)，使得 $E = \cup E_n$ 且對於所有 n，我們有 $\mu(E_n) < \infty$，那麼我們說 μ 是個 σ 有限測度 (σ-finite measure) 。
- 令 $x \in E$ 並假設 $\{x\} \in \mathcal{A}$。若 $\mu(\{x\}) > 0$，我們說 x 是 μ 的原子 (atom) ，並說 μ 是個原子測度 (atomic measure) ；反之，則說 μ 是個無原子測度 (atomless measure) 。

範例 1.1.13 ：

(1) 如果 μ 是個在 $(E, \mathcal{A})$ 上的有限測度，那麼 $\frac{\mu}{\mu(E)}$ 在相同可測空間上是個機率測度。

(2) 在可測空間 $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ 上，勒貝格測度 μ 是滿足下列條件的唯一測度：對於任意 $a < b$，我們有 $\mu([a, b]) = \mu((a, b)) = b - a$。勒貝格測度的唯一性是單調類引理的直接結果，細

class lemma, see Corollary 1.1.19 and Remark 1.1.20 for details. Its existence can be derived from Theorem 1.2.29.

(3) (Dirac measure) Given a measurable space $(E, \mathcal{A})$. Let $x \in E$ and suppose $\{x\} \in \mathcal{A}$. The measure $\mu = \delta_x$ is the *Dirac measure* (狄拉克測度) at x.

Proposition 1.1.14 (Upper and lower continuity) **:** *Let $(E, \mathcal{A})$ be a measurable space. Define $\mu : \mathcal{A} \longrightarrow [0, \infty]$ satisfying*

- $\mu(\varnothing) = 0$,
- *(Finite additivity) For any* finite *disjoint sequence of $(A_n)_{1 \leqslant n \leqslant N}$ with values in $\mathcal{A}$, we have*

$$\mu\Big(\bigcup_{n=1}^{N} A_n\Big) = \sum_{n=1}^{N} \mu(A_n).$$

Then, the following properties are equivalent.

(i) (σ-additivity) μ satisfies σ-additivity mentioned in Eq. (1.1), that is, $(E, \mathcal{A}, \mu)$ is a measure space.

(ii) (Lower continuity) For any increasing sequence $(A_n)_{n \geqslant 1}$ with values in $\mathcal{A}$, we have

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n \to \infty} \uparrow \mu(A_n).$$

(iii) (Upper continuity) For any decreasing sequence $(A_n)_{n \geqslant 1}$ with values in $\mathcal{A}$, if there exists n_0 such that $\mu(A_{n_0}) < \infty$, then we have

$$\mu\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n \to \infty} \downarrow \mu(A_n).$$

Proof : Let us prove (i) $\Longrightarrow$ (ii). Given an increasing sequence $(A_n)_{n \geqslant 1}$, define the sequence $(B_n)_{n \geqslant 1}$ as follows. Set $B_1 := A_1$ and $B_n := A_n \backslash A_{n-1}$ for all $n \geqslant 2$. Under this construction, the elements B_n are pairwise disjoint and for all $n \geqslant 1$, we have

$$\bigcup_{k=1}^{n} B_k = A_n = \bigcup_{k=1}^{n} A_k \quad \text{and} \quad \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k.$$

As a consequence of σ-additivity, we find

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n \to \infty} \mu\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n \to \infty} \mu(A_n).$$

節請見系理 1.1.19 以及註解 1.1.20 。他的存在性可以由定理 1.2.29 推得。

(3) 【狄拉克測度】給定可測空間 $(E, \mathcal{A})$。令 $x \in E$ 並假設 $\{x\} \in \mathcal{A}$。測度 $\mu = \delta_x$ 是在 x 的狄拉克測度 (Dirac measure) 。

下列命題告訴我們，σ 可加性的假設是重要的，因為他與有限測度的連續性等價。

命題 1.1.14 【上連續與下連續】 **：** 令 $(E, \mathcal{A})$ 為可測空間，並且定義 $\mu : \mathcal{A} \longrightarrow [0, \infty]$ 滿足：

