9 Brownian Motion

第九章 布朗運動

9.1 Limit of Random Walks

In physics, we interprete the Brownian motion as the movement of a tiny particle under the shocks of the other particles. The trajectory of such a particle is quite irregular and highly unpredictable; moreover, the direction of the movement can change drastically at any time. From the point of view of mathematics, we can modelize this using the trajectory of a particle following a random walk on $\mathbb{Z}^d$ after a proper scaling and time-change.

More precisely speaking, let us consider a random walk $(S_n)_{n\geqslant 0}$ on $\mathbb{Z}^d$ started from 0, defined as below,

$$S_n = Y_1 + \cdots + Y_n, \qquad \forall n \geqslant 0,$$

where $(Y_n)_{n\geqslant 1}$ is an i.i.d. sequence of random variables with values in $\mathbb{Z}^d$ with distribution μ, Besides, suppose that μ satisfies the following properties.

(1) Centered and in L^2,

$$\sum_{x\in\mathbb{Z}^d} x\mu(x) = 0 \qquad \text{and} \qquad \sum_{x\in\mathbb{Z}^d} |x|^2\mu(x) < \infty.$$

(2) Isotropic: there exists $\sigma > 0$ such that for all $1 \leqslant i,j \leqslant d$, we have

$$\sum_{x\in\mathbb{Z}^d} x_i x_j \mu(x) = \sigma^2\delta_{i,j}.$$

Theorem 4.5.4 tells us that the rescaled random walk $\frac{1}{\sqrt{n}}S_n$ converges in distribution to the d-dimensional Gaussian variable with covariance matrix given by $\sigma^2\mathrm{Id}$. It is not hard to check that the symmetric simple random walk on $\mathbb{Z}^d$ satisfies the above properties.

We are interested in the *global* behavior of this random walk both in time and in space, that is the behavior of the trajectory $n \mapsto S_n$ for large time. Hence, we introduce the following scaling,

$$S_t^{(n)} = \frac{1}{\sqrt{n}} S_{\lfloor nt\rfloor}, \qquad \forall t \geqslant 0, \quad \forall n \geqslant 1.$$

Alternatively speaking, the new time-scale t corresponds to the original random walk at time $\lfloor nt\rfloor$; besides, due to the central limit theorem, we know that $S_{\lfloor nt\rfloor}$ is of order $\sqrt{n}$ when n is large, explaining the division by $\sqrt{n}$ for a potentially non trivial limit.

Proposition 9.1.1 : *For any positive integer $p \geqslant 1$ and real numbers $0 = t_0 < t_1 < \cdots < t_p$, the following convergence holds,*

$$S_{t_1}^{(n)}, \ldots, S_{t_p}^{(n)} \xrightarrow{\mathcal{L}} (U_{t_1}, \ldots, U_{t_p}), \tag{9.1}$$

where the limiting distribution is characterized by the following properties,

第一節 隨機漫步的收斂

在物理上理解布朗運動的方式，是將他看作一顆非常細小的粒子，在受到其他粒子碰撞後得到的軌跡，這樣的軌跡非常不規律且無法預測，且在每個時間點，都有可能會被碰撞而改變方向。以數學的角度來看，我們可以用一顆在 $\mathbb{Z}^d$ 上做隨機漫步的粒子來模擬，他經過適當的尺度縮放及時間變換後得到的軌跡便會是布朗運動。

更確切的說，我們考慮在 $\mathbb{Z}^d$ 上從 0 出發的隨機漫步 $(S_n)_{n\geqslant 0}$：

$$S_n = Y_1 + \cdots + Y_n, \qquad \forall n \geqslant 0,$$

其中 $(Y_n)_{n\geqslant 1}$ 為取值在 $\mathbb{Z}^d$ 中的 i.i.d. 隨機變數，他們共同的分佈為 μ 並滿足下列性質：

(1) 置中且在 L^2 之中：

$$\sum_{x\in\mathbb{Z}^d} x\mu(x) = 0 \qquad \text{且} \qquad \sum_{x\in\mathbb{Z}^d} |x|^2\mu(x) < \infty.$$

(2) 方向均勻性：存在 $\sigma > 0$ 使得對於所有 $1 \leqslant i,j \leqslant d$，我們有

$$\sum_{x\in\mathbb{Z}^d} x_i x_j \mu(x) = \sigma^2\delta_{i,j}.$$

從定理 4.5.4 ，我們可以得知，經過適當縮放的隨機漫步 $\frac{1}{\sqrt{n}}S_n$ 會分佈收斂至 d 維度的高斯分佈，其共變異數矩陣為 $\sigma^2\mathrm{Id}$。我們可以輕易驗證，在 $\mathbb{Z}^d$ 上的對稱簡單隨機漫步會滿足以上的性質。

我們有興趣的是此隨機漫步在時空上全域的行為，也就是說，在時間夠大的情況下，此隨機漫步軌跡函數 $n \mapsto S_n$ 的表現。因此我們引進下列尺度變換：

$$S_t^{(n)} = \frac{1}{\sqrt{n}} S_{\lfloor nt\rfloor}, \qquad \forall t \geqslant 0, \quad \forall n \geqslant 1.$$

也就是說，在新的時間尺度 t 下，我們可以觀察到原本隨機漫步在時間 $\lfloor nt\rfloor$ 的行為，但由於中央極限定理告訴我們，當 n 夠大時，$S_{\lfloor nt\rfloor}$ 的表現會是 $\sqrt{n}$ 數量級的，因此必須除掉 $\sqrt{n}$ 才有可能得到一個不平凡的極限。

命題 9.1.1 ： 對於任意正整數 $p \geqslant 1$ 及實數 $0 = t_0 < t_1 < \cdots < t_p$，我們有下列收斂：

$$S_{t_1}^{(n)}, \ldots, S_{t_p}^{(n)} \xrightarrow{\mathcal{L}} (U_{t_1}, \ldots, U_{t_p}), \tag{9.1}$$

(1) *the sequence* $(U_{t_j} - U_{t_{j-1}})_{1 \leqslant j \leqslant p}$ *of random variables is independent* ($U_0 = 0$);

(2) *for any* $1 \leqslant j \leqslant p$, $U_{t_j} - U_{t_{j-1}}$ *is a centered Gaussian vector whose covariance matrix writes* $\sigma^2(t_j - t_{j-1})\mathrm{Id}$.

