6 Discrete Time Martingales

6.1 Definitions and Examples

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A sequence of random variables $(X_n)_{n\geqslant 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *random process* (隨機過程) . Note that the index set of $(X_n)_{n\geqslant 0}$ is given by non-negative integers, that is $\mathbb{N}_0 = \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \dots\}$, which can be intepreted by the fact that our random process is indexed by discrete time steps. In this course, we may also assume that our random processes take their values in $\mathbb{R}$.

Definition 6.1.1 : In $(\Omega, \mathcal{F}, \mathbb{P})$, a non-decreasing sequence $(\mathcal{F}_n)_{n\geqslant 0}$ of sub-σ-algebras is called a *filtration* (濾鏈) if

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}.$$

We then call $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0}, \mathbb{P})$ a *filtered probability space* (濾鏈機率空間) .

In the above definition, we can interprete n as time and $\mathcal{F}_n$ can be understood as the information that we are disposed of up to time n.

Example 6.1.2 : Below are two examples of filtration.

(1) If $(X_n)_{n\geqslant 0}$ is a random process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, let

$$\forall n \geqslant 0, \quad \mathcal{F}_n^X := \sigma(X_0, \dots, X_n),$$

be the smallest σ-algebra making the random variables $X_0, \dots, X_n$ measurable. Then, we say that $(\mathcal{F}_n^X)_{n\geqslant 0}$ is the *canonical filtration* (正則濾鏈) of the random process $(X_n)_{n\geqslant 0}$.

(2) Assume that $\Omega = [0, 1)$ and $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel σ-algebra and $\mathbb{P}$ be the Lebesgue measure. Let

$$\forall n \geqslant 0, \quad \mathcal{F}_n := \sigma([\tfrac{i-1}{2^n}, \tfrac{i}{2^n}) : 1 \leqslant i \leqslant 2^n).$$

Then, we say that $(\mathcal{F}_n)_{n\geqslant 0}$ is the *dyadic filtration* (二元濾鏈) on $[0, 1)$. Alternatively, we may also see the dyadic filtration as the canonical filtration of the random process $(X_n)_{n\geqslant 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, where $X_n(\omega)$ gives the n-th digit in the (finite) dyadic expansion of ω.

Definition 6.1.3 : Given a random process $(X_n)_{n\geqslant 0}$. If the random variable X_n is $\mathcal{F}_n$-measurable for all $n \geqslant 0$, then we say that $(X_n)_{n\geqslant 0}$ is a $(\mathcal{F}_n)_{n\geqslant 0}$-*adapted* (適應) random process.

Remark 6.1.4 : The canonical filtration can be understood as the smallest (for inclusion) filtration making the random process adapted.

In what follows, we fix a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0}, \mathbb{P})$ and define the following notions below.

Definition 6.1.5 : Let $(X_n)_{n\geqslant 0}$ be an adapted random process and assume that we have $\mathbb{E}[|X_n|] < \infty$ for all $n \geqslant 0$.

(1) If for all $n \geqslant 0$, we have,
$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n,$$
then we say that the random process $(X_n)_{n\geqslant 0}$ is a *martingale* (鞅).

(2) If for all $n \geqslant 0$, we have,
$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leqslant X_n,$$
then we say that the random process $(X_n)_{n\geqslant 0}$ is a *supermartingale* (上鞅).

(3) If for all $n \geqslant 0$, we have,
$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geqslant X_n,$$
then we say that the random process $(X_n)_{n\geqslant 0}$ is a *submartingale* (下鞅).

Remark 6.1.6 : From the above definition, a martingale can be easily checked to satisfy the following property: for all $0 \leqslant n \leqslant m$,
$$\mathbb{E}[X_m \mid \mathcal{F}_n] = X_n. \tag{6.1}$$
This also implies $\mathbb{E}[X_m] = \mathbb{E}[X_n] = \mathbb{E}[X_0]$. In the case of a supermartingale or a submartingale, we can also deduce similar properties and the equality becomes the corresponding inequality.

