6 Discrete Time Martingales

離散時間鞅

第六章

6.1 Definitions and Examples

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A sequence of random variables $(X_n)_{n\geqslant 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *random process* (隨機過程). Note that the index set of $(X_n)_{n\geqslant 0}$ is given by non-negative integers, that is $\mathbb{N}_0 = \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \dots\}$, which can be intepreted by the fact that our random process is indexed by discrete time steps. In this course, we may also assume that our random processes take their values in $\mathbb{R}$.

Definition 6.1.1 : In $(\Omega, \mathcal{F}, \mathbb{P})$, a non-decreasing sequence $(\mathcal{F}_n)_{n\geqslant 0}$ of sub-σ-algebras is called a *filtration* (濾鏈) if

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}.$$

We then call $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0}, \mathbb{P})$ a *filtered probability space* (濾鏈機率空間).

In the above definition, we can interprete n as time and $\mathcal{F}_n$ can be understood as the information that we are disposed of up to time n.

Example 6.1.2 : Below are two examples of filtration.

(1) If $(X_n)_{n\geqslant 0}$ is a random process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, let

$$\forall n \geqslant 0, \quad \mathcal{F}_n^X := \sigma(X_0, \dots, X_n),$$

be the smallest σ-algebra making the random variables $X_0, \dots, X_n$ measurable. Then, we say that $(\mathcal{F}_n^X)_{n\geqslant 0}$ is the *canonical filtration* (正則濾鏈) of the random process $(X_n)_{n\geqslant 0}$.

(2) Assume that $\Omega = [0, 1)$ and $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel σ-algebra and $\mathbb{P}$ be the Lebesgue measure. Let

$$\forall n \geqslant 0, \quad \mathcal{F}_n := \sigma([\tfrac{i-1}{2^n}, \tfrac{i}{2^n}) : 1 \leqslant i \leqslant 2^n).$$

Then, we say that $(\mathcal{F}_n)_{n\geqslant 0}$ is the *dyadic filtration* (二元濾鏈) on $[0, 1)$. Alternatively, we may also see the dyadic filtration as the canonical filtration of the random process $(X_n)_{n\geqslant 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, where $X_n(\omega)$ gives the n-th digit in the (finite) dyadic expansion of ω.

第一節 定義及例子

我們固定機率空間 $(\Omega, \mathcal{F}, \mathbb{P})$，我們稱在 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的隨機變數序列 $(X_n)_{n\geqslant 0}$ 為隨機過程 (random process)。我們可以注意到，$(X_n)_{n\geqslant 0}$ 的下標集合為非負整數，也就是 $\mathbb{N}_0 = \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \dots\}$，我們可以把此集合理解為我們的隨機過程是由離散時間所標記的。在這裡，我們假設所有的隨機過程值域皆為 $\mathbb{R}$。

定義 6.1.1 ： 在 $(\Omega, \mathcal{F}, \mathbb{P})$ 中，非遞減的子 σ 代數序列 $(\mathcal{F}_n)_{n\geqslant 0}$ 稱作濾鏈 (filtration)，也就是

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}.$$

我們稱 $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0}, \mathbb{P})$ 為濾鏈機率空間 (filtered probability space)。

在上述定義中，我們可以把 n 理解為時間，這樣的話，可以把 $\mathcal{F}_n$ 視作在時間 n 所擁有的資訊。

範例 6.1.2 ： 下面我們給出兩個濾鏈的例子。

(1) 若 $(X_n)_{n\geqslant 0}$ 是個定義在 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的隨機過程，我們定義

$$\forall n \geqslant 0, \quad \mathcal{F}_n^X := \sigma(X_0, \dots, X_n),$$

為使得隨機變數 $X_0, \dots, X_n$ 可測的最小 σ 代數，則我們稱 $(\mathcal{F}_n^X)_{n\geqslant 0}$ 為隨機過程 $(X_n)_{n\geqslant 0}$ 的正則濾鏈 (canonical filtration)。

(2) 假設 $\Omega = [0, 1)$ 且 $\mathcal{F} = \mathcal{B}(\Omega)$ 為布雷爾 σ 代數，$\mathbb{P}$ 為勒貝格測度。設

$$\forall n \geqslant 0, \quad \mathcal{F}_n := \sigma([\tfrac{i-1}{2^n}, \tfrac{i}{2^n}) : 1 \leqslant i \leqslant 2^n).$$

則我們稱 $(\mathcal{F}_n)_{n\geqslant 0}$ 為 $[0, 1)$ 上的二元濾鏈 (dyadic filtration)。換個方法看，我們可以把二元濾鏈看作是在 $(\Omega, \mathcal{F}, \mathbb{P})$ 上隨機過程 $(X_n)_{n\geqslant 0}$ 的正則濾鍊，其中 $X_n(\omega)$ 的值為 ω 的（有限）二元展開中的第 n 個小數點值。

Definition 6.1.3 : Given a random process $(X_n)_{n\geqslant 0}$. If the random variable X_n is $\mathcal{F}_n$-measurable for all $n \geqslant 0$, then we say that $(X_n)_{n\geqslant 0}$ is a $(\mathcal{F}_n)_{n\geqslant 0}$-*adapted* (適應) random process.

Remark 6.1.4 : The canonical filtration can be understood as the smallest (for inclusion) filtration making the random process adapted.

In what follows, we fix a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0}, \mathbb{P})$ and define the following notions below.

Definition 6.1.5 : Let $(X_n)_{n\geqslant 0}$ be an adapted random process and assume that we have $\mathbb{E}[|X_n|] < \infty$ for all $n \geqslant 0$.

(1) If for all $n \geqslant 0$, we have,
$$\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] = X_n,$$
then we say that the random process $(X_n)_{n\geqslant 0}$ is a *martingale* (鞅).

(2) If for all $n \geqslant 0$, we have,
$$\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] \leqslant X_n,$$
then we say that the random process $(X_n)_{n\geqslant 0}$ is a *supermartingale* (上鞅).

(3) If for all $n \geqslant 0$, we have,
$$\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] \geqslant X_n,$$
then we say that the random process $(X_n)_{n\geqslant 0}$ is a *submartingale* (下鞅).

Remark 6.1.6 : From the above definition, a martingale can be easily checked to satisfy the following property: for all $0 \leqslant n \leqslant m$,
$$\mathbb{E}[X_m \,|\, \mathcal{F}_n] = X_n. \tag{6.1}$$
This also implies $\mathbb{E}[X_m] = \mathbb{E}[X_n] = \mathbb{E}[X_0]$. In the case of a supermartingale or a submartingale, we can also deduce similar properties and the equality becomes the corresponding inequality.

In general, we illustrate the notion of martingale using a fair game: X_n represents the total asset of a player at time n, $\mathcal{F}_n$ the information at his / her disposition at time n (and before). The martingale property $\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] = X_n$ can be understood by the fact that the expected asset of the player at time $n+1$, knowing what happened up to time n, is equal to his / her asset at time n. In other words, in average, the player does not win or lose any money.

Using this interpretation, we can view a supermartingale as a game where the banker has advantages.

We can also note that, if $(X_n)_{n\geqslant 0}$ is a supermartingale, then $(-X_n)_{n\geqslant 0}$ is a submartingale. As a consequence, later when we state results on super/submartingales, we only take care of either the case of super- or submartingales.