Remark 9.1.2 :

(1) If we define

$$p_a(x) = \frac{1}{(2\pi a)^{d/2}} \exp\Big(-\frac{|x|^2}{2a}\Big), \qquad x \in \mathbb{R}^d,$$

then the density function of $U_{t_j} - U_{t_{j-1}}$ is $p_{\sigma^2(t_j - t_{j-1})}(x)$. Therefore, the density function of the limiting distribution $(U_{t_1}, \dots, U_{t_p})$ given above can be written as

$$f(y_1, \dots, y_p) = \prod_{j=1}^{p} p_{\sigma^2(t_j - t_{j-1})}(y_j - y_{j-1}).$$

(2) The convergence in Eq. (9.1) is the finite-dimensional convergence in distribution of the random process $(S_t^{(n)})_{t \geqslant 0}$, which does not imply the convergence in distribution of the whole process. For instance, the below convergence cannot be seen as a consequence of the finite-dimension convergence in distribution,

$$\sup_{t \geqslant 0} S_t^{(n)} = \sup_{k \in \frac{1}{n}\mathbb{Z}_{\geqslant 0}} S_k^{(n)} \xrightarrow[n\to\infty]{?} \sup_{t \geqslant 0} U_t.$$

The main reason is that the regularity on the limiting path (the infinite-dimensional distribution) cannot be obtained from the knowledge of all the finite-dimensional distributions. Later in Section 9.2 we will construct the Brownian motion as a continuous path, and the Donsker's theorem (not discussed in this lecture) allows us to make sense of the convergence of random walk paths to the Brownian motion under a proper topological space.

Proof : By Lévy's continuity theorem (Theorem 4.4.15), we only need to prove that the following convergence holds for all $\xi_1, \dots, \xi_p \in \mathbb{R}^d$,

$$\mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \xi_j \cdot S_{t_j}^{(n)}\Big)\Big] \xrightarrow[n\to\infty]{} \mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \xi_j \cdot U_{t_j}\Big)\Big].$$

In other words, we need the following convergence to hold for all $\eta_1, \dots, \eta_p \in \mathbb{R}^d$,

$$\mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \eta_j \cdot (S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})\Big)\Big] \xrightarrow[n\to\infty]{} \mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \eta_j \cdot (U_{t_j} - U_{t_{j-1}})\Big)\Big]. \tag{9.2}$$

Let us start with the computation of the above term on the left. First, we have

$$S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} = \frac{1}{\sqrt{n}} \sum_{k=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j \rfloor} Y_k,$$

其中極限分佈是被下列性質所刻劃的：

(1) 隨機變數序列 $(U_{t_j} - U_{t_{j-1}})_{1 \leqslant j \leqslant p}$ 是獨立的（$U_0 = 0$）；

(2) 對任意 $1 \leqslant j \leqslant p$，$U_{t_j} - U_{t_{j-1}}$ 是個置中高斯向量，且其共變異矩陣寫作 $\sigma^2(t_j - t_{j-1})\mathrm{Id}$。

註解 9.1.2 ：

(1) 若定義

$$p_a(x) = \frac{1}{(2\pi a)^{d/2}} \exp\Big(-\frac{|x|^2}{2a}\Big), \qquad x \in \mathbb{R}^d,$$

則 $U_{t_j} - U_{t_{j-1}}$ 的密度函數是 $p_{\sigma^2(t_j - t_{j-1})}(x)$，因此上述極限分佈 $(U_{t_1}, \dots, U_{t_p})$ 的密度函數是可以這樣表示的：

$$f(y_1, \dots, y_p) = \prod_{j=1}^{p} p_{\sigma^2(t_j - t_{j-1})}(y_j - y_{j-1}).$$

(2) 式 (9.1) 中的收斂是整個隨機過程 $(S_t^{(n)})_{t \geqslant 0}$ 有限維度的分佈收斂，但並不代表整個隨機過程的分佈收斂。如下列收斂，無法由有限維度的分佈收斂所推得：

$$\sup_{t \geqslant 0} S_t^{(n)} = \sup_{k \in \frac{1}{n}\mathbb{Z}_{\geqslant 0}} S_k^{(n)} \xrightarrow[n\to\infty]{?} \sup_{t \geqslant 0} U_t.$$

最主要的原因是，在已知所有有限維度分佈的情況之下，我們無法得知極限路徑（無限維度分佈）的規律性。稍後在第 9.2 節之中，我們會構造軌跡連續的布朗運動，然後 Donsker 定理（這門課不討論）會告訴我們在適當的拓撲空間中，我們可以討論隨機漫步軌跡收斂至布朗運動。

證明：根據 Lévy 連續定理（定理 4.4.15 ），我們只需要證明對於所有 $\xi_1, \dots, \xi_p \in \mathbb{R}^d$，下列收斂會成立即可：

$$\mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \xi_j \cdot S_{t_j}^{(n)}\Big)\Big] \xrightarrow[n\to\infty]{} \mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \xi_j \cdot U_{t_j}\Big)\Big].$$