In general, we illustrate the notion of martingale using a fair game: X_n represents the total asset of a player at time n, $\mathcal{F}_n$ the information at his / her disposition at time n (and before). The martingale property $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ can be understood by the fact that the expected asset of the player at time $n+1$, knowing what happened up to time n, is equal to his / her asset at time n. In other words, in average, the player does not win or lose any money.

Using this interpretation, we can view a supermartingale as a game where the banker has advantages.

We can also note that, if $(X_n)_{n\geqslant 0}$ is a supermartingale, then $(-X_n)_{n\geqslant 0}$ is a submartingale. As a consequence, later when we state results on super/submartingales, we only take care of either the case of super- or submartingales.

Example 6.1.7 : Here we give a few examples of martingales, supermartingales and submartingales and define some additional notions. Let us fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a filtration $(\mathcal{F}_n)_{n\geqslant 0}$.

(1) If $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, let
$$\forall n \geqslant 0, \quad X_n = \mathbb{E}[X \mid \mathcal{F}_n]. \tag{6.2}$$
Then $(X_n)_{n\geqslant 0}$ is a martingale,
$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] = \mathbb{E}[X \mid \mathcal{F}_n] = X_n.$$

A martingale satisfying Eq. (6.2) is called a *closed martingale* (封閉鞅).

(2) If $(X_n)_{n\geqslant 0}$ is a non-increasing sequence of adapted and integrable random variables, then $(X_n)_{n\geqslant 0}$ is a supermartingale. Indeed, we have

$$\mathbb{E}[X_{n+1}\,|\,\mathcal{F}_n] \leqslant \mathbb{E}[X_n\,|\,\mathcal{F}_n] = X_n.$$

(3) We look at a random walk on $\mathbb{R}$. Let $x \in \mathbb{R}$ and $(Y_n)_{n\geqslant 1}$ be an i.i.d. sequence of random variables with distribution μ and $\mathbb{E}[|Y_1|] < \infty$. Let

$$X_0 = x \quad \text{and} \quad \forall n \geqslant 1, \quad X_n = x + Y_1 + \cdots + Y_n.$$

The filtration $(\mathcal{F}_n)_{n\geqslant 0}$ is defined as,

$$\mathcal{F}_0 = \{\varnothing, \Omega\} \quad \text{and} \quad \forall n \geqslant 1, \quad \mathcal{F}_n = \sigma(Y_1, \ldots, Y_n),$$

which is the canonical filtration of $(X_n)_{n\geqslant 0}$.

(a) If $\mathbb{E}[Y_1] = 0$, then $(X_n)_{n\geqslant 0}$ is a martingale.
(b) If $\mathbb{E}[Y_1] \leqslant 0$, then $(X_n)_{n\geqslant 0}$ is a supermartingale.
(c) If $\mathbb{E}[Y_1] \geqslant 0$, then $(X_n)_{n\geqslant 0}$ is a submartingale.

We say that $(X_n)_{n\geqslant 0}$ is a random walk on $\mathbb{R}$ started at x with jump distribution given by μ.

(4) Let us come back to (2) in Example 6.1.2 and remind that the filtration of our interest is the dyadic filtration $(\mathcal{F}_n)_{n\geqslant 0}$. Let μ be a finite measure on $[0,1)$, $\mathbb{P} = \lambda$ be the Lebesgue measure on $[0,1)$. For all $n \geqslant 0$, let

$$f_n = \Big(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\Big)_{|\mathcal{F}_n},$$

where the Radon–Nikodym derivative is defined by considering μ and λ as measures restricted on the σ-algebra $\mathcal{F}_n$. The Radon–Nikodym derivative is well defined because all the finite measures on $\mathcal{F}_n$ are absolutely continuous with respect to $\lambda_{|\mathcal{F}_n}$. Hence, we obtain,

$$f_n(\omega) = \sum_{i=1}^{2^n} \frac{\mu([(i-1)2^{-n}, i2^{-n}))}{2^{-n}} \mathbb{1}_{[(i-1)2^{-n}, i2^{-n})}(\omega).$$

Then, $(f_n)_{n\geqslant 1}$ is a martingale since for all $A \in \mathcal{F}_n$, we have,

$$\mathbb{E}[\mathbb{1}_A f_{n+1}] = \int \mathbb{1}_A(\omega) f_{n+1}(\omega)\,\mathrm{d}\omega = \mu(A) = \int \mathbb{1}_A(\omega) f_n(\omega)\,\mathrm{d}\omega = \mathbb{E}[\mathbb{1}_A f_n],$$

implying $f_n = \mathbb{E}[f_{n+1}\,|\,\mathcal{F}_n]$.