- $\mu(\varnothing) = 0$；
- 【有限加法性】對於任何取值在 $\mathcal{A}$ 中的有限互斥序列 $(A_n)_{1 \leqslant n \leqslant N}$，我們有

$$\mu\Big(\bigcup_{n=1}^{N} A_n\Big) = \sum_{n=1}^{N} \mu(A_n).$$

則下列性質等價。

(i) 【σ 可加性】μ 滿足式 (1.1) 中的 σ 可加性，也就是說 $(E, \mathcal{A}, \mu)$ 是個測度空間。

(ii) 【下連續】對於任意取值在 $\mathcal{A}$ 中的遞增序列 $(A_n)_{n \geqslant 1}$，我們有

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n \to \infty} \uparrow \mu(A_n).$$

(iii) 【上連續】對於任意取值在 $\mathcal{A}$ 中的遞減序列 $(A_n)_{n \geqslant 1}$，若存在 n_0 使得 $\mu(A_{n_0}) < \infty$，則我們有

$$\mu\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n \to \infty} \downarrow \mu(A_n).$$

證明： 證明 (i) $\Longrightarrow$ (ii)。給定遞增序列 $(A_n)_{n \geqslant 1}$，由下列方式來定義序列 $(B_n)_{n \geqslant 1}$：設 $B_1 := A_1$，對於所有 $n \geqslant 2$，設 $B_n := A_n \backslash A_{n-1}$。這樣的構造下，所有的 B_n 為兩兩互斥的，且對於所有 $n \geqslant 1$，我們有

$$\bigcup_{k=1}^{n} B_k = A_n = \bigcup_{k=1}^{n} A_k \quad \text{以及} \quad \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k.$$

因此根據 σ 可加性，我們有

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n \to \infty} \mu\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n \to \infty} \mu(A_n).$$

Let us prove (ii) $\Longrightarrow$ (i). Consider a sequence $(A_n)_{n\geqslant 1}$ of disjoint elements and define the sequence $(B_n)_{n\geqslant 1}$ as follows,

$$\forall n \geqslant 1, \qquad B_n := \bigcup_{k=1}^{n} A_k.$$

We note that $(B_n)_{n\geqslant 1}$ is an increasing sequence and using the property (ii), we find

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(A_k) = \sum_{k=1}^{\infty} \mu(A_k).$$

Let us prove (ii) $\Longrightarrow$ (iii). Consider a decreasing sequence $(A_n)_{n\geqslant 1}$ with n_0 such that $\mu(A_{n_0}) < \infty$. Set $B_n := A_n^c \cap A_{n_0}$ for all $n \geqslant 1$. Then, the sequence $(B_n)_{n\geqslant 1}$ is increasing. Using the property (ii), we find

$$\begin{aligned}
\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcap_{n=n_0}^{\infty} (A_n \cap A_{n_0})\right) &= \mu\left(A_{n_0} \backslash \Big(\bigcup_{n=n_0}^{\infty} B_n\Big)\right) \\
&= \mu(A_{n_0}) - \mu\left(\bigcup_{n=n_0}^{\infty} B_n\right) \\
&= \mu(A_{n_0}) - \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_n).
\end{aligned}$$

To finish the proof, we use a similar method to prove (iii) $\Longrightarrow$ (ii). □

1.1.3 Monotone Classes

In the previous subsections, we defined the notion of σ-algebras and measures. However, σ-algebras are not easy to manipulate directly, since in general, one does not have a way to write down a generic element from them. Apart from this, in the definition of a measure, we talk about countable unions of *disjoint* sequences of subsets; but in the definition of a σ-algebra, this can be countable unions of *any* sequence of subsets. As a consequence, when we need to construct a measure on a measurable space, or to prove some specific conditions such as the uniqueness of the measure, it can be a bit tricky. In this subsection, we will introduce the notion of *monotone classes*, state the monotone class lemma (Theorem 1.1.18) and discuss about the uniqueness of the measure (Corollary 1.1.19).

Definition 1.1.15 : Let $\mathcal{M}$ be a subset of $\mathcal{P}(E)$. We call $\mathcal{M}$ a *monotone class* (單調類) if the following conditions are satisfied.

(a) $E \in \mathcal{M}$.