定義 6.1.3 ： 給定隨機過程 $(X_n)_{n\geqslant 0}$，若對於所有 $n \geqslant 0$，隨機變數 X_n 是 $\mathcal{F}_n$ 可測的，則我們說 $(X_n)_{n\geqslant 0}$ 是個適應 (adapted) 濾鏈 $(\mathcal{F}_n)_{n\geqslant 0}$ 的隨機過程。

註解 6.1.4 ： 正則濾鏈可以看作是使隨機過程適應的最小（在包含的意義下）濾鏈。

再來，我們固定濾鏈機率空間 $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0}, \mathbb{P})$，並在上面定義下列概念。

定義 6.1.5 ： 令 $(X_n)_{n\geqslant 0}$ 為適應隨機過程，並假設對於所有 $n \geqslant 0$，我們有 $\mathbb{E}[|X_n|] < \infty$。

(1) 若對於所有 $n \geqslant 0$，我們有
$$\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] = X_n,$$
則我們稱隨機過程 $(X_n)_{n\geqslant 0}$ 為鞅 (martingale)。

(2) 若對於所有 $n \geqslant 0$，我們有
$$\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] \leqslant X_n,$$
則我們稱隨機過程 $(X_n)_{n\geqslant 0}$ 為上鞅 (supermartingale)。

(3) 若對於所有 $n \geqslant 0$，我們有
$$\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] \geqslant X_n,$$
則我們稱隨機過程 $(X_n)_{n\geqslant 0}$ 為下鞅 (submartingale)。

註解 6.1.6 ： 根據上述鞅的定義，我們可以輕易驗證下列性質：對於所有 $0 \leqslant n \leqslant m$，
$$\mathbb{E}[X_m \,|\, \mathcal{F}_n] = X_n. \tag{6.1}$$
這同時也告訴我們，$\mathbb{E}[X_m] = \mathbb{E}[X_n] = \mathbb{E}[X_0]$。在上鞅或下鞅的情況下，我們可以得到類似的性質且等式會變成相對應的不等式。

通常，我們可以將鞅的概念用公平遊戲來詮釋：X_n 代表玩家在時間 n 的總資產，$\mathcal{F}_n$ 是他在時間 n 所擁有的資訊（包含在此時間點以前的遊戲過程），鞅的性質 $\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] = X_n$ 可以詮釋為此玩家在時間 $n+1$ 總資產的期望值，若我們只有時間 n 以前的資訊的話，會是他在時間 n 的總資產，也就是說，平均來講，他不會賺也不會賠。

用這樣的詮釋方式，上鞅代表的是有利莊家的遊戲。我們還可以注意到，若 $(X_n)_{n\geqslant 0}$ 是個上鞅，則 $(-X_n)_{n\geqslant 0}$ 會是個下鞅，因此在後續我們討論的上下鞅時，通常只會給出上鞅或下鞅的結果，因為反之亦然。

Example 6.1.7 : Here we give a few examples of martingales, supermartingales and submartingales and define some additional notions. Let us fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a filtration $(\mathcal{F}_n)_{n\geqslant 0}$.

(1) If $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, let

$$\forall n \geqslant 0, \quad X_n = \mathbb{E}[X \mid \mathcal{F}_n]. \tag{6.2}$$

Then $(X_n)_{n\geqslant 0}$ is a martingale,

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] = \mathbb{E}[X \mid \mathcal{F}_n] = X_n.$$

A martingale satisfying Eq. (6.2) is called a *closed martingale* (封閉鞅) .

(2) If $(X_n)_{n\geqslant 0}$ is a non-increasing sequence of adapted and integrable random variables, then $(X_n)_{n\geqslant 0}$ is a supermartingale. Indeed, we have

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leqslant \mathbb{E}[X_n \mid \mathcal{F}_n] = X_n.$$

(3) We look at a random walk on $\mathbb{R}$. Let $x \in \mathbb{R}$ and $(Y_n)_{n\geqslant 1}$ be an i.i.d. sequence of random variables with distribution μ and $\mathbb{E}[|Y_1|] < \infty$. Let

$$X_0 = x \quad \text{and} \quad \forall n \geqslant 1, \quad X_n = x + Y_1 + \cdots + Y_n.$$

The filtration $(\mathcal{F}_n)_{n\geqslant 0}$ is defined as,

$$\mathcal{F}_0 = \{\varnothing, \Omega\} \quad \text{and} \quad \forall n \geqslant 1, \quad \mathcal{F}_n = \sigma(Y_1, \ldots, Y_n),$$

which is the canonical filtration of $(X_n)_{n\geqslant 0}$.

(a) If $\mathbb{E}[Y_1] = 0$, then $(X_n)_{n\geqslant 0}$ is a martingale.

(b) If $\mathbb{E}[Y_1] \leqslant 0$, then $(X_n)_{n\geqslant 0}$ is a supermartingale.

(c) If $\mathbb{E}[Y_1] \geqslant 0$, then $(X_n)_{n\geqslant 0}$ is a submartingale.

We say that $(X_n)_{n\geqslant 0}$ is a random walk on $\mathbb{R}$ started at x with jump distribution given by μ.

(4) Let us come back to (2) in Example 6.1.2 and remind that the filtration of our interest is the dyadic filtration $(\mathcal{F}_n)_{n\geqslant 0}$. Let μ be a finite measure on $[0,1)$, $\mathbb{P} = \lambda$ be the Lebesgue measure on $[0,1)$. For all $n \geqslant 0$, let

$$f_n = \Big(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\Big)_{|\mathcal{F}_n},$$

where the Radon–Nikodym derivative is defined by considering μ and λ as measures restricted on the σ-algebra $\mathcal{F}_n$. The Radon–Nikodym derivative is well defined because all the finite measures on $\mathcal{F}_n$ are absolutely continuous with respect to $\lambda_{|\mathcal{F}_n}$. Hence, we obtain,

$$f_n(\omega) = \sum_{i=1}^{2^n} \frac{\mu([(i-1)2^{-n}, i2^{-n}))}{2^{-n}} \mathbb{1}_{[(i-1)2^{-n}, i2^{-n})}(\omega).$$

Then, $(f_n)_{n\geqslant 1}$ is a martingale since for all $A \in \mathcal{F}_n$, we have,

$$\mathbb{E}[\mathbb{1}_A f_{n+1}] = \int \mathbb{1}_A(\omega) f_{n+1}(\omega)\,\mathrm{d}\omega = \mu(A) = \int \mathbb{1}_A(\omega) f_n(\omega)\,\mathrm{d}\omega = \mathbb{E}[\mathbb{1}_A f_n],$$

implying $f_n = \mathbb{E}[f_{n+1} \mid \mathcal{F}_n]$.

範例 6.1.7 ： 這裡我們給出幾個鞅、上鞅、下鞅的例子，以及定義一些新的概念。我們固定機率空間 $(\Omega, \mathcal{A}, \mathbb{P})$ 及濾鏈 $(\mathcal{F}_n)_{n\geqslant 0}$。

(1) 若 $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$，設

$$\forall n \geqslant 0, \quad X_n = \mathbb{E}[X \mid \mathcal{F}_n]. \tag{6.2}$$

則 $(X_n)_{n\geqslant 0}$ 是個鞅：

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] = \mathbb{E}[X \mid \mathcal{F}_n] = X_n.$$

我們稱滿足式 (6.2) 的鞅為封閉鞅 (closed martingale) 。

(2) 若 $(X_n)_{n\geqslant 0}$ 是個適應且由可積隨機變數構成的非遞增序列，則 $(X_n)_{n\geqslant 0}$ 是個上鞅：

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leqslant \mathbb{E}[X_n \mid \mathcal{F}_n] = X_n.$$

(3) 我們討論 $\mathbb{R}$ 上的隨機漫步。令 $x \in \mathbb{R}$ 以及 $(Y_n)_{n\geqslant 1}$ 為 i.i.d. 隨機變數序列，且假設其分佈為 μ 以及 $\mathbb{E}[|Y_1|] < \infty$。設

$$X_0 = x \quad \text{且} \quad \forall n \geqslant 1, \quad X_n = x + Y_1 + \cdots + Y_n.$$

我們將濾鏈 $(\mathcal{F}_n)_{n\geqslant 0}$ 定義作

$$\mathcal{F}_0 = \{\varnothing, \Omega\} \quad \text{且} \quad \forall n \geqslant 1, \quad \mathcal{F}_n = \sigma(Y_1, \ldots, Y_n),$$