If $\mu \ll \nu$ on $\mathcal{F}$, we write f for Radon–Nikodym derivative of μ with respect to ν, then we have a closed martingale,

$$\forall n \geqslant 1, \quad f_n = \mathbb{E}[f\,|\,\mathcal{F}_n].$$

Proposition 6.1.8 : *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_{\geqslant 0}$ be a non-negative convex function. Let $(X_n)_{n\geqslant 0}$ be an adapted process and suppose that for all $n \geqslant 0$, we have, $\mathbb{E}[\varphi(X_n)] < \infty$.*

(1) If (X_n) is a martingale, then $(\varphi(X_n))$ is a submartingale.

(2) If (X_n) is a submartingale and φ is non-decreasing, then $(\varphi(X_n))$ is also a submartingale.

Remark 6.1.9 : From this proposition, we know that if $(X_n)_{n\geqslant 0}$ is a martingale, then $(|X_n|)_{n\geqslant 0}$ is a submartingale; if for all n, X_n is square-integrable, then $(X_n^2)_{n\geqslant 0}$ is also a submartingale. If $(X_n)_{n\geqslant 0}$ is a submartingale, then so is $(X_n^+)_{n\geqslant 0}$.

Proof :

(1) We use Jensen's inequality of conditional expectations ((6) in Proposition 5.2.8) to obtain,

$$\forall n \geqslant 0, \quad \mathbb{E}[\varphi(X_{n+1}) \,|\, \mathcal{F}_n] \geqslant \varphi(\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n]) = \varphi(X_n).$$

(2) The proof is similar since $X_n \leqslant \mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n]$ and φ is non-decreasing, we have,

$$\forall n \geqslant 0, \quad \mathbb{E}[\varphi(X_{n+1}) \,|\, \mathcal{F}_n] \geqslant \varphi(\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n]) \geqslant \varphi(X_n).$$

□

Definition 6.1.10 : Given a real sequence $(H_n)_{n\geqslant 1}$ of random variables. We say that $(H_n)_{n\geqslant 1}$ is a *predictable* (可預測) sequence if for all $n \geqslant 1$, H_n is bounded and is $\mathcal{F}_{n-1}$-measurable.

Proposition 6.1.11 : *Given an adapted process $(X_n)_{n\geqslant 0}$ and a predictable sequence $(H_n)_{n\geqslant 1}$. Let $(H \cdot X)_0 = 0$ and*

$$\forall n \geqslant 1, \quad (H \cdot X)_n = H_1(X_1 - X_0) + \cdots + H_n(X_n - X_{n-1}).$$

Then, we have the two following properties,

(1) If (X_n) is a martingale, then $((H \cdot X)_n)$ is also a martingale.

(2) If (X_n) is a supermartingale (resp. a submartingale) and for all $n \geqslant 1$, $H_n \geqslant 0$, then $((H \cdot X)_n)$ is also a supermartingale (resp. a submartingale).

Proof :

(1) Since the random variables H_n are are bounded, we can check that all the random variables $(H \cdot X)_n$ are still integrable; and from construction, the random process $((H \cdot X)_n)$ is still an adapted process. Hence, we only need to check that for all $n \geqslant 0$, we have,

$$\mathbb{E}[(H \cdot X)_{n+1} - (H \cdot X)_n \,|\, \mathcal{F}_n] = 0.$$

Since $(H \cdot X)_{n+1} - (H \cdot X)_n = H_{n+1}(X_{n+1} - X_n)$ and H_{n+1} is $\mathcal{F}_n$-measurable, we have,

$$\mathbb{E}[H_{n+1}(X_{n+1} - X_n) \,|\, \mathcal{F}_n] = H_{n+1}\,\mathbb{E}[X_{n+1} - X_n \,|\, \mathcal{F}_n] = 0.$$

The proof is complete.