(b) If $A, B \in \mathcal{M}$ and $A \subseteq B$, then $B\backslash A \in \mathcal{M}$.

(c) If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of elements in $\mathcal{M}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$.

證明 (ii) $\Longrightarrow$ (i)。考慮互斥序列 $(A_n)_{n\geqslant 1}$ 並由下列方式來定義序列 $(B_n)_{n\geqslant 1}$：

$$\forall n \geqslant 1, \qquad B_n := \bigcup_{k=1}^{n} A_k.$$

我們注意到，$(B_n)_{n\geqslant 1}$ 是個遞增序列，因此根據 (ii) 的性質，我們有

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(A_k) = \sum_{k=1}^{\infty} \mu(A_k).$$

證明 (ii) $\Longrightarrow$ (iii)。考慮遞減的序列 $(A_n)_{n\geqslant 1}$ 以及 n_0 使得 $\mu(A_{n_0}) < \infty$。對所有 $n \geqslant 1$，設 $B_n := A_n^c \cap A_{n_0}$，則序列 $(B_n)_{n\geqslant 1}$ 會是遞增的。利用 (ii) 的性質，我們有

$$\begin{aligned}
\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcap_{n=n_0}^{\infty} (A_n \cap A_{n_0})\right) &= \mu\left(A_{n_0} \backslash \Big(\bigcup_{n=n_0}^{\infty} B_n\Big)\right) \\
&= \mu(A_{n_0}) - \mu\left(\bigcup_{n=n_0}^{\infty} B_n\right) \\
&= \mu(A_{n_0}) - \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_n).
\end{aligned}$$

最後，我們可以用類似的手法來證明 (iii) $\Longrightarrow$ (ii)。 □

第三小節 單調類

在前面，給定一個空間，我們定義了 σ 代數以及測度的概念。但 σ 代數並不是一個好操作的定義，因為一般來講，要寫出 σ 代數中的元素並不容易。除此之外，在測度的定義中，我們討論的是互斥可數序列的聯集，但 σ 代數的條件卻是任意可數序列，所以當我們要在一個可測空間之上建構測度，或是證明在某些條件下，測度的唯一性，會比較棘手。也因此，在這小節中，我們要引進單調類的概念，並且敘述單調類引理（定理 1.1.18），以及討論測度的唯一性（系理 1.1.19）。

定義 1.1.15 ： 令 $\mathcal{M}$ 為 $\mathcal{P}(E)$ 的子集合，若 $\mathcal{M}$ 滿足下列條件：

(a) $E \in \mathcal{M}$。

(b) 若 $A, B \in \mathcal{M}$ 及 $A \subseteq B$，則 $B\backslash A \in \mathcal{M}$。

(c) 若 $(A_n)_{n\in\mathbb{N}}$ 為在 $\mathcal{M}$ 之上的遞增集合序列，則 $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$。

則稱之為單調類 (monotone class)。

Remark 1.1.16 : Any σ-algebra is also a monotone class.

As in the case of σ-algebras, we can also define the notion of a *generated monoton class* (生成單調類) .

Definition 1.1.17 : For any $\mathcal{C} \subseteq \mathcal{P}(E)$, there exists a smallest monotone class such that it contains $\mathcal{C}$, denoted

$$\mathcal{M}(\mathcal{C}) = \bigcap_{\substack{\mathcal{M} \text{ is a monotone class} \\ \text{s.t. } \mathcal{C} \subseteq \mathcal{M}}} \mathcal{M}.$$

The following lemma tells us the conditions under which a generated monotone class and a generated σ-algebra are equal.

Theorem 1.1.18 (Monotone class lemma) **:** *If $\mathcal{C} \subseteq \mathcal{P}(E)$ is closed under finite intersection, then $\mathcal{M}(\mathcal{C}) = \sigma(\mathcal{C})$.*

This is one of the most important statements in Measure Theory. We only provide the necessary steps for the proof, the complete proof being an exercise.

Proof : From the above remark, we know that $\mathcal{M}(\mathcal{C}) \subseteq \sigma(\mathcal{C})$, so we only need to prove the other inclusion. We need to prove the two following points.