也就是 $(X_n)_{n\geqslant 0}$ 的正則濾鏈。

(a) 若 $\mathbb{E}[Y_1] = 0$，則 $(X_n)_{n\geqslant 0}$ 是個鞅。

(b) 若 $\mathbb{E}[Y_1] \leqslant 0$，則 $(X_n)_{n\geqslant 0}$ 是個上鞅。

(c) 若 $\mathbb{E}[Y_1] \geqslant 0$，則 $(X_n)_{n\geqslant 0}$ 是個下鞅。

我們稱 $(X_n)_{n\geqslant 0}$ 為在 $\mathbb{R}$ 上由 x 出發，跳躍分佈為 μ 的隨機漫步。

(4) 我們回到範例 6.1.2 中 (2) 的例子，並且別忘記我們有興趣的濾鍊是二元濾鍊 $(\mathcal{F}_n)_{n\geqslant 0}$。令 μ 是個在 $[0,1)$ 上的有限測度，$\mathbb{P} = \lambda$ 是在 $[0,1)$ 上的勒貝格測度。對於所有 $n \geqslant 0$，令

$$f_n = \Big(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\Big)_{|\mathcal{F}_n},$$

其中 Radon–Nikodym 微分定義方式是將 μ 及 λ 視為在 σ 代數 $\mathcal{F}_n$ 上的測度。此 Radon–Nikodym 微分定義良好，因為在 $\mathcal{F}_n$ 之上，所有有限測度對於 $\lambda_{|\mathcal{F}_n}$ 皆是絕對連續的。因此我們可以得到

$$f_n(\omega) = \sum_{i=1}^{2^n} \frac{\mu([(i-1)2^{-n}, i2^{-n}))}{2^{-n}} \mathbb{1}_{[(i-1)2^{-n}, i2^{-n})}(\omega).$$

If $\mu \ll \nu$ on $\mathcal{F}$, we write f for Radon–Nikodym derivative of μ with respect to ν, then we have a closed martingale,

$$\forall n \geqslant 1, \quad f_n = \mathbb{E}[f \mid \mathcal{F}_n].$$

Proposition 6.1.8 : *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_{\geqslant 0}$ be a non-negative convex function. Let $(X_n)_{n\geqslant 0}$ be an adapted process and suppose that for all $n \geqslant 0$, we have, $\mathbb{E}[\varphi(X_n)] < \infty$.*

(1) If (X_n) is a martingale, then $(\varphi(X_n))$ is a submartingale.

(2) If (X_n) is a submartingale and φ is non-decreasing, then $(\varphi(X_n))$ is also a submartingale.

Remark 6.1.9 : From this proposition, we know that if $(X_n)_{n\geqslant 0}$ is a martingale, then $(|X_n|)_{n\geqslant 0}$ is a submartingale; if for all n, X_n is square-integrable, then $(X_n^2)_{n\geqslant 0}$ is also a submartingale. If $(X_n)_{n\geqslant 0}$ is a submartingale, then so is $(X_n^+)_{n\geqslant 0}$.

Proof :

(1) We use Jensen's inequality of conditional expectations ((6) in Proposition 5.2.8) to obtain,

$$\forall n \geqslant 0, \quad \mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] \geqslant \varphi(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]) = \varphi(X_n).$$

(2) The proof is similar since $X_n \leqslant \mathbb{E}[X_{n+1} \mid \mathcal{F}_n]$ and φ is non-decreasing, we have,

$$\forall n \geqslant 0, \quad \mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] \geqslant \varphi(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]) \geqslant \varphi(X_n).$$

□

Definition 6.1.10 : Given a real sequence $(H_n)_{n\geqslant 1}$ of random variables. We say that $(H_n)_{n\geqslant 1}$ is a *predictable* (可預測) sequence if for all $n \geqslant 1$, H_n is bounded and is $\mathcal{F}_{n-1}$-measurable.

Proposition 6.1.11 : *Given an adapted process $(X_n)_{n\geqslant 0}$ and a predictable sequence $(H_n)_{n\geqslant 1}$. Let $(H \cdot X)_0 = 0$ and*

$$\forall n \geqslant 1, \quad (H \cdot X)_n = H_1(X_1 - X_0) + \cdots + H_n(X_n - X_{n-1}).$$

Then, we have the two following properties,

則 $(f_n)_{n\geqslant 1}$ 是個鞅：因為對於所有 $A \in \mathcal{F}_n$，我們有

$$\mathbb{E}[\mathbb{1}_A f_{n+1}] = \int \mathbb{1}_A(\omega) f_{n+1}(\omega)\, \mathrm{d}\omega = \mu(A) = \int \mathbb{1}_A(\omega) f_n(\omega)\, \mathrm{d}\omega = \mathbb{E}[\mathbb{1}_A f_n],$$

也就是說 $f_n = \mathbb{E}[f_{n+1} \mid \mathcal{F}_n]$。

若在 $\mathcal{F}$ 上 $\mu \ll \nu$，我們記 f 為 μ 對 ν 的 Radon–Nikodym 微分，則我們得到封閉鞅：

$$\forall n \geqslant 1, \quad f_n = \mathbb{E}[f \mid \mathcal{F}_n].$$

命題 6.1.8 ： 令 $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_{\geqslant 0}$ 為非負凸函數，令 $(X_n)_{n\geqslant 0}$ 為適應過程且假設對於所有 $n \geqslant 0$，我們有 $\mathbb{E}[\varphi(X_n)] < \infty$。

(1) 若 (X_n) 是個鞅，則 $(\varphi(X_n))$ 是個下鞅。

(2) 若 (X_n) 是個下鞅，且 φ 非遞減，則 $(\varphi(X_n))$ 也是個下鞅。

註解 6.1.9 ： 從此命題，我們得知若 $(X_n)_{n\geqslant 0}$ 為鞅，$(|X_n|)_{n\geqslant 0}$ 會是個下鞅；若對於所有 n，X_n 皆為平方可積，則 $(X_n^2)_{n\geqslant 0}$ 也是個下鞅。若 $(X_n)_{n\geqslant 0}$ 是個下鞅，則 $(X_n^+)_{n\geqslant 0}$ 也是。

證明：

(1) 我們用條件期望值中的琴生不等式（命題 5.2.8 中的 (6)），而得到

$$\forall n \geqslant 0, \quad \mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] \geqslant \varphi(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]) = \varphi(X_n).$$

(2) 同理，由於 $X_n \leqslant \mathbb{E}[X_{n+1} \mid \mathcal{F}_n]$ 且 φ 非遞減，

$$\forall n \geqslant 0, \quad \mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] \geqslant \varphi(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]) \geqslant \varphi(X_n).$$

□

定義 6.1.10 ： 給定實隨機變數序列 $(H_n)_{n\geqslant 1}$。若對於所有 $n \geqslant 1$，H_n 有界且對於 $\mathcal{F}_{n-1}$ 可測，則我們說 $(H_n)_{n\geqslant 1}$ 為可預測 (predictable) 的序列。

命題 6.1.11 ： 給定適應過程 $(X_n)_{n\geqslant 0}$ 以及可預測序列 $(H_n)_{n\geqslant 1}$。令 $(H \cdot X)_0 = 0$ 以及

$$\forall n \geqslant 1, \quad (H \cdot X)_n = H_1(X_1 - X_0) + \cdots + H_n(X_n - X_{n-1}).$$

則我們有下列兩個性質

(1) *If (X_n) is a martingale, then $((H \cdot X)_n)$ is also a martingale.*

(2) *If (X_n) is a supermartingale (resp. a submartingale) and for all $n \geqslant 1$, $H_n \geqslant 0$, then $((H \cdot X)_n)$ is also a supermartingale (resp. a submartingale).*

Proof :

(1) Since the random variables H_n are are bounded, we can check that all the random variables $(H \cdot X)_n$ are still integrable; and from construction, the random process $((H \cdot X)_n)$ is still an adapted process. Hence, we only need to check that for all $n \geqslant 0$, we have,

$$\mathbb{E}[(H \cdot X)_{n+1} - (H \cdot X)_n \,|\, \mathcal{F}_n] = 0.$$

Since $(H \cdot X)_{n+1} - (H \cdot X)_n = H_{n+1}(X_{n+1} - X_n)$ and H_{n+1} is $\mathcal{F}_n$-measurable, we have,

$$\mathbb{E}[H_{n+1}(X_{n+1} - X_n) \,|\, \mathcal{F}_n] = H_{n+1}\,\mathbb{E}[X_{n+1} - X_n \,|\, \mathcal{F}_n] = 0.$$

The proof is complete.

(2) The proof is similar to (1).