(2) The proof is similar to (1).

□

In the case of (1), if X_n represents the total asset of a player at time n, then $X_{n+1} - X_n$ is the variation of his his / her total asset between time n and $n+1$. We can think of H_{n+1} as a modification of strategy from the player at time n, by multiplying the bet by H_{n+1} which is a $\mathcal{F}_n$-measurable strategy. In such a case, a fair game stays fair but the variation of the total asset of the player between time n and $n+1$ becomes $H_{n+1}(X_{n+1} - X_n)$.

6.2 Stopping Time

Definition 6.2.1 : Given a random variable $T : \Omega \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ and a filtration $(\mathcal{F}_n)_{n\geqslant 0}$. We say that T is a *stopping time* (停止時間) with respect to the filtration $(\mathcal{F}_n)_{n\geqslant 0}$ if

$$\forall n \in \mathbb{N}_0 \cup \{\infty\}, \quad \{T = n\} \in \mathcal{F}_n. \tag{6.3}$$

Remark 6.2.2 : Note that T is allowed to take the value of $+\infty$ and we can write,

$$\{T = +\infty\} = \Omega \backslash \bigcup_{n\geqslant 0} \{T = n\}.$$

Hence, if we define,

$$\mathcal{F}_\infty = \bigvee_{n\geqslant 0} \mathcal{F}_n := \sigma(\bigcup_{n\geqslant 0} \mathcal{F}_n),$$

then $\{T = +\infty\} \in \mathcal{F}_\infty$.

Remark 6.2.3 : We can easily check that the definition in Eq. (6.3) is equivalent to the following,

$$\forall n \in \mathbb{N}_0 \cup \{\infty\}, \quad \{T \leqslant n\} \in \mathcal{F}_n. \tag{6.4}$$

Therefore, depending on the circumstance, sometimes we use the definition in Eq. (6.3) and sometimes the one in Eq. (6.4).

If we come back to the interpretation of a game, the stopping time is the time at wich a player decides to quit the game. Thus, to make this decision at time n, the only available information is what has been accumulated up to time n, which is $\mathcal{F}_n$. This being said, we cannot decide to sell a share in the stock market when it is at its highest level, since we do not know anything about its future evolution.

Example 6.2.4 : Below are examples of stopping times.

(1) Given $k \in \mathbb{N}_0$, the the constant time $T = k$ is a stopping time.

(2) If $(Y_n)_{n\geqslant 0}$ is an adapted process, then for any Borel set $A \in \mathcal{B}(\mathbb{R})$,

$$T_A := \inf\{n \in \mathbb{N}_0 : Y_n \in A\}$$

is a stopping time, called the *entry time in A*. It can be checked as follows,

$$\{T_A = n\} = \{Y_0 \notin A, \dots, Y_{n-1} \notin A, Y_n \in A\} \in \mathcal{F}_n.$$

Note that we use the following convention: $\inf \varnothing = +\infty$.

(3) For a fixed $N > 0$, we let

$$L_A := \sup\{n \leqslant N : Y_n \in A\} \quad (\sup \varnothing = 0).$$

Then, L_A is not necessarily a stopping time since for $1 \leqslant n \leqslant N-1$, we have,

$$\{L_A = n\} = \{Y_n \in A, Y_{n+1} \notin A, \dots, Y_N \notin A\},$$

which is not necessarily $\mathcal{F}_n$-measurable.

Proposition 6.2.5 : *Stopping times have the following properties.*

(1) If S and T are both stopping times, then $S \vee T$ and $S \wedge T$ are also stopping times.