(1) If $\mathcal{M}$ is a monotone class and is closed under finite intersections, then $\mathcal{M}$ is a σ-algebra.

(2) Prove that $\mathcal{M}(\mathcal{C})$ is closed under finite intersections in two steps:
   (a) Given $A \in \mathcal{C}$, let $\mathcal{M}_1 = \{B \in \mathcal{M}(C) : A \cap B \in \mathcal{M}(\mathcal{C})\}$ and prove $\mathcal{M}_1 = \mathcal{M}(\mathcal{C})$.
   (b) Given $B \in \mathcal{M}(\mathcal{C})$, let $\mathcal{M}_2 = \{A \in \mathcal{M}(\mathcal{C}) : A \cap B \in \mathcal{M}(\mathcal{C})\}$ and prove $\mathcal{M}_2 = \mathcal{M}(\mathcal{C})$. □

The monotone class lemma allows us to show the uniqueness of the measure.

Corollary 1.1.19 : *Let μ and ν be two measures on the measurable space $(E, \mathcal{A})$. Assume that there exists a subset $\mathcal{C} \subseteq \mathcal{A}$ which is closed under finite intersections satisfying $\sigma(\mathcal{C}) = \mathcal{A}$ and such that for all $A \in \mathcal{C}$, we have $\mu(A) = \nu(A)$.*

(1) (Finite measure) If $\mu(E) = \nu(E) < \infty$, then $\mu = \nu$.

(2) (σ-finite measure) If there exists a increasing sequence (E_n) in $\mathcal{C}$ such that $E = \cup E_n$ and that for all n, we have $\mu(E_n) = \nu(E_n)$, then $\mu = \nu$.

Proof :

(1) Let $\mathcal{G} = \{A \in \mathcal{A} : \mu(A) = \nu(A)\}$. From the hypothesis, we have $\mathcal{C} \subseteq \mathcal{G}$. Using the properties of a measure, we can show that $\mathcal{G}$ is a monotone class. Moreover, since $\mathcal{C}$ is closed under finite

註解 1.1.16 ： 任何一個 σ 代數也是個單調類。

如同 σ 代數的情況，我們也可以定義生成單調類 (generated monotone class) 的概念。

定義 1.1.17 ： 對於任意 $\mathcal{C} \subseteq \mathcal{P}(E)$，存在一個最小的單調類，使得他包含 $\mathcal{C}$ ，記作

$$\mathcal{M}(\mathcal{C}) = \bigcap_{\substack{\mathcal{M} \text{ 為單調類} \\ \text{使得 } \mathcal{C} \subseteq \mathcal{M}}} \mathcal{M}.$$

下列引理告訴我們在怎樣的情況下，生成單調類和生成 σ 代數是相等的。

定理 1.1.18 【單調類引理】 ： 若 $\mathcal{C} \subseteq \mathcal{P}(E)$ 在有限交集下是封閉的，那麼 $\mathcal{M}(\mathcal{C}) = \sigma(\mathcal{C})$。

這是測度論中重要定理之一，我們這裡不提供完整的證明，只給出需要證明的步驟。

證明：從上面的註解，我們知道 $\mathcal{M}(\mathcal{C}) \subseteq \sigma(\mathcal{C})$，因此只需要證明另一個包含關係。我們需要證明下列兩點：

(1) 若 $\mathcal{M}$ 為單調類，且在有限交集下是封閉的，則 $\mathcal{M}$ 為 σ 代數。

(2) 分兩步驟證明 $\mathcal{M}(\mathcal{C})$ 在有限交集下是封閉的：
   (a) 給定 $A \in \mathcal{C}$，令 $\mathcal{M}_1 = \{B \in \mathcal{M}(C) : A \cap B \in \mathcal{M}(\mathcal{C})\}$，證明 $\mathcal{M}_1 = \mathcal{M}(\mathcal{C})$。
   (b) 給定 $B \in \mathcal{M}(\mathcal{C})$，令 $\mathcal{M}_2 = \{A \in \mathcal{M}(\mathcal{C}) : A \cap B \in \mathcal{M}(\mathcal{C})\}$，證明 $\mathcal{M}_2 = \mathcal{M}(\mathcal{C})$。 □