也就是說，對於所有的 $\eta_1, \dots, \eta_p \in \mathbb{R}^d$，下列收斂要成立：

$$\mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \eta_j \cdot (S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})\Big)\Big] \xrightarrow[n\to\infty]{} \mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \eta_j \cdot (U_{t_j} - U_{t_{j-1}})\Big)\Big]. \tag{9.2}$$

我們針對上式的左邊做計算。首先，我們有

$$S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} = \frac{1}{\sqrt{n}} \sum_{k=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j \rfloor} Y_k,$$

that is, the sequence $(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})_{1\leqslant j\leqslant p}$ of random variables is independent. Additionally, we have, for a fixed j,

$$S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} \overset{(d)}{=} \frac{1}{\sqrt{n}} S_{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor} = \sqrt{\frac{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}{n}} \frac{S_{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}}{\sqrt{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}}.$$

The central limit theorem states that the above random variable converges in distribution to $\sqrt{t_j - t_{j-1}}N$, where N is a Gaussian vector with covariance matrix $\sigma^2\mathrm{Id}$. Hence, the below convergence holds for all $1 \leqslant j \leqslant p$,

$$\mathbb{E}\left[\exp\left(\mathrm{i}\,\eta_j \cdot (S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})\right)\right] \xrightarrow[n\to\infty]{} \mathbb{E}[\exp(\mathrm{i}\sqrt{t_j - t_{j-1}}\eta_j \cdot N)] = \exp\left(-\frac{\sigma^2|\eta_j|^2(t_j - t_{j-1})}{2}\right).$$

The last equality in the above formula is the Laplace transform of the Gaussian distribution, computed in Lemma 2.4.14. Finally, using the independence of $(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})_{1\leqslant j\leqslant p}$, we can get Eq. (9.2) and conclude the proof. □

Definition 9.1.3 : Given a sequence $(B_t)_{t\geqslant 0}$ of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$. We say that $(B_t)_{t\geqslant 0}$ is a d-dimensional *standard Brownian motion* (標準布朗運動) started from 0 if the following properties are satisfied.

(BM1) $B_0 = 0$ a.s. and for all positive integer $p \geqslant 1$ and real numbers $0 = t_0 < t_1 < \cdots < t_p$, the sequence $(B_{t_j} - B_{t_{j-1}})_{1\leqslant j\leqslant p}$ of random variables is independent and for all $1 \leqslant j \leqslant p$, the Gaussian vector $B_{t_j} - B_{t_{j-1}}$ is centered with covariance matrix $(t_j - t_{j-1})\mathrm{Id}$.

(BM2) For all $\omega \in \Omega$, the function $t \mapsto B_t(\omega)$ is continuous.

Remark 9.1.4 :

(1) The most of the time, we will say Brownian motion instead of standard Brownian motion.

(2) In the below Section 9.2, we will construct a standard Brownian motion, giving its existence. In consequence, the statement in Proposition 9.1.1 can be reformulated as follows. For any $0 \leqslant t_1 < \cdots < t_p$, the following convergence in distribution holds,

$$(S_{t_1}^{(n)}, \ldots, S_{t_p}^{(n)}) \xrightarrow{(d)} (\sigma B_{t_1}, \ldots, \sigma B_{t_p}).$$

This tells us that a random walk convergence to a Brownian motion after a proper space-time scaling. From the point of view of physics, this can be interpreted as the macroscopic random trajectory arised from the microscopic shocks from particles.

(3) As we have seen above, the distribution of the random vector $(B_{t_1}, \ldots, B_{t_p})$ is characterized by

$$\mathbb{P}\left((B_{t_1}, \ldots, B_{t_p}) \in A\right) = \int_A \prod_{j=1}^p p_{t_j - t_{j-1}}(y_{t_j} - y_{t_{j-1}}) \prod_{j=1}^p \mathrm{d}y_j, \qquad \forall A \in \mathcal{B}((\mathbb{R}^d)^p) = \mathcal{B}(\mathbb{R}^d)^{\otimes p}. \tag{9.3}$$

也就是說，隨機變數序列 $(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})_{1\leqslant j\leqslant p}$ 是獨立的。此外，給定 j 時，我們有

$$S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} \overset{(d)}{=} \frac{1}{\sqrt{n}} S_{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor} = \sqrt{\frac{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}{n}} \frac{S_{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}}{\sqrt{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}}.$$

根據中央極限定理，上述隨機變數會分佈收斂至 $\sqrt{t_j - t_{j-1}}N$，其中 N 是個共變異矩陣為 $\sigma^2\mathrm{Id}$ 的高斯向量。因此對每個 $1 \leqslant j \leqslant p$，我們有下列收斂

$$\mathbb{E}\left[\exp\left(\mathrm{i}\,\eta_j \cdot (S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})\right)\right] \xrightarrow[n\to\infty]{} \mathbb{E}[\exp(\mathrm{i}\sqrt{t_j - t_{j-1}}\eta_j \cdot N)] = \exp\left(-\frac{\sigma^2|\eta_j|^2(t_j - t_{j-1})}{2}\right).$$

上式中的最後一個等是是根據高斯分佈的 Laplace 變換計算（引理 2.4.14）而得到的。最後，我們用 $(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})_{1\leqslant j\leqslant p}$ 的獨立性，得到式 (9.2) 而完成證明。 □

定義 9.1.3 ： 給定取值在 $\mathbb{R}^d$ 中，定義在機率空間 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的隨機變數序列 $(B_t)_{t\geqslant 0}$。若 $(B_t)_{t\geqslant 0}$ 滿足：

(BM1) $B_0 = 0$ a.s. 且對於所有正整數 $p \geqslant 1$ 及實數 $0 = t_0 < t_1 < \cdots < t_p$，隨機變數序列 $(B_{t_j} - B_{t_{j-1}})_{1\leqslant j\leqslant p}$ 是獨立的且對於所有 $1 \leqslant j \leqslant p$，$B_{t_j} - B_{t_{j-1}}$ 是個置中且變異數矩陣為 $(t_j - t_{j-1})\mathrm{Id}$ 的高斯向量；