(2) If $(T_k)_{k\geqslant 1}$ is a sequence of stopping times, then

$$\inf_{k\geqslant 1} T_k, \qquad \liminf_{k\to\infty} T_k,$$
$$\sup_{k\geqslant 1} T_k, \qquad \limsup_{k\to\infty} T_k$$

are all stopping times.

Proof :

(1) Given $n \in \mathbb{N}_0$, we have,

$$\{S \wedge T \leqslant n\} = \{S \leqslant n\} \cup \{T \leqslant n\},$$
$$\{S \vee T \leqslant n\} = \{S \leqslant n\} \cap \{T \leqslant n\}.$$

(2) Given $n \in \mathbb{N}_0$, we have,

$$\left\{\inf_{k\geqslant 1} T_k \leqslant n\right\} = \bigcup_{k\geqslant 1} \{T_k \leqslant n\},$$
$$\left\{\liminf_{k\geqslant 1} T_k \leqslant n\right\} = \bigcap_{m\geqslant 1} \Big(\bigcup_{k\geqslant m} \{T_k \leqslant n\}\Big).$$

□

Definition 6.2.6 : If T is a stopping time, then the *stopped σ-algebra* (停止 σ 代數) or the *σ-algebra of T-past* (T 的過往 σ 代數) is defined by

$$\mathcal{F}_T = \{A \in \mathcal{F} : \forall n \in \mathbb{N}_0, A \cap \{T = n\} \in \mathcal{F}_n\}.$$

Question 6.2.7: Check that $\mathcal{F}_T$ is a σ-algebra and that $\mathcal{F}_T = \mathcal{F}_n$ for a constant stopping time $T = n$.

Proposition 6.2.8 : *If S and T are both stopping times and $S \leqslant T$, then we have, $\mathcal{F}_S \subseteq \mathcal{F}_T$.*

Proof : Let $A \in \mathcal{F}_S$. Then, for all $n \geqslant 0$, we have,

$$A \cap \{T = n\} = \bigcup_{k=0}^{n} (A \cap \{S = k\}) \cap \{T = n\} \in \mathcal{F}_n.$$

□

Proposition 6.2.9 : *Let $(Y_n)_{n \geqslant 0}$ be an adapted process and T be a stopping time, then the random variable defined below is $\mathcal{F}_T$-measurable,*

$$\mathbb{1}_{T<\infty} Y_T(\omega) = \begin{cases} Y_n(\omega) & \text{if } T(\omega) = n \in \mathbb{N}_0, \\ 0 & \text{if } T(\omega) = +\infty. \end{cases} \tag{6.5}$$

Proof : Let $B \in \mathcal{B}(\mathbb{R})$. Assume that $0 \notin B$ and that for all $n \in \mathbb{N}_0$, we have,

$$\{\mathbb{1}_{T<\infty} Y_T \in B\} \cap \{T = n\} = \{Y_n \in B\} \cap \{T = n\} \in \mathcal{F}_n.$$

This means that $\{\mathbb{1}_{T<\infty} Y_T \in B\} \in \mathcal{F}_T$. If $0 \in B$, then for all $n \in \mathbb{N}_0$, we have,

$$\{\mathbb{1}_{T<\infty} Y_T \in B\} \cap \{T = n\} = \{\mathbb{1}_{T<\infty} Y_T \in B^c\}^c \cap \{T = n\} \in \mathcal{F}_n.$$

We may conclude in a similar way. □

When the stopping time T is almost surely finite, we can simply write Eq. (6.5) as Y_T. Moreover, for any stopping time T and any non-negative integer n, $T \wedge n$ is also a stopping time (Proposition 6.2.5). Thus, from Proposition 6.2.9, $Y_{T\wedge n}$ is $\mathcal{F}_{T\wedge n}$-measurable, so also $\mathcal{F}_n$-measurable.