單調類引理可以讓我們證明測度的唯一性。

系理 1.1.19 ： 令 μ 及 ν 為兩個在 $(E, \mathcal{A})$ 上的測度。假設存在一個子集合 $\mathcal{C} \subseteq \mathcal{A}$，在有限交集下是封閉的，滿足 $\sigma(\mathcal{C}) = \mathcal{A}$ 且對於所有的 $A \in \mathcal{C}$，我們有 $\mu(A) = \nu(A)$。

(1) 【有限測度】若 $\mu(E) = \nu(E) < \infty$，我們有 $\mu = \nu$。

(2) 【σ 有限測度】若存在 $\mathcal{C}$ 中遞增序列 (E_n)，使得 $E = \cup E_n$ 且對於所有 n，我們有 $\mu(E_n) = \nu(E_n)$，那麼 $\mu = \nu$。

證明：

(1) 令 $\mathcal{G} = \{A \in \mathcal{A} : \mu(A) = \nu(A)\}$。根據假設，我們有 $\mathcal{C} \subseteq \mathcal{G}$，且根據測度的性質，我們可以證明 $\mathcal{G}$ 是個單調類。此外，由於 $\mathcal{C}$ 在有限交集下是封閉的，根據單調類引理，我們有

intersections, according to the monotone class lemma, we have $\mathcal{M}(\mathcal{C}) = \sigma(\mathcal{C}) = \mathcal{A}$. This proves that $\mathcal{G} = \mathcal{A}$, which is $\mu = \nu$.

(2) For all n, we can define the restricted measures of μ and ν on E_n,

$$\forall A \in \mathcal{A}, \quad \mu_n(A) = \mu(A \cap E_n), \quad \nu_n(A) = \nu(A \cap E_n).$$

From the first part of the proof, we obtain $\mu_n = \nu_n$, then from the property (3) of a measure, we get

$$\forall A \in \mathcal{A}, \quad \mu(A) = \lim \uparrow \mu_n(A \cap E_n), \quad \nu(A) = \lim \uparrow \nu_n(A \cap E_n).$$

This proves the desired statement. □

Remark 1.1.20 : The above corollary states that if there exists two measures λ and λ' on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that for all intervals (a, b), we have $\lambda((a, b)) = \lambda'((a, b)) = b - a$, then they must be the same measure $\lambda = \lambda'$.

Question 1.1.21: Please explain why it is important to assume that μ and ν are finite measures or σ-finite measures in Corollary 1.1.19. In other words, is it possible to find infinite measures μ and ν such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{C}$, but $\mu \neq \nu$?

$\mathcal{M}(\mathcal{C}) = \sigma(\mathcal{C}) = \mathcal{A}$。由此得證 $\mathcal{G} = \mathcal{A}$，也就是 $\mu = \nu$。

(2) 對於所有 n，我們可以定義 μ 及 ν 設限在 E_n 上的測度：

$$\forall A \in \mathcal{A}, \quad \mu_n(A) = \mu(A \cap E_n), \quad \nu_n(A) = \nu(A \cap E_n).$$

根據第一部份，我們有 $\mu_n = \nu_n$，接著使用測度的性質三，我們有

$$\forall A \in \mathcal{A}, \quad \mu(A) = \lim \uparrow \mu_n(A \cap E_n), \quad \nu(A) = \lim \uparrow \nu_n(A \cap E_n).$$

由此得證。□

註解 1.1.20 ： 上述系理告訴我們若存在 $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ 上存在兩個測度 λ 及 λ'，使得對於所有開區間 (a, b)，我們有 $\lambda((a, b)) = \lambda'((a, b)) = b - a$，則他們必定為同一個測度 $\lambda = \lambda'$。

問題 1.1.21：請解釋為什麼在系理 1.1.19 中，有限測度或是 σ 有限測度的假設是重要的。換句話說，是否能夠找到無限測度 μ 及 ν，使得對於所有 $A \in \mathcal{C}$，我們有 $\mu(A) = \nu(A)$，但卻會有 $\mu \neq \nu$？