(BM2) 對於所有 $\omega \in \Omega$，函數 $t \mapsto B_t(\omega)$ 是連續的，

則我稱 $(B_t)_{t\geqslant 0}$ 為 d 維度由 0 出發的標準布朗運動 (standard Brownian motion)。

註解 9.1.4 ：

(1) 大部分的時候，我們會直接將標準布朗運動稱作布朗運動。

(2) 在下面的第 9.2 節中，我們會構造標準布朗運動，進而得到存在性，因此命題 9.1.1 的敘述可以重新解釋為，對於任意的 $0 \leqslant t_1 < \cdots < t_p$，我們有分佈收斂

$$(S_{t_1}^{(n)}, \ldots, S_{t_p}^{(n)}) \xrightarrow{(d)} (\sigma B_{t_1}, \ldots, \sigma B_{t_p}).$$

這告訴我們，隨機漫步在經過正確的時空尺度變換後會收斂至布朗運動；從物理的觀點來理解，可以將此現象視為微觀的粒子碰撞現象及巨觀的隨機運動軌跡。

(3) 如同上面所看到的，隨機向量 $(B_{t_1}, \ldots, B_{t_p})$ 的分佈可以由下列性質所描繪：

$$\mathbb{P}\left((B_{t_1}, \ldots, B_{t_p}) \in A\right) = \int_A \prod_{j=1}^p p_{t_j - t_{j-1}}(y_{t_j} - y_{t_{j-1}}) \prod_{j=1}^p \mathrm{d}y_j, \qquad \forall A \in \mathcal{B}((\mathbb{R}^d)^p) = \mathcal{B}(\mathbb{R}^d)^{\otimes p}. \tag{9.3}$$

9.2 Construction of Brownian Motion

In this section, we want to construct a Brownian motion as a random continuous function. More precisely, we need to construct an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we can define a random variable with values in $\mathcal{C}([0,1], \mathbb{R})$. Then, since the Brownian motion we want to define is a random process taking values in $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$, we need to glue i.i.d. copies of the random process constructed above together, then repeat the same operation on all the d coordinates to obtain the Brownian motion we looked for.

9.2.1 Lévy's Construction

The construction we present below is from Lévy. The idea is to start with an appropriate increasing sets $\mathcal{D}_n$ of $[0,1]$ such that $\cup\mathcal{D}_n$ is dense in $[0,1]$, then construct a random process with property (BM1) satisfied on $\mathcal{D}_n$. Next, we extend this random process by linear interpolation to points in $[0,1]\backslash\mathcal{D}_n$. In the end, we show that, when n tends to infinity, such a construction converges uniformly on $[0,1]$, so the limit satisfies the property (BM2) as well.

Before describing Lévy's construction in details, we start with the following lemma, used as the induction step in the construction, and is also the key idea of this method.

Lemma 9.2.1 : *Given S, T and Z three random varibles with normal distributions $\mathcal{N}(0,s), \mathcal{N}(0,t)$ and $\mathcal{N}(0,1)$ and suppose that $S, T-S$ and Z are independent. Define*

$$U = \frac{S+T}{2} + \frac{\sqrt{t-s}}{2} Z.$$

Then, $S, U-S$ and $T-U$ are independent random variables with normal distributions $\mathcal{N}(0,s), \mathcal{N}(0,u-s)$ and $\mathcal{N}(0,t-u)$, where $u = \frac{s+t}{2}$.

Proof : By definition, the covariance matrix of $(S, T-S, Z)$ writes

$$K_{(S,T-S,Z)} = \mathrm{Diag}(s, t-s, 1).$$

Besides, if we define the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{t-s}}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{t-s}}{2} \end{pmatrix},$$

then we have $(S, U-S, T-U)^T = A(S, T-S, Z)^T$. By Exercise 2.21, we can write

$$K_{(S,U-S,T-U)} = AK_{(S,T-S,Z)}A^T = \mathrm{Diag}(s, \tfrac{t-s}{2}, \tfrac{t-s}{2}).$$

Since $(S, U-S, T-U)$ is a Gaussian vector with diagonal covariance matrix, Proposition 3.4.1 tells us that its components are independent. □

第二節 布朗運動的構造

在此章節中，我們想要把布朗運動當作一個隨機的連續函數來構造，更確切的說，我們想要建構一個合適的機率空間 $(\Omega, \mathcal{F}, \mathbb{P})$ 讓我們可以在上面定義取值在 $C([0,1], \mathbb{R})$ 上隨機變數。接著，因為我們的布朗運動是個取值在 $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ 中的隨機過程，我們可以考慮前述構造 i.i.d. 定義在 $C([0,1], \mathbb{R})$ 上的隨機變數，將他們黏在一起，並且對 d 維每個座標都做相同操作，即可得到我們要的布朗運動。

第一小節 Lévy 的構造

我們要介紹的是 Lévy 提出的構造方式：想法是先在 $[0,1]$ 上找出合適的遞增集合 $\mathcal{D}_n$ 使得 $\cup\mathcal{D}_n$ 是個在 $[0,1]$ 中的稠密集合，接著在 $\mathcal{D}_n$ 上構造好滿足性質 (BM1) 的隨機過程，並且以線性內插法將此隨機過程拓展到 $[0,1]\backslash\mathcal{D}_n$ 上的點，最後證明這樣的構造方式在 n 趨近無窮大時，會在 $[0,1]$ 上均勻收斂，因此極限也會滿足性質 (BM2)。