Theorem 6.2.10 (Stopping theorem) **:** *Let $(X_n)_{n \geqslant 0}$ be a martingale (resp. a supermartingale). Let T be a stopping time. Then, $(X_{T\wedge n})_{n \geqslant 0}$ is also a martingale (resp. a supermartingale). If the stopping time T is bounded, then we have $X_T \in L^1$ and,*

$$\mathbb{E}[X_T] = \mathbb{E}[X_0] \quad (\text{or } \mathbb{E}[X_T] \leqslant \mathbb{E}[X_0]).$$

Proof : Let us see how to construct the random process $(X_{T\wedge n})_{n\geqslant 0}$ from $(X_n)_{n\geqslant 0}$ in the sense of Proposition 6.1.11, so that we can apply directly the proposition. We note that for any $n\geqslant 0$, we have

$$X_{T\wedge(n+1)} - X_{T\wedge n} = \mathbb{1}_{T\geqslant n+1}(X_{n+1} - X_n).$$

Therefore, let us define

$$\forall n\geqslant 1,\quad H_n = \mathbb{1}_{T\geqslant n} = 1 - \mathbb{1}_{T\leqslant n-1},$$

Then, $(H_n)_{n\geqslant 1}$ is a predictable sequence and we have

$$\forall n\geqslant 1,\quad X_{T\wedge n} = X_0 + (H\cdot X)_n.$$

This allows us to conclude for the first part of the theorem.

Next, if T is bounded, let N be its upper bound, then $\mathbb{E}[X_T] = \mathbb{E}[X_{T\wedge N}] = \mathbb{E}[X_0]$ (resp. $\leqslant \mathbb{E}[X_0]$).
□

Question 6.2.11: Consider an i.i.d. sequence $(Y_n)_{n\geqslant 1}$ of random variables with distribution $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$. Define $X_0 = 0$ and for all $n\geqslant 1$, define $X_n = Y_1 + \cdots + Y_n$, that is to say $(X_n)_{n\geqslant 0}$ is a random walk on $\mathbb{Z}$ started at 0 with symmetric distribution. Let

$$T = \inf\{n\geqslant 0 : X_n = 1\}.$$

(1) Show that $T<\infty$ a.s.

(2) Show that $\mathbb{E}[X_T] = 1$ and $\mathbb{E}[X_0] = 0$.

(3) Explain.

6.3 Almost Sure Convergence of Martingales

In this section, we discuss the almost sure convergence of martingales (resp. submartingales).

We fisrt consider a sequence of real numbers $(\alpha_n)_{n\geqslant 0}$. For all real numbers $a<b$, if we consider two *time series* (時間序列) $(S_k(\alpha))$ and $(T_k(\alpha))$ with values in $\overline{\mathbb{N}}$ defined as follows,

$$\begin{aligned} S_1(\alpha) &= \inf\{n\geqslant 0 : \alpha_n \leqslant a\},\\ T_1(\alpha) &= \inf\{n\geqslant S_1(\alpha) : \alpha_n \geqslant b\}, \end{aligned}$$

and by induction, for all $k\geqslant 1$, define

$$\begin{aligned} S_{k+1}(\alpha) &= \inf\{n\geqslant T_k(\alpha) : \alpha_n \leqslant a\},\\ T_{k+1}(\alpha) &= \inf\{n\geqslant S_{k+1}(\alpha) : \alpha_n \geqslant b\}. \end{aligned}$$

In the above definition, we take the convention $\inf\varnothing = +\infty$. Next, for all $n\geqslant 1$, we define

$$\begin{aligned} N_n([a,b],\alpha) &= \sum_{k=1}^{\infty} \mathbb{1}_{T_k(\alpha)\leqslant n},\\ N_\infty([a,b],\alpha) &= \sum_{k=1}^{\infty} \mathbb{1}_{T_k(\alpha)<\infty}, \end{aligned}$$

meaning that $N_n([a,b],\alpha)$ denotes the number of upcrossings of the sequence $(\alpha_k)_{k\geqslant 0}$ on the interval $[a,b]$ before time n.

The lemma below is a fundamental property in analysis.