□

In the case of (1), if X_n represents the total asset of a player at time n, then $X_{n+1} - X_n$ is the variation of his his / her total asset between time n and $n+1$. We can think of H_{n+1} as a modification of strategy from the player at time n, by multiplying the bet by H_{n+1} which is a $\mathcal{F}_n$-measurable strategy. In such a case, a fair game stays fair but the variation of the total asset of the player between time n and $n+1$ becomes $H_{n+1}(X_{n+1} - X_n)$.

6.2 Stopping Time

Definition 6.2.1 : Given a random variable $T : \Omega \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ and a filtration $(\mathcal{F}_n)_{n \geqslant 0}$. We say that T is a *stopping time* (停止時間) with respect to the filtration $(\mathcal{F}_n)_{n \geqslant 0}$ if

$$\forall n \in \mathbb{N}_0 \cup \{\infty\}, \quad \{T = n\} \in \mathcal{F}_n. \tag{6.3}$$

Remark 6.2.2 : Note that T is allowed to take the value of $+\infty$ and we can write,

$$\{T = +\infty\} = \Omega \backslash \bigcup_{n \geqslant 0} \{T = n\}.$$

Hence, if we define,

$$\mathcal{F}_\infty = \bigvee_{n \geqslant 0} \mathcal{F}_n := \sigma\Big(\bigcup_{n \geqslant 0} \mathcal{F}_n\Big),$$

then $\{T = +\infty\} \in \mathcal{F}_\infty$.

(1) 若 (X_n) 是個鞅，則 $((H \cdot X)_n)$ 也是個鞅。

(2) 若 (X_n) 是個上鞅（或下鞅），且對於所有 $n \geqslant 1$，$H_n \geqslant 0$，則 $((H \cdot X)_n)$ 也是個上鞅（或下鞅）。

證明：

(1) 由於所有的隨機變數 H_n 皆為有界，我們可以檢查所有的隨機變數 $(H \cdot X)_n$ 仍然是可積的，而且根據構造，隨機過程 $((H \cdot X)_n)$ 是個適應過程。因此我們只需要檢查，對於所有的 $n \geqslant 0$，我們有

$$\mathbb{E}[(H \cdot X)_{n+1} - (H \cdot X)_n \,|\, \mathcal{F}_n] = 0.$$

由於 $(H \cdot X)_{n+1} - (H \cdot X)_n = H_{n+1}(X_{n+1} - X_n)$ 且 H_{n+1} 為 $\mathcal{F}_n$ 可測，我們有

$$\mathbb{E}[H_{n+1}(X_{n+1} - X_n) \,|\, \mathcal{F}_n] = H_{n+1}\,\mathbb{E}[X_{n+1} - X_n \,|\, \mathcal{F}_n] = 0.$$

因此得證。

(2) 證明與 (1) 相似。

□

在 (1) 的情況下，若我們將 X_n 當作玩家在時間 n 的總資產，則 $X_{n+1} - X_n$ 為其在時間 n 及 $n+1$ 之間總資產的變化。我們可以把 H_{n+1} 想做玩家在時間 n 改變遊戲策略，而將下注的金額乘上 H_{n+1}，注意，此策略必須要是 $\mathcal{F}_n$ 可測的。這樣的情況下，公平遊戲仍然維持他的公平性，只是玩家總資產在時間 n 及 $n+1$ 的改變量為 $H_{n+1}(X_{n+1} - X_n)$。

第二節 停止時間

定義 6.2.1 ： 給定隨機變數 $T : \Omega \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ 以及濾鏈 $(\mathcal{F}_n)_{n \geqslant 0}$。若下列條件成立，

$$\forall n \in \mathbb{N}_0 \cup \{\infty\}, \quad \{T = n\} \in \mathcal{F}_n, \tag{6.3}$$

則我們說 T 是個（對於濾鏈 $(\mathcal{F}_n)$ 的）停止時間 (stopping time)。

註解 6.2.2 ： 注意，T 是可以取 $+\infty$ 這個值的，而且我們可以寫作

$$\{T = +\infty\} = \Omega \backslash \bigcup_{n \geqslant 0} \{T = n\}.$$

因此，若我們定義

$$\mathcal{F}_\infty = \bigvee_{n \geqslant 0} \mathcal{F}_n := \sigma\Big(\bigcup_{n \geqslant 0} \mathcal{F}_n\Big),$$

Remark 6.2.3 : We can easily check that the definition in Eq. (6.3) is equivalent to the following,

$$\forall n \in \mathbb{N}_0 \cup \{\infty\}, \quad \{T \leqslant n\} \in \mathcal{F}_n. \tag{6.4}$$

Therefore, depending on the circumstance, sometimes we use the definition in Eq. (6.3) and sometimes the one in Eq. (6.4).

If we come back to the interpretation of a game, the stopping time is the time at wich a player decides to quit the game. Thus, to make this decision at time n, the only available information is what has been accumulated up to time n, which is $\mathcal{F}_n$. This being said, we cannot decide to sell a share in the stock market when it is at its highest level, since we do not know anything about its future evolution.

Example 6.2.4 : Below are examples of stopping times.

(1) Given $k \in \mathbb{N}_0$, the the constant time $T = k$ is a stopping time.

(2) If $(Y_n)_{n\geqslant 0}$ is an adapted process, then for any Borel set $A \in \mathcal{B}(\mathbb{R})$,

$$T_A := \inf\{n \in \mathbb{N}_0 : Y_n \in A\}$$

is a stopping time, called the *entry time in A*. It can be checked as follows,

$$\{T_A = n\} = \{Y_0 \notin A, \ldots, Y_{n-1} \notin A, Y_n \in A\} \in \mathcal{F}_n.$$

Note that we use the following convention: $\inf \varnothing = +\infty$.

(3) For a fixed $N > 0$, we let

$$L_A := \sup\{n \leqslant N : Y_n \in A\} \quad (\sup \varnothing = 0).$$

Then, L_A is not necessarily a stopping time since for $1 \leqslant n \leqslant N-1$, we have,

$$\{L_A = n\} = \{Y_n \in A, Y_{n+1} \notin A, \ldots, Y_N \notin A\},$$

which is not necessarily $\mathcal{F}_n$-measurable.

Proposition 6.2.5 : *Stopping times have the following properties.*

(1) If S and T are both stopping times, then $S \vee T$ and $S \wedge T$ are also stopping times.

(2) If $(T_k)_{k\geqslant 1}$ is a sequence of stopping times, then

$$\begin{aligned} &\inf_{k\geqslant 1} T_k, && \liminf_{k\to\infty} T_k, \\ &\sup_{k\geqslant 1} T_k, && \limsup_{k\to\infty} T_k \end{aligned}$$

則 $\{T = +\infty\} \in \mathcal{F}_\infty$。

註解 6.2.3 ： 我們可以輕易檢查，式 (6.3) 定義與下列等價：

$$\forall n \in \mathbb{N}_0 \cup \{\infty\}, \quad \{T \leqslant n\} \in \mathcal{F}_n. \tag{6.4}$$

因此，根據情況，有時候我們會使用式 (6.3) 中的定義，有時候會用式 (6.4) 中的定義。

若我們回到遊戲玩家的詮釋，停止時間可以視為玩家決定撤出遊戲的時間，因此，若我們決定要在時間 n 停止，我們只能夠依靠在時間 n 之前所累積的資訊，也就是 $\mathcal{F}_n$。若我們拿股市來做例子，我們無法決定說要在股票最高點時賣出，因為我們不知道股票未來的走向。

範例 6.2.4 ： 以下是停止時間的例子。

(1) 給定 $k \in \mathbb{N}_0$，則常數時間 $T = k$ 是個停止時間。

(2) 若 $(Y_n)_{n\geqslant 0}$ 是個適應過程，對於任意布雷爾集合 $A \in \mathcal{B}(\mathbb{R})$，

$$T_A := \inf\{n \in \mathbb{N}_0 : Y_n \in A\}$$

是個停止時間，我們稱之為進入 A 的時間。我們可以做下列驗證：

$$\{T_A = n\} = \{Y_0 \notin A, \ldots, Y_{n-1} \notin A, Y_n \in A\} \in \mathcal{F}_n.$$

注意到，這裡我們必須要用下列慣例：$\inf \varnothing = +\infty$。

(3) 固定 $N > 0$ 且令

$$L_A := \sup\{n \leqslant N : Y_n \in A\} \quad (\sup \varnothing = 0),$$

則 L_A 未必會是個停止時間，因為對於 $1 \leqslant n \leqslant N-1$，我們有

$$\{L_A = n\} = \{Y_n \in A, Y_{n+1} \notin A, \ldots, Y_N \notin A\},$$

也就是說不一定會是 $\mathcal{F}_n$ 可測。

命題 6.2.5 ： 停止時間之間有下列性質。

(1) 若 S 及 T 兩者皆為停止時間，則 $S \vee T$ 及 $S \wedge T$ 也是停止時間。

are all stopping times.