在詳細敘述上述 Lévy 的構造前，由於我們會使用數學歸納法來構造，我們先來證明下面的引理，這也是 Lévy 構造的精髓所在。

引理 9.2.1 ：給定 S、T 及 Z 三個常態分佈隨機變數 $\mathcal{N}(0,s)$、$\mathcal{N}(0,t)$ 及 $\mathcal{N}(0,1)$，我們假設 S、$T-S$ 及 Z 獨立。定義

$$U = \frac{S+T}{2} + \frac{\sqrt{t-s}}{2} Z,$$

則 S、$U-S$ 及 $T-U$ 為獨立的常態分佈隨機變數 $\mathcal{N}(0,s)$、$\mathcal{N}(0,u-s)$ 及 $\mathcal{N}(0,t-u)$，其中 $u = \frac{s+t}{2}$。

證明：根據定義，我們知道 $(S, T-S, Z)$ 的共變異數矩陣寫作

$$K_{(S,T-S,Z)} = \mathrm{Diag}(s, t-s, 1).$$

此外，若定義矩陣

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{t-s}}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{t-s}}{2} \end{pmatrix},$$

則我們有 $(S, U-S, T-U)^T = A(S, T-S, Z)^T$。根據習題 2.21 ，我們得知

$$K_{(S,U-S,T-U)} = AK_{(S,T-S,Z)}A^T = \mathrm{Diag}(s, \tfrac{t-s}{2}, \tfrac{t-s}{2}).$$

由於 $(S, U-S, T-U)$ 是個高斯向量且其共變異數矩陣為對角矩陣，命題 3.4.1 告訴我們獨立向量中的元素是獨立的。 □

Theorem 9.2.2 : *A standard Brownian motion exists, that is, we can find an appropriate probability space and a random process $(B_t)_{t\geqslant 0}$ such that the two properties in Definition 9.1.3 are satisfied.*

Proof : To begin with, let us define the following dyadic sets

$$\forall n \geqslant 0, \quad \mathcal{D}_n = \{\tfrac{k}{2^n} : 0 \leqslant k \leqslant 2^n\}.$$

Next, we will construct, for each n, a random continuous function on $[0,1]$ such that its marginal distributions satisfy the property required by Definition 9.1.3. And in the end, we will show that such random continuous functions will converge uniformly to a random continuous function, which is the Brownian motion we are looking for.

Write $\mathcal{D} = \bigcup_{n\geqslant 0} \mathcal{D}_n$ and consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we can define an i.i.d. sequence $(Z_d)_{d\in\mathcal{D}}$ of standard normal random varibales. We construct by induction the sequence $(B_d)_{d\in\mathcal{D}}$ of random processes satisfying

(a) for any $s < t$ in $\mathcal{D}_n$, the distribution of the random variable $B_t - B_s$ is $\mathcal{N}(0, t-s)$ and is independent of B_s;

(b) for any non-negative integer $n \geqslant 0$, the random process $(B_d)_{d\in\mathcal{D}_n}$ and $(Z_t)_{t\in\mathcal{D}\setminus\mathcal{D}_n}$ are independent.

Our construction goes as follows.

(1) Define $B_0 = 0$ and $B_1 = Z_1$. The properties (a) and (b) clearly hold.

(2) Given $n \geqslant 1$ and assume that B_d for $d \in \mathcal{D}_{n-1}$ is well-defined and that the properties (a) and (b) are satified for any $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$. Next, we define

$$B_d = \frac{1}{2}(B_{d-2^{-n}} + B_{d+2^{-n}}) + \frac{Z_d}{2^{(n+1)/2}}.$$

We need to check that in the induction step (2), the properties (a) and (b) are still satisfied, which is a direct consequence of Lemma 9.2.1.

Now that we have defined a random process on dyadic points in $\mathcal{D}$, we interpolate its value outside of $\mathcal{D}$ linearly. To be more precise, let us define

$$F_0(t) = \begin{cases} 0 & t = 0, \\ Z_1 & t = 1, \end{cases}$$

and interpolate $F_0(t)$, for $t \in (0,1)$, linearly between $t = 0$ and $t = 1$. Next, we define

$$\forall n \geqslant 1, \quad F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t & t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}, \\ 0 & t \in \mathcal{D}_{n-1}, \end{cases}$$

and define $F_n(t)$, for $t \notin \mathcal{D}_n$, to be the linear interpolation between the two closent points from $\mathcal{D}_n$. All the above functions are continuous on $[0,1]$ and for all $n \geqslant 0$ and $d \in \mathcal{D}_n$, we have

$$B_d = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d).$$

定理 9.2.2 ： 標準布朗運動存在，也就是說，我們可以找到合適的機率空間及隨機過程 $(B_t)_{t\geqslant 0}$ 使得定義 9.1.3 中的兩條件皆滿足。

證明：我們先定義下列二元集合：

$$\forall n \geqslant 0, \quad \mathcal{D}_n = \{\tfrac{k}{2^n} : 0 \leqslant k \leqslant 2^n\}.$$

再來我們將對於所有 n，構造在 $[0,1]$ 上的隨機連續函數，使得他在 $\mathcal{D}_n$ 上的邊緣分佈會滿足定義 9.1.3 中所要求的，最後再證明這樣構造出來的隨機函數，會均勻收斂至一個隨機連續函數，就是我們的布朗運動。

我們記 $\mathcal{D} = \bigcup_{n\geqslant 0} \mathcal{D}_n$ 並考慮機率空間 $(\Omega, \mathcal{F}, \mathbb{P})$ 使得我們可以在上面定義 i.i.d. 的標準常態分佈序列 $(Z_d)_{d\in\mathcal{D}}$。我們以數學歸納法的方式來構造隨機過程序列 $(B_d)_{d\in\mathcal{D}}$ 使得下列性質能被滿足：