Lemma 6.3.1 : *The sequence $(\alpha_n)_{n\geqslant 0}$ converges in $\overline{\mathbb{R}}$ if and only if for all real nubmers $a<b$, we have $N_\infty([a,b],\alpha)<\infty$.*

Question 6.3.2: Given an adapted process $(X_n)_{n\geqslant 0}$, then for all $k\geqslant 1$, $S_k(X)$ and $T_k(X)$ are random variables with values in $\mathbb{N}_0\cup\{+\infty\}$. Check the following points:

(1) For all $k\geqslant 1$, $S_k(X)$ and $T_k(X)$ are both stopping times.

(2) $N_n([a,b],X)$ is $\mathcal{F}_n$-measurable.

Lemma 6.3.3 (Doob's upcrossing inequality) **:** *Let $(X_n)_{n\geqslant 0}$ be a submartingale. Then for all real numbers $a<b$ and positive integer $n\geqslant 1$, we have,*

$$(b-a)\,\mathbb{E}[N_n([a,b],X)]\leqslant \mathbb{E}[(X_n-a)^+-(X_0-a)^+].$$

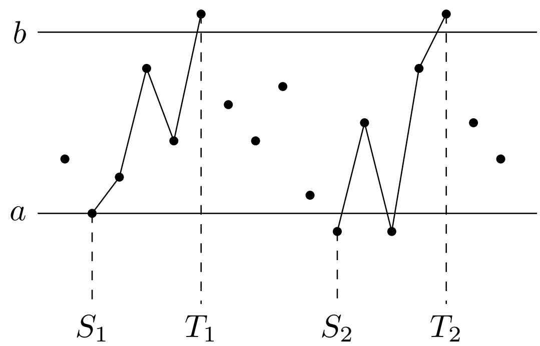

Figure 6.1: An illustration of the values taken by a submartingale $X=(X_n)_{n\geqslant 0}$ and the corresponding stopping times $(S_k)_{k\geqslant 1}$ and $(T_k)_{k\geqslant 1}$. Solid lines represent the increments that are included in the random process $(H\cdot X)_{n\geqslant 0}$. To compute $(H\cdot Y)_{n\geqslant 0}$, we move up the points below a to a and consider the corresponding increments.

Proof : Given $a<b$. Let $Y_n=(X_n-a)^+$ for all $n\geqslant 0$. We know from Proposition 6.1.8 that $(Y_n)_{n\geqslant 0}$ is still a submartingale. To simplify the notation, we replace $S_k(X)$, $T_k(X)$ and $N_n([a,b],X)$ with S_k, T_k and N_n. Define the predictable sequence $(H_n)_{n\geqslant 1}$ of random variables,

$$H_n=\sum_{k=1}^{\infty}\mathbb{1}_{S_k<n\leqslant T_k}\leqslant 1.$$

(Check that the measurable event $\{S_k<n\leqslant T_k\}$ is indeed in $\mathcal{F}_{n-1}$.) We have,

$$(H\cdot Y)_n=\sum_{k=1}^{N_n}(Y_{T_k}-Y_{S_k})+\mathbb{1}_{S_{N_n+1}<n}(Y_n-Y_{S_{N_n+1}})\geqslant\sum_{k=1}^{N_n}(Y_{T_k}-Y_{S_k})\geqslant N_n(b-a).$$

The first inequality in the above formula holds, since on the event $\{S_{N_n+1} < \infty\}$, we have $Y_{S_{N_n+1}} = 0$ and $Y_n \geqslant 0$. Thus, we obtain,

$$\mathbb{E}[(H \cdot Y)_n] \geqslant (b-a)\,\mathbb{E}[N_n].$$

Moreover, for all $n \geqslant 1$, let $K_n = 1 - H_n$. The non-negative sequence $(K_n)_{n\geqslant 1}$ is also predictable. We know from Proposition 6.1.11 that $(K \cdot Y)$ is still a submartingale, implying $\mathbb{E}[(K \cdot Y)_n] \geqslant \mathbb{E}[(K \cdot Y)_0] = 0$. Besides, we also have,

$$(H \cdot Y)_n + (K \cdot Y)_n = Y_n - Y_0, \quad \forall n \geqslant 0.$$

Therefore,

$$(b-a)\,\mathbb{E}[N_n] \leqslant \mathbb{E}[(H \cdot Y)_n] \leqslant \mathbb{E}[(H \cdot Y)_n + (K \cdot Y)_n] = \mathbb{E}[Y_n - Y_0].$$