Proof :

(1) Given $n \in \mathbb{N}_0$, we have,

$$\{S \wedge T \leqslant n\} = \{S \leqslant n\} \cup \{T \leqslant n\},$$
$$\{S \vee T \leqslant n\} = \{S \leqslant n\} \cap \{T \leqslant n\}.$$

(2) Given $n \in \mathbb{N}_0$, we have,

$$\Big\{\inf_{k\geqslant 1} T_k \leqslant n\Big\} = \bigcup_{k\geqslant 1}\{T_k \leqslant n\},$$
$$\Big\{\liminf_{k\geqslant 1} T_k \leqslant n\Big\} = \bigcap_{m\geqslant 1}\Big(\bigcup_{k\geqslant m}\{T_k \leqslant n\}\Big).$$

□

Definition 6.2.6 : If T is a stopping time, then the *stopped σ-algebra* (停止 σ 代數) or the *σ-algebra of T-past* (T 的過往 σ 代數) is defined by

$$\mathcal{F}_T = \{A \in \mathcal{F} : \forall n \in \mathbb{N}_0, A \cap \{T = n\} \in \mathcal{F}_n\}.$$

Question 6.2.7: Check that $\mathcal{F}_T$ is a σ-algebra and that $\mathcal{F}_T = \mathcal{F}_n$ for a constant stopping time $T = n$.

Proposition 6.2.8 : *If S and T are both stopping times and $S \leqslant T$, then we have, $\mathcal{F}_S \subseteq \mathcal{F}_T$.*

Proof : Let $A \in \mathcal{F}_S$. Then, for all $n \geqslant 0$, we have,

$$A \cap \{T = n\} = \bigcup_{k=0}^{n}(A \cap \{S = k\}) \cap \{T = n\} \in \mathcal{F}_n.$$

□

Proposition 6.2.9 : *Let $(Y_n)_{n\geqslant 0}$ be an adapted process and T be a stopping time, then the random*

(2) 若 $(T_k)_{k\geqslant 1}$ 是個停止時間序列，則

$$\inf_{k\geqslant 1} T_k, \quad \liminf_{k\to\infty} T_k,$$
$$\sup_{k\geqslant 1} T_k, \quad \limsup_{k\to\infty} T_k$$

皆為停止時間。

證明：

(1) 給定 $n \in \mathbb{N}_0$，我們有

$$\{S \wedge T \leqslant n\} = \{S \leqslant n\} \cup \{T \leqslant n\},$$
$$\{S \vee T \leqslant n\} = \{S \leqslant n\} \cap \{T \leqslant n\}.$$

(2) 給定 $n \in \mathbb{N}_0$，我們有

$$\Big\{\inf_{k\geqslant 1} T_k \leqslant n\Big\} = \bigcup_{k\geqslant 1}\{T_k \leqslant n\},$$
$$\Big\{\liminf_{k\geqslant 1} T_k \leqslant n\Big\} = \bigcap_{m\geqslant 1}\Big(\bigcup_{k\geqslant m}\{T_k \leqslant n\}\Big).$$

□

定義 6.2.6 ： 若 T 為停止時間，則我們可以定義停止 σ 代數 (stopped σ-algebra) ，又稱為 T 的過往 σ 代數 (σ-algebra of T-past) 為

$$\mathcal{F}_T = \{A \in \mathcal{F} : \forall n \in \mathbb{N}_0, A \cap \{T = n\} \in \mathcal{F}_n\}.$$

問題 6.2.7： 檢查 $\mathcal{F}_T$ 為 σ 代數且當 $T = n$ 為常數停止時間時，$\mathcal{F}_T = \mathcal{F}_n$。

命題 6.2.8 ： 若 S 以及 T 兩者皆為停止時間，且 $S \leqslant T$，則我們有 $\mathcal{F}_S \subseteq \mathcal{F}_T$。

證明： 令 $A \in \mathcal{F}_S$，則對於所有 $n \geqslant 0$，我們有

$$A \cap \{T = n\} = \bigcup_{k=0}^{n}(A \cap \{S = k\}) \cap \{T = n\} \in \mathcal{F}_n.$$

□

命題 6.2.9 ： 令 $(Y_n)_{n\geqslant 0}$ 為適應過程以及 T 為停止時間，則下面定義的隨機變數對 $\mathcal{F}_T$ 會是可

variable defined below is $\mathcal{F}_T$-measurable,

$$\mathbb{1}_{T<\infty}Y_T(\omega) = \begin{cases} Y_n(\omega) & \text{if } T(\omega) = n \in \mathbb{N}_0, \\ 0 & \text{if } T(\omega) = +\infty. \end{cases} \tag{6.5}$$

Proof : Let $B \in \mathcal{B}(\mathbb{R})$. Assume that $0 \notin B$ and that for all $n \in \mathbb{N}_0$, we have,

$$\{\mathbb{1}_{T<\infty}Y_T \in B\} \cap \{T = n\} = \{Y_n \in B\} \cap \{T = n\} \in \mathcal{F}_n.$$

This means that $\{\mathbb{1}_{T<\infty}Y_T \in B\} \in \mathcal{F}_T$. If $0 \in B$, then for all $n \in \mathbb{N}_0$, we have,

$$\{\mathbb{1}_{T<\infty}Y_T \in B\} \cap \{T = n\} = \{\mathbb{1}_{T<\infty}Y_T \in B^c\}^c \cap \{T = n\} \in \mathcal{F}_n.$$

We may conclude in a similar way. □

When the stopping time T is almost surely finite, we can simply write Eq. (6.5) as Y_T. Moreover, for any stopping time T and any non-negative integer n, $T \wedge n$ is also a stopping time (Proposition 6.2.5). Thus, from Proposition 6.2.9, $Y_{T\wedge n}$ is $\mathcal{F}_{T\wedge n}$-measurable, so also $\mathcal{F}_n$-measurable.

Theorem 6.2.10 (Stopping theorem) **:** *Let $(X_n)_{n\geqslant 0}$ be a martingale (resp. a supermartingale). Let T be a stopping time. Then, $(X_{T\wedge n})_{n\geqslant 0}$ is also a martingale (resp. a supermartingale). If the stopping time T is bounded, then we have $X_T \in L^1$ and,*

$$\mathbb{E}[X_T] = \mathbb{E}[X_0] \quad (\text{or } \mathbb{E}[X_T] \leqslant \mathbb{E}[X_0]).$$

Proof : Let us see how to construct the random process $(X_{T\wedge n})_{n\geqslant 0}$ from $(X_n)_{n\geqslant 0}$ in the sense of Proposition 6.1.11, so that we can apply directly the proposition. We note that for any $n \geqslant 0$, we have

$$X_{T\wedge(n+1)} - X_{T\wedge n} = \mathbb{1}_{T\geqslant n+1}(X_{n+1} - X_n).$$

Therefore, let us define

$$\forall n \geqslant 1, \quad H_n = \mathbb{1}_{T\geqslant n} = 1 - \mathbb{1}_{T\leqslant n-1},$$

Then, $(H_n)_{n\geqslant 1}$ is a predictable sequence and we have

$$\forall n \geqslant 1, \quad X_{T\wedge n} = X_0 + (H \cdot X)_n.$$

This allows us to conclude for the first part of the theorem.