(a) 對於任意在 $\mathcal{D}_n$ 中的元素 $s < t$，隨機變數 $B_t - B_s$ 的分佈為 $\mathcal{N}(0, t-s)$ 且與 B_s 獨立；

(b) 對於任意非負整數 $n \geqslant 0$，隨機過程 $(B_d)_{d\in\mathcal{D}_n}$ 與 $(Z_t)_{t\in\mathcal{D}\setminus\mathcal{D}_n}$ 互為獨立。

我們的構造如下：

(1) 我們定義 $B_0 = 0$ 及 $B_1 = Z_1$，顯然性質 (a) 及 (b) 成立。

(2) 給定 $n \geqslant 1$ 並且假設 B_d 在 $d \in \mathcal{D}_{n-1}$ 上有定義且上述性質 (a) 及 (b) 成立，對於任意 $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$，我們定義

$$B_d = \frac{1}{2}(B_{d-2^{-n}} + B_{d+2^{-n}}) + \frac{Z_d}{2^{(n+1)/2}}.$$

我們需要驗證，在遞迴構造 (2) 中，性質 (a) 及 (b) 仍然有被滿足，這是引理 9.2.1 的直接結果。

我們現在已經在二元數 $\mathcal{D}$ 上面定義了隨機過程，在其他非二元數的點上，我們就以線性內插的方式定義。正確來說，我們定義

$$F_0(t) = \begin{cases} 0 & t = 0, \\ Z_1 & t = 1, \end{cases}$$

並將 $F_0(t)$ 在 $t \in (0,1)$ 上定義為在 $t = 0$ 及 $t = 1$ 之間做線性內插；接著定義

$$\forall n \geqslant 1, \quad F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t & t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}, \\ 0 & t \in \mathcal{D}_{n-1}, \end{cases}$$

並將 $F_n(t)$ 在 $t \notin \mathcal{D}_n$ 上定義為在 $\mathcal{D}_n$ 中最接近的相鄰兩點上所取的值的線性內插。以上函數皆在 $[0,1]$ 上連續，且對於所有 $n \geqslant 0$ 及 $d \in \mathcal{D}_n$，我們有

$$B_d = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d).$$

We want to show that $\sum_{n\geqslant 0} F_n$ converges uniformly on $[0,1]$, meaning that we need to estimate the tail of Z_d. Exercise 2.14 states that for any $c>1$ and large enough n, the following inequality holds,

$$\mathbb{P}(|Z_d| \geqslant c\sqrt{n}) \leqslant \exp\big(-\frac{c^2 n}{2}\big).$$

This implies that

$$\sum_{n\geqslant 0} \mathbb{P}(\exists d \in \mathcal{D}_n : |Z_d| \geqslant c\sqrt{n}) \leqslant \sum_{n\geqslant 0}\sum_{d\in\mathcal{D}_n} \mathbb{P}(|Z_d| \geqslant c\sqrt{n}) \leqslant \sum_{n\geqslant 0} (2^n+1)\exp\big(-\frac{c^2 n}{2}\big),$$

where the right side converge as long as $c > \sqrt{2\ln 2}$. Borel–Cantelli lemma then states that, there exists a random variable $N<\infty$ a.s. such that for all $n \geqslant N$ and $d \in \mathcal{D}_n$, we have $|Z_d| < c\sqrt{n}$, that is, we have for all $n \geqslant N$,

$$\|F_n\|_\infty < c\sqrt{n}2^{-(n+1)/2},$$

and in consequence, $\sum_{n\geqslant 0} F_n$ indeed converges uniformly on $[0,1]$ to a (random) continuous function, denoted B.

We can easily check that $(B_t)_{t\in[0,1]}$ satisfies the property (BM1) on the dense set $\mathcal{D}$, and by interpolation, this property is also satisfied on $[0,1]$. Also, by construction, the property (BM2) clearly holds. In conclusion, we have constructed a one-dimensional Brownian motion on the time interval $[0,1]$. If want to extend the time interval to $\mathbb{R}_{\geqslant 0}$, we may consider an i.i.d. sequence of Brownian motion constructed above $(B^{(k)})_{k\geqslant 0}$, where each random variable $B^{(k)}$ is a random function in $C([0,1],\mathbb{R})$. We may glue them together to obtain a Brownian motion defined on the time interval $\mathbb{R}_{\geqslant 0}$. To define a Brownian motion in dimension d, we take d i.i.d. components and we are done with the construction. □

Remark 9.2.3 : There is a more algebraic perspective of this construction.

(1) We may consider a countably infinite family $(Z_n)_{n\geqslant 0}$ of i.i.d. $\mathcal{N}(0,1)$ random variables on a probability space $L^2(\Omega,\mathcal{F},\mathbb{P})$. The probability space $L^2(\Omega,\mathcal{F},\mathbb{P})$ is a Hilbert space and the random variables $(Z_n)_{n\geqslant 0}$ form an orthonormal family on it.