□

Theorem 6.3.4 : *Let $(X_n)_{n\geqslant 0}$ be a submartingale satisfying one of the equivalent properties*

(1) $(X_n^+)_{n\geqslant 0}$ is bounded in L^1, that is

$$\sup_{n\geqslant 0} \mathbb{E}[(X_n)^+] < \infty. \tag{6.6}$$

(2) $(X_n)_{n\geqslant 0}$ is bounded in L^1, that is

$$\sup_{n\geqslant 0} \mathbb{E}[|X_n|] < \infty. \tag{6.7}$$

Then X_n converges almost surely and its limit X_∞ satisfies $\mathbb{E}[|X_\infty|] < \infty$ and $|X_\infty| < \infty$ a.s.

Remark 6.3.5 : We first note that we have $\mathbb{E}[X_n] = \mathbb{E}[(X_n)^+] - \mathbb{E}[(X_n)^-]$. Since the submartingale $(X_n)_{n\geqslant 0}$ satisfies $\mathbb{E}[X_n] \geqslant \mathbb{E}[X_0]$, then for all $k \geqslant 0$, we have,

$$\mathbb{E}[(X_k)^-] \leqslant \left(\sup_{n\geqslant 0} \mathbb{E}[(X_n)^+]\right) - \mathbb{E}[X_0].$$

Thus, we deduce that Eq. (6.6) implies Eq. (6.7), and the converse holds trivially.

Proof : Given $a, b \in \mathbb{R}$ and $a < b$, from Lemma 6.3.3, for all $n \geqslant 1$, we have,

$$\begin{aligned}(b-a)\,\mathbb{E}[N_n([a,b],X)] &\leqslant \mathbb{E}[(X_n - a)^+] \leqslant |a| + \mathbb{E}[(X_n)^+] \\ &\leqslant |a| + \sup_{k\geqslant 0} \mathbb{E}[(X_k)^+] < \infty.\end{aligned}$$

Thus, when $n \to \infty$, we have,

$$(b-a)\,\mathbb{E}[N_\infty([a,b],X)] < \infty,$$

meaning that $N_\infty([a,b],X)$ is almost surely finite: $\mathbb{P}(N_\infty([a,b],X) < \infty) = 1$.

If we consider the countable set $\{(a,b) \in \mathbb{Q}^2 : a < b\}$, then we have,

$$\mathbb{P}\left(N_\infty([a,b],X) < \infty, \forall a, b \in \mathbb{Q} : a < b\right) = 1.$$

Finally we use Lemma 6.3.1 to conclude that X_n converges almost surely in $\overline{\mathbb{R}}$.

For the integrability of the almost sure limit X_∞, we apply Fatou's lemma,

$$\mathbb{E}[|X_\infty|] \leqslant \liminf_{n\to\infty} \mathbb{E}[|X_n|] \leqslant \sup_{n\geqslant 0} \mathbb{E}[|X_n|] < \infty.$$

Hence, $|X_\infty|$ is finite almost surely. □

Corollary 6.3.6 : *Let (X_n) be a non-negative supermartingale. Then, X_n converges almost surely and its limit X_∞ is integrable and satisfies $X_n \geqslant \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ for all $n \geqslant 0$.*

Proof : This is an application of Theorem 6.3.4. Let $Y_n = -X_n$, then (Y_n) satisfies the assumptions in Eq. (6.6). Finally, we apply the Fatou's lemma for conditional expectations, which is the point (4) from Proposition 5.2.8,

$$X_n \geqslant \liminf_{m\to\infty} \mathbb{E}[X_m \mid \mathcal{F}_n] \geqslant \mathbb{E}[\liminf_{m\to\infty} X_m \mid \mathcal{F}_n] = \mathbb{E}[X_\infty \mid \mathcal{F}_n].$$

□