Next, if T is bounded, let N be its upper bound, then $\mathbb{E}[X_T] = \mathbb{E}[X_{T\wedge N}] = \mathbb{E}[X_0]$ (resp. $\leqslant \mathbb{E}[X_0]$). □

測的：

$$\mathbb{1}_{T<\infty}Y_T(\omega) = \begin{cases} Y_n(\omega) & \text{若 } T(\omega) = n \in \mathbb{N}_0, \\ 0 & \text{若 } T(\omega) = +\infty. \end{cases} \tag{6.5}$$

證明：令 $B \in \mathcal{B}(\mathbb{R})$，假設 $0 \notin B$，則對於所有 $n \in \mathbb{N}_0$，我們有

$$\{\mathbb{1}_{T<\infty}Y_T \in B\} \cap \{T = n\} = \{Y_n \in B\} \cap \{T = n\} \in \mathcal{F}_n.$$

也就是說，$\{\mathbb{1}_{T<\infty}Y_T \in B\} \in \mathcal{F}_T$。若 $0 \in B$，則對於所有 $n \in \mathbb{N}_0$，我們有

$$\{\mathbb{1}_{T<\infty}Y_T \in B\} \cap \{T = n\} = \{\mathbb{1}_{T<\infty}Y_T \in B^c\}^c \cap \{T = n\} \in \mathcal{F}_n.$$

同理得證。□

當停止時間 T 殆必有限時，我們可以直接將式 (6.5) 寫作 Y_T。此外，對於任意停止時間 T 以及任意非負整數 n，$T \wedge n$ 也是個停止時間（命題 6.2.5），因此根據命題 6.2.9，$Y_{T\wedge n}$ 對於 $\mathcal{F}_{T\wedge n}$ 是可測的，也就是對於 $\mathcal{F}_n$ 可測。

定理 6.2.10 【停止定理】**：**令 $(X_n)_{n\geqslant 0}$ 為鞅（或上鞅），令 T 為停止時間，則 $(X_{T\wedge n})_{n\geqslant 0}$ 也是個鞅（或上鞅）。若停止時間 T 有界，我們有 $X_T \in L^1$，而且

$$\mathbb{E}[X_T] = \mathbb{E}[X_0] \quad (\text{或 } \mathbb{E}[X_T] \leqslant \mathbb{E}[X_0]).$$

證明：我們使用命題 6.1.11 中的方法，由 $(X_n)_{n\geqslant 0}$ 出發，構造隨機過程 $(X_{T\wedge n})_{n\geqslant 0}$，進而使用命題來總結。我們注意到對於所有 $n \geqslant 0$，我們有

$$X_{T\wedge(n+1)} - X_{T\wedge n} = \mathbb{1}_{T\geqslant n+1}(X_{n+1} - X_n).$$

因此，我們定義

$$\forall n \geqslant 1, \quad H_n = \mathbb{1}_{T\geqslant n} = 1 - \mathbb{1}_{T\leqslant n-1},$$

則 $(H_n)_{n\geqslant 1}$ 為可預測序列且

$$\forall n \geqslant 1, \quad X_{T\wedge n} = X_0 + (H \cdot X)_n.$$

這讓我們可以對定理的第一部份總結。

接著，若 T 有界，令其上界為 N，則 $\mathbb{E}[X_T] = \mathbb{E}[X_{T\wedge N}] = \mathbb{E}[X_0]$（或 $\leqslant \mathbb{E}[X_0]$）。□

Question 6.2.11: Consider an i.i.d. sequence $(Y_n)_{n\geqslant 1}$ of random variables with distribution $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$. Define $X_0 = 0$ and for all $n \geqslant 1$, define $X_n = Y_1 + \cdots + Y_n$, that is to say $(X_n)_{n\geqslant 0}$ is a random walk on $\mathbb{Z}$ started at 0 with symmetric distribution. Let

$$T = \inf\{n \geqslant 0 : X_n = 1\}.$$

(1) Show that $T < \infty$ a.s.

(2) Show that $\mathbb{E}[X_T] = 1$ and $\mathbb{E}[X_0] = 0$.

(3) Explain.

6.3 Almost Sure Convergence of Martingales

In this section, we discuss the almost sure convergence of martingales (resp. submartingales).

We fisrt consider a sequence of real numbers $(\alpha_n)_{n\geqslant 0}$. For all real numbers $a < b$, if we consider two *time series* (時間序列) $(S_k(\alpha))$ and $(T_k(\alpha))$ with values in $\overline{\mathbb{N}}$ defined as follows,

$$S_1(\alpha) = \inf\{n \geqslant 0 : \alpha_n \leqslant a\},$$
$$T_1(\alpha) = \inf\{n \geqslant S_1(\alpha) : \alpha_n \geqslant b\},$$

and by induction, for all $k \geqslant 1$, define

$$S_{k+1}(\alpha) = \inf\{n \geqslant T_k(\alpha) : \alpha_n \leqslant a\},$$
$$T_{k+1}(\alpha) = \inf\{n \geqslant S_{k+1}(\alpha) : \alpha_n \geqslant b\}.$$

In the above definition, we take the convention $\inf \varnothing = +\infty$. Next, for all $n \geqslant 1$, we define

$$N_n([a,b],\alpha) = \sum_{k=1}^{\infty} \mathbb{1}_{T_k(\alpha)\leqslant n},$$
$$N_\infty([a,b],\alpha) = \sum_{k=1}^{\infty} \mathbb{1}_{T_k(\alpha)<\infty},$$

meaning that $N_n([a,b],\alpha)$ denotes the number of upcrossings of the sequence $(\alpha_k)_{k\geqslant 0}$ on the interval $[a,b]$ before time n.

The lemma below is a fundamental property in analysis.

Lemma 6.3.1 : *The sequence $(\alpha_n)_{n\geqslant 0}$ converges in $\overline{\mathbb{R}}$ if and only if for all real nubmers $a < b$, we have $N_\infty([a,b],\alpha) < \infty$.*

Question 6.3.2: Given an adapted process $(X_n)_{n\geqslant 0}$, then for all $k \geqslant 1$, $S_k(X)$ and $T_k(X)$ are random variables with values in $\mathbb{N}_0 \cup \{+\infty\}$. Check the following points:

(1) For all $k \geqslant 1$, $S_k(X)$ and $T_k(X)$ are both stopping times.

(2) $N_n([a,b],X)$ is $\mathcal{F}_n$-measurable.

問題 6.2.11：考慮 $(Y_n)_{n\geqslant 1}$ 為 i.i.d. 隨機變數序列，使得 $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$。定義 $X_0 = 0$ 以及對於所有 $n \geqslant 1$，$X_n = Y_1 + \cdots + Y_n$，也就是說 $(X_n)_{n\geqslant 0}$ 是個從 0 出發，在 $\mathbb{Z}$ 上且分佈對稱的隨機漫步。設

$$T = \inf\{n \geqslant 0 : X_n = 1\}.$$

(1) 證明 $T < \infty$ a.s.。

(2) 證明 $\mathbb{E}[X_T] = 1$ 且 $\mathbb{E}[X_0] = 0$。

(3) 解釋。

第三節 鞅的殆必收斂

在這個章節中，我們要討論鞅（或下鞅）殆必收斂的性質。

首先考慮實數數列 $(\alpha_n)_{n\geqslant 0}$。對於所有實數 $a < b$，我們考慮兩個在 $\overline{\mathbb{N}}$ 中取值的時間序列 (time series) $(S_k(\alpha))$ 及 $(T_k(\alpha))$：

$$S_1(\alpha) = \inf\{n \geqslant 0 : \alpha_n \leqslant a\},$$
$$T_1(\alpha) = \inf\{n \geqslant S_1(\alpha) : \alpha_n \geqslant b\},$$

接著透過遞迴，對於所有 $k \geqslant 1$，定義

$$S_{k+1}(\alpha) = \inf\{n \geqslant T_k(\alpha) : \alpha_n \leqslant a\},$$
$$T_{k+1}(\alpha) = \inf\{n \geqslant S_{k+1}(\alpha) : \alpha_n \geqslant b\}.$$

在上述的定義中，我們固定慣例取 $\inf \varnothing = +\infty$。接著，對於所有整數 $n \geqslant 1$，我們定義

$$N_n([a,b],\alpha) = \sum_{k=1}^{\infty} \mathbb{1}_{T_k(\alpha)\leqslant n},$$
$$N_\infty([a,b],\alpha) = \sum_{k=1}^{\infty} \mathbb{1}_{T_k(\alpha)<\infty},$$