Moreover, $L^2([0,1],\mathcal{B}([0,1]),\lambda)$ is also a Hilbert space. which has an orthonormal basis given by the family of functions given by $h_0 \equiv 1$, and

$$\forall n \geqslant 0, \quad \forall k = 0,\dots,2^n-1, \quad h_n^k = 2^{n/2}\mathbb{1}_{[2k/2^{n+1},(2k+1)/2^{n+1})} - 2^{n/2}\mathbb{1}_{[(2k+1)/2^{n+1},(2k+2)/2^{n+1})}.$$

Since both Hilbert spaces $L^2(\Omega,\mathcal{F},\mathbb{P})$ and $L^2([0,1],\mathcal{B}([0,1]),\lambda)$ are of infinite dimension, we can define a linear isometry between $L^2([0,1],\mathcal{B}([0,1]),\lambda)$ and a subspace of $L^2(\Omega,\mathcal{F},\mathbb{P})$. First, we reorder the elements of $(Z_n)_{n\geqslant 0}$ into $Z_0, (Z_n^k)_{n\geqslant 0, 0\leqslant k\leqslant 2^n-1}$ and we set $\Phi(h_0) = Z_0$, and $\Phi(h_n^k) = Z_n^k$ for all $n\geqslant 0$ and $0 \leqslant k \leqslant 2^n-1$. This extends naturally Φ on the whole space by setting

$$\Phi(f) = \Big(\int_0^1 f(t)h_0(t)\,\mathrm{d}t\Big)Z_0 + \sum_{n\geqslant 0}\sum_{0\leqslant k\leqslant 2^n-1}\Big(\int_0^1 f(t)h_n^k(t)\,\mathrm{d}t\Big)Z_n^k,$$

for all $f \in L^2([0,1],\mathcal{B}([0,1]),\lambda)$. We can also easily check that

$$\mathbb{E}[\Phi(f)\Phi(g)] = \|fg\|_2, \qquad \forall f,g \in L^2([0,1],\mathcal{B}([0,1]),\lambda).$$

Then, we define $B_t = \Phi(\mathbb{1}_{[0,t]})$ for $t\in[0,1]$. This definition allows us to check (BM1) easily for the

我們希望證明 $\sum_{n\geqslant 0} F_n$ 會在 $[0,1]$ 上均勻收斂，因此我們需要對 Z_d 做尾端估計。根據習題 2.14 ，我們知道對於 $c>1$ 及夠大的 n，下列不等式成立：

$$\mathbb{P}(|Z_d| \geqslant c\sqrt{n}) \leqslant \exp\big(-\frac{c^2 n}{2}\big).$$

也就是說，我們有

$$\sum_{n\geqslant 0} \mathbb{P}(\exists d \in \mathcal{D}_n : |Z_d| \geqslant c\sqrt{n}) \leqslant \sum_{n\geqslant 0}\sum_{d\in\mathcal{D}_n} \mathbb{P}(|Z_d| \geqslant c\sqrt{n}) \leqslant \sum_{n\geqslant 0} (2^n+1)\exp\big(-\frac{c^2 n}{2}\big),$$

其中右方只要 $c > \sqrt{2\ln 2}$ 時便會收斂，因此從 Borel–Cantelli 引理得知，隨在隨機變數 $N<\infty$ a.s. 使得對於所有 $n\geqslant N$ 及 $d\in\mathcal{D}_n$，我們有 $|Z_d| < c\sqrt{n}$，也就是說，對於所有 $n\geqslant N$，我們有

$$\|F_n\|_\infty < c\sqrt{n}2^{-(n+1)/2},$$

因此 $\sum_{n\geqslant 0} F_n$ 的確會在 $[0,1]$ 上均勻收斂至一（隨機）連續函數，我們將其定義做 B。

我們可以輕易檢查 $(B_t)_{t\in[0,1]}$ 在稠密集合 $\mathcal{D}$ 上滿足定義 9.1.3 中的性質 (1)，因此透過適當逼近，此性質也會在 $[0,1]$ 上成立；另外根據構造，性質 (2) 顯然成立。因此我們構造出了在時間 $[0,1]$ 上的布朗運動，若想要將時間拓展到 $\mathbb{R}_{\geqslant 0}$ 上，我們可以考慮 i.i.d. 的布朗運動序列 $(B^{(k)})_{k\geqslant 0}$，其中每個隨機變數 $B^{(k)}$ 皆是取值在 $C([0,1],\mathbb{R})$ 中的隨機函數，我們將他們全部黏接在一起，得到的隨機過程便是時間在 $\mathbb{R}_{\geqslant 0}$ 上的布朗運動。若要得到 d 維度的構造，我們可以考慮 d 個 i.i.d. 布朗運動作為分量，完成我們的構造。 □

註解 9.2.3 ： 這個構造方式可以使用比較代數的方法來理解。

(1) 我們可以考慮定義在一個機率空間 $L^2(\Omega,\mathcal{F},\mathbb{P})$ 上，可數無窮多個隨機變數 $(Z_n)_{n\geqslant 0}$ 構成的 i.i.d. $\mathcal{N}(0,1)$ 序列。機率空間 $L^2(\Omega,\mathcal{F},\mathbb{P})$ 是個希爾伯特空間，且隨機變數 $(Z_n)_{n\geqslant 0}$ 構成正規序列。

此外，$L^2([0,1],\mathcal{B}([0,1]),\lambda)$ 也是個希爾伯特空間，且 $h_0\equiv 1$ 與下列函數會構成他的正規基底：

$$\forall n \geqslant 0, \quad \forall k = 0,\dots,2^n-1, \quad h_n^k = 2^{n/2}\mathbb{1}_{[2k/2^{n+1},(2k+1)/2^{n+1})} - 2^{n/2}\mathbb{1}_{[(2k+1)/2^{n+1},(2k+2)/2^{n+1})}.$$

由於 $L^2(\Omega,\mathcal{F},\mathbb{P})$ 及 $L^2([0,1],\mathcal{B}([0,1]),\lambda)$ 兩個希爾伯特空間皆是無窮維度的，我們可以定義一個從 $L^2([0,1],\mathcal{B}([0,1]),\lambda)$ 出發，映射至 $L^2(\Omega,\mathcal{F},\mathbb{P})$ 子空間的線性等距變換。