也就是說，$N_n([a,b],\alpha)$ 計算序列 $(\alpha_n)_{n\geqslant 0}$ 在 $[a,b]$ 區間中上升的次數。

在基礎分析中，我們有下列引理。

引理 6.3.1 ：若且唯若實序列 $(\alpha_n)_{n\geqslant 0}$ 在 $\overline{\mathbb{R}}$ 中收斂，則對於所有實數 $a < b$，我們有 $N_\infty([a,b],\alpha) < \infty$。

問題 6.3.2：給定一個適應過程 $(X_n)_{n\geqslant 0}$，則對於所有 $k \geqslant 1$，$S_k(X)$ 及 $T_k(X)$ 為值域為 $\mathbb{N}_0 \cup \{+\infty\}$ 的隨機變數。驗證下列兩點：

(1) 對於所有 $k \geqslant 1$，$S_k(X)$ 及 $T_k(X)$ 為停止時間。

(2) $N_n([a,b],X)$ 是 $\mathcal{F}_n$ 可測的。

Lemma 6.3.3 (Doob's upcrossing inequality) : *Let $(X_n)_{n\geqslant 0}$ be a submartingale. Then for all real numbers $a<b$ and positive integer $n\geqslant 1$, we have,*

$$(b-a)\,\mathbb{E}[N_n([a,b],X)]\leqslant \mathbb{E}[(X_n-a)^+-(X_0-a)^+].$$

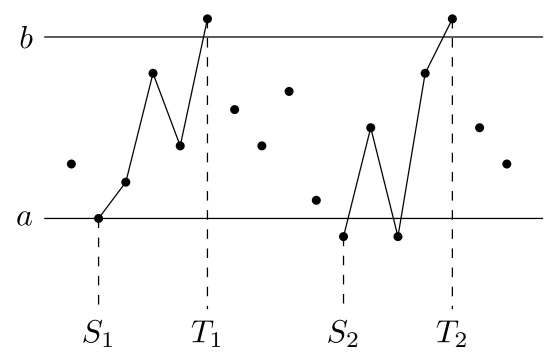

Figure 6.1: An illustration of the values taken by a submartingale $X=(X_n)_{n\geqslant 0}$ and the corresponding stopping times $(S_k)_{k\geqslant 1}$ and $(T_k)_{k\geqslant 1}$. Solid lines represent the increments that are included in the random process $(H\cdot X)_{n\geqslant 0}$. To compute $(H\cdot Y)_{n\geqslant 0}$, we move up the points below a to a and consider the corresponding increments.

Proof : Given $a<b$. Let $Y_n=(X_n-a)^+$ for all $n\geqslant 0$. We know from Proposition 6.1.8 that $(Y_n)_{n\geqslant 0}$ is still a submartingale. To simplify the notation, we replace $S_k(X)$, $T_k(X)$ and $N_n([a,b],X)$ with S_k, T_k and N_n. Define the predictable sequence $(H_n)_{n\geqslant 1}$ of random variables,

$$H_n=\sum_{k=1}^{\infty}\mathbb{1}_{S_k<n\leqslant T_k}\leqslant 1.$$

(Check that the measurable event $\{S_k<n\leqslant T_k\}$ is indeed in $\mathcal{F}_{n-1}$.) We have,

$$(H\cdot Y)_n=\sum_{k=1}^{N_n}(Y_{T_k}-Y_{S_k})+\mathbb{1}_{S_{N_n+1}<n}(Y_n-Y_{S_{N_n+1}})\geqslant\sum_{k=1}^{N_n}(Y_{T_k}-Y_{S_k})\geqslant N_n(b-a).$$

The first inequality in the above formula holds, since on the event $\{S_{N_n+1}<\infty\}$, we have $Y_{S_{N_n+1}}=0$ and $Y_n\geqslant 0$. Thus, we obtain,

$$\mathbb{E}[(H\cdot Y)_n]\geqslant(b-a)\,\mathbb{E}[N_n].$$

Moreover, for all $n\geqslant 1$, let $K_n=1-H_n$. The non-negative sequence $(K_n)_{n\geqslant 1}$ is also predictable. We know from Proposition 6.1.11 that $(K\cdot Y)$ is still a submartingale, implying $\mathbb{E}[(K\cdot Y)_n]\geqslant\mathbb{E}[(K\cdot Y)_0]=0$. Besides, we also have,

$$(H\cdot Y)_n+(K\cdot Y)_n=Y_n-Y_0,\quad\forall n\geqslant 0.$$

引理 6.3.3【Doob 上升不等式】：令 $(X_n)_{n\geqslant 0}$ 為下鞅，則對於所有實數 $a<b$ 以及整數 $n\geqslant 1$，我們有

$$(b-a)\,\mathbb{E}[N_n([a,b],X)]\leqslant \mathbb{E}[(X_n-a)^+-(X_0-a)^+].$$

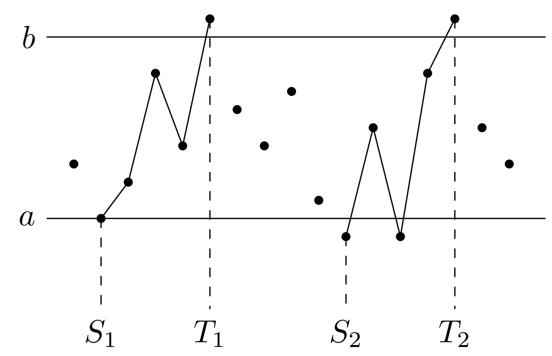

圖 6.1: 下鞅 $X=(X_n)_{n\geqslant 0}$ 的示意圖，以及所對應的停止時間 $(S_k)_{k\geqslant 1}$ 和 $(T_k)_{k\geqslant 1}$。實線所對應到的是隨機過程 $(H\cdot X)_{n\geqslant 0}$ 中考慮的增量。如果我們要計算 $(H\cdot Y)_{n\geqslant 0}$，我們會把在 a 以下的點上移至 a，並且考慮相對應的增量。

證明：給定 $a<b$。對於所有 $n\geqslant 0$，令 $Y_n=(X_n-a)^+$，根據命題 6.1.8 ，$(Y_n)_{n\geqslant 0}$ 還是個下鞅。我們簡化記號，將 $S_k(X)$，$T_k(X)$ 以及 $N_n([a,b],X)$ 分別記作 S_k，T_k 以及 N_n。定義可預測隨機變數序列 $(H_n)_{n\geqslant 1}$：

$$H_n=\sum_{k=1}^{\infty}\mathbb{1}_{S_k<n\leqslant T_k}\leqslant 1.$$

（請自行檢查可測事件 $\{S_k<n\leqslant T_k\}$ 的確是在 $\mathcal{F}_{n-1}$ 之中。）我們有，

$$(H\cdot Y)_n=\sum_{k=1}^{N_n}(Y_{T_k}-Y_{S_k})+\mathbb{1}_{S_{N_n+1}<n}(Y_n-Y_{S_{N_n+1}})\geqslant\sum_{k=1}^{N_n}(Y_{T_k}-Y_{S_k})\geqslant N_n(b-a).$$

上式中的第一個不等式成立，因為在事件 $\{S_{N_n+1}<\infty\}$ 之上，$Y_{S_{N_n+1}}=0$ 且 $Y_n\geqslant 0$。因此，我們得到

$$\mathbb{E}[(H\cdot Y)_n]\geqslant(b-a)\,\mathbb{E}[N_n].$$

此外，對於所有 $n\geqslant 1$，令 $K_n=1-H_n$，則非負序列 $(K_n)_{n\geqslant 1}$ 也會是個可測序列。從命題 6.1.11 得知，$(K\cdot Y)$ 仍然會是個下鞅，也就是說 $\mathbb{E}[(K\cdot Y)_n]\geqslant\mathbb{E}[(K\cdot Y)_0]=0$。另外，我們還有

$$(H\cdot Y)_n+(K\cdot Y)_n=Y_n-Y_0,\quad\forall n\geqslant 0.$$

Therefore,

$$(b-a)\,\mathbb{E}[N_n] \leqslant \mathbb{E}[(H\cdot Y)_n] \leqslant \mathbb{E}[(H\cdot Y)_n + (K\cdot Y)_n] = \mathbb{E}[Y_n - Y_0].$$

□

Theorem 6.3.4 : *Let $(X_n)_{n\geqslant 0}$ be a submartingale satisfying one of the equivalent properties*

(1) $(X_n^+)_{n\geqslant 0}$ is bounded in L^1, that is

$$\sup_{n\geqslant 0} \mathbb{E}[(X_n)^+] < \infty. \tag{6.6}$$

(2) $(X_n)_{n\geqslant 0}$ is bounded in L^1, that is

$$\sup_{n\geqslant 0} \mathbb{E}[|X_n|] < \infty. \tag{6.7}$$

Then X_n converges almost surely and its limit X_∞ satisfies $\mathbb{E}[|X_\infty|] < \infty$ and $|X_\infty| < \infty$ a.s.

Remark 6.3.5 : We first note that we have $\mathbb{E}[X_n] = \mathbb{E}[(X_n)^+] - \mathbb{E}[(X_n)^-]$. Since the submartingale $(X_n)_{n\geqslant 0}$ satisfies $\mathbb{E}[X_n] \geqslant \mathbb{E}[X_0]$, then for all $k \geqslant 0$, we have,

$$\mathbb{E}[(X_k)^-] \leqslant \left(\sup_{n\geqslant 0} \mathbb{E}[(X_n)^+]\right) - \mathbb{E}[X_0].$$

Thus, we deduce that Eq. (6.6) implies Eq. (6.7), and the converse holds trivially.

Proof : Given $a, b \in \mathbb{R}$ and $a < b$, from Lemma 6.3.3, for all $n \geqslant 1$, we have,

$$\begin{aligned}(b-a)\,\mathbb{E}[N_n([a,b],X)] &\leqslant \mathbb{E}[(X_n - a)^+] \leqslant |a| + \mathbb{E}[(X_n)^+] \\ &\leqslant |a| + \sup_{k\geqslant 0}\mathbb{E}[(X_k)^+] < \infty.\end{aligned}$$

Thus, when $n \to \infty$, we have,

$$(b-a)\,\mathbb{E}[N_\infty([a,b],X)] < \infty,$$

meaning that $N_\infty([a,b],X)$ is almost surely finite: $\mathbb{P}(N_\infty([a,b],X) < \infty) = 1$.

If we consider the countable set $\{(a,b) \in \mathbb{Q}^2 : a < b\}$, then we have,

$$\mathbb{P}\left(N_\infty([a,b],X) < \infty, \forall a, b \in \mathbb{Q} : a < b\right) = 1.$$

Finally we use Lemma 6.3.1 to conclude that X_n converges almost surely in $\overline{\mathbb{R}}$.

For the integrability of the almost sure limit X_∞, we apply Fatou's lemma,

$$\mathbb{E}[|X_\infty|] \leqslant \liminf_{n\to\infty} \mathbb{E}[|X_n|] \leqslant \sup_{n\geqslant 0} \mathbb{E}[|X_n|] < \infty.$$

Hence, $|X_\infty|$ is finite almost surely. □

因此，

$$(b-a)\,\mathbb{E}[N_n] \leqslant \mathbb{E}[(H\cdot Y)_n] \leqslant \mathbb{E}[(H\cdot Y)_n + (K\cdot Y)_n] = \mathbb{E}[Y_n - Y_0].$$

□

定理 6.3.4 ： 令 $(X_n)_{n\geqslant 0}$ 為下鞅且滿足下列其中一個等價性質：

(1) $(X_n^+)_{n\geqslant 0}$ 在 L^1 中有界，也就是

$$\sup_{n\geqslant 0} \mathbb{E}[(X_n)^+] < \infty. \tag{6.6}$$

(2) $(X_n)_{n\geqslant 0}$ 在 L^1 中有界，也就是

$$\sup_{n\geqslant 0} \mathbb{E}[|X_n|] < \infty. \tag{6.7}$$

則 X_n 殆必收斂，且其極限 X_∞ 滿足 $\mathbb{E}[|X_\infty|] < \infty$ 以及 $|X_\infty| < \infty$ a.s.。

註解 6.3.5 ： 首先注意到，我們有 $\mathbb{E}[X_n] = \mathbb{E}[(X_n)^+] - \mathbb{E}[(X_n)^-]$。由於下鞅 $(X_n)_{n\geqslant 0}$ 滿足 $\mathbb{E}[X_n] \geqslant \mathbb{E}[X_0]$，則對於所有 $k \geqslant 0$，我們有

$$\mathbb{E}[(X_k)^-] \leqslant \left(\sup_{n\geqslant 0} \mathbb{E}[(X_n)^+]\right) - \mathbb{E}[X_0].$$

因此，我們能夠推得式 (6.6) 蘊含式 (6.7)，其逆命題顯然成立。

證明：給定 $a, b \in \mathbb{R}$ 且 $a < b$，根據引理 6.3.3 ，對於所有 $n \geqslant 1$，我們有

$$\begin{aligned}(b-a)\,\mathbb{E}[N_n([a,b],X)] &\leqslant \mathbb{E}[(X_n - a)^+] \leqslant |a| + \mathbb{E}[(X_n)^+] \\ &\leqslant |a| + \sup_{k\geqslant 0}\mathbb{E}[(X_k)^+] < \infty.\end{aligned}$$

因此當 $n \to \infty$，我們有

$$(b-a)\,\mathbb{E}[N_\infty([a,b],X)] < \infty,$$

也就是說，$N_\infty([a,b],X)$ 殆必有限：$\mathbb{P}(N_\infty([a,b],X) < \infty) = 1$。

若我們考慮 $\{(a,b) \in \mathbb{Q}^2 : a < b\}$，由於此數對集合仍然可數，我們有

$$\mathbb{P}\left(N_\infty([a,b],X) < \infty, \forall a, b \in \mathbb{Q} : a < b\right) = 1.$$

最後我們使用引理 6.3.1 ，得證 X_n 在 $\overline{\mathbb{R}}$ 中殆必收斂。

關於 a.s. 極限 X_∞ 的可積性質，我們利用 Fatou 引理而得到

$$\mathbb{E}[|X_\infty|] \leqslant \liminf_{n\to\infty} \mathbb{E}[|X_n|] \leqslant \sup_{n\geqslant 0} \mathbb{E}[|X_n|] < \infty.$$

因此，我們也有 $|X_\infty|$ 殆必有限。 □

Corollary 6.3.6 : *Let (X_n) be a non-negative supermartingale. Then, X_n converges almost surely and its limit X_∞ is integrable and satisfies $X_n \geqslant \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ for all $n \geqslant 0$.*

Proof : This is an application of Theorem 6.3.4. Let $Y_n = -X_n$, then (Y_n) satisfies the assumptions in Eq. (6.6). Finally, we apply the Fatou's lemma for conditional expectations, which is the point (4) from Proposition 5.2.8,

$$X_n \geqslant \liminf_{m\to\infty} \mathbb{E}[X_m \mid \mathcal{F}_n] \geqslant \mathbb{E}[\liminf_{m\to\infty} X_m \mid \mathcal{F}_n] = \mathbb{E}[X_\infty \mid \mathcal{F}_n].$$

□

系理 6.3.6 ： 令 (X_n) 為非負上鞅，則 X_n 殆必收斂且其極限 X_∞ 可積而且滿足對於所有整數 $n \geqslant 0$，$X_n \geqslant \mathbb{E}[X_\infty \mid \mathcal{F}_n]$。

證明：這是定理 6.3.4 的應用：令 $Y_n = -X_n$，則式 (6.6) 中的假設對於 (Y_n) 顯然成立。最後我們使用條件期望值的 Fatou 引理（命題 5.2.8 中的 (4)）：

$$X_n \geqslant \liminf_{m\to\infty} \mathbb{E}[X_m \mid \mathcal{F}_n] \geqslant \mathbb{E}[\liminf_{m\to\infty} X_m \mid \mathcal{F}_n] = \mathbb{E}[X_\infty \mid \mathcal{F}_n].$$

□