首先，我們將 $(Z_n)_{n\geqslant 0}$ 中的元素重新排序為 $Z_0, (Z_n^k)_{n\geqslant 0, 0\leqslant k\leqslant 2^n-1}$，設 $\Phi(h_0) = Z_0$ 並對於所有 $n\geqslant 0$ 及 $0\leqslant k\leqslant 2^n-1$，設 $\Phi(h_n^k) = Z_n^k$。這讓我們將 Φ 的定義自然拓展到整個空間：對於所有 $f\in L^2([0,1],\mathcal{B}([0,1]),\lambda)$，我們有

$$\Phi(f) = \Big(\int_0^1 f(t)h_0(t)\,\mathrm{d}t\Big)Z_0 + \sum_{n\geqslant 0}\sum_{0\leqslant k\leqslant 2^n-1}\Big(\int_0^1 f(t)h_n^k(t)\,\mathrm{d}t\Big)Z_n^k.$$

且我們可以驗證：

$$\mathbb{E}[\Phi(f)\Phi(g)] = \|fg\|_2, \qquad \forall f,g \in L^2([0,1],\mathcal{B}([0,1]),\lambda).$$

random process $B := (B_t)_{t\geqslant 0}$. We may also check that the random process B agrees with the one given by Lévy constructed in Theorem 9.2.2, giving us the continuity property (BM2). Alternatively, we may also apply Exercise 9.4 to regularize the trajectories of B, giving the continuity as an easy consequence.

(2) The very first construction of Brownian motion was given by Norbert Wiener in 1923 in "Differential Space". It writes

$$B_t = Z_0 t + \sqrt{2}\sum_{n\geqslant 1} Z_n \frac{\sin(\pi n t)}{\pi n}, \qquad \forall t \in [0,1].$$

If we adapt the linear isometry perspective described above, this construction corresponds to another choice of the orthonormal basis on $L^2([0,1],\mathcal{B}([0,1]),\lambda)$, which is given by trigonometric functions,

$$\{1\} \cup \{t \mapsto \sqrt{2}\cos(\pi n t)\}_{n\geqslant 1}.$$

9.2.2 Wiener's Process

Let $\mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d)$ be the space of continuous functions from $\mathbb{R}_{\geqslant 0}$ to $\mathbb{R}^d$ equipped with the σ-algebra $\mathcal{C}$ which is the smallest σ-algebra making all the coordinate functions $w \mapsto w(t)$, for all $t \geqslant 0$, measurable.

Lemma 9.2.4 : *The σ-algebra $\mathcal{C}$ conincides with the Borel σ-algebra when $\mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d)$ is equipped with the topology of the uniform convergence on all compact sets.*

Proof : First, let us note that the topology of the uniform convergence on all compact sets is metrizable by the following distance,

$$d(w,w') = \sum_{n=1}^{\infty} 2^{-n} \sup_{0\leqslant t\leqslant n} (|w(t)-w'(t)| \wedge 1).$$

We write $\mathcal{B}$ for the Borel σ-algebra given by d.

On one hand, the coordinate functions $w \mapsto w(t)$ are clearly continuous from $(\mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d), d)$ to $\mathbb{R}^d$, so they are also $\mathcal{B}$-measurable, giving $\mathcal{C} \subseteq \mathcal{B}$.

On the other hand, the metric space $(\mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d), d)$ is separable (可分), so any open set of the space can be written as a countable union of open balls. In consequence, in order to show $\mathcal{B} \subseteq \mathcal{C}$, it is sufficient to prove that any open ball of $\mathcal{B}$ is in $\mathcal{C}$. That is, for given any $w_0 \in \mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d)$, we need to show that the function $w \mapsto d(w_0,w)$ is $\mathcal{C}$-measurable. If we rewrite sup in the definition of the distance d as follows,

$$\sup_{t\in[0,n]} (|w(t)-w_0(t)| \wedge 1) = \sup_{t\in[0,n]\cap\mathbb{Q}} (|w(t)-w_0(t)| \wedge 1)$$

then clearly we obtain the $\mathcal{C}$-measurability. □

再來，對於所有 $t \in [0,1]$ 定義 $B_t = \Phi(\mathbb{1}_{[0,t]})$。這個定義讓我們可以對隨機過程 $B := (B_t)_{t\geqslant 0}$ 輕易驗證性質 (BM1)。同時，我們也可以檢驗隨機過程 B 與定理 9.2.2 中 Lévy 的構造相同，因此我們得到連續性質 (BM2)。若不使用此方法，我們也可以利用習題 9.4 來探討隨機過程 B 的規律性，也會給出他的連續性。

(2) 第一個布朗運動的構造是在 1923 年由 Norbert Wiener 在「Differential Space」中給出，寫作

$$B_t = Z_0 t + \sqrt{2}\sum_{n\geqslant 1} Z_n \frac{\sin(\pi n t)}{\pi n}, \qquad \forall t \in [0,1].$$

若我們使用上述描述的線性等距變換的角度來理解，此構造對應於在 $L^2([0,1],\mathcal{B}([0,1]),\lambda)$ 上另一個正規基底的選擇，也就是由三角函數所構成的：

$$\{1\} \cup \{t \mapsto \sqrt{2}\cos(\pi n t)\}_{n\geqslant 1}.$$

第二小節 Wiener 過程

令 $\mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d)$ 為由 $\mathbb{R}_{\geqslant 0}$ 至 $\mathbb{R}^d$ 中的連續函數構成的空間。在此空間上，我們配置 σ 代數 $\mathcal{C}$ 定義為使所有座標函數 $w \mapsto w(t)$，對於所有 $t \geqslant 0$ 皆為可測的最小 σ 代數。

引理 9.2.4 ： σ 代數 $\mathcal{C}$ 與 $\mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d)$ 之上，由在所有緊緻集上均勻收斂的拓撲所給出的布雷爾 σ 代數相同。

證明：首先，我們知道由在所有緊緻集上均勻收斂的拓撲可以被下列距離所描述：

$$d(w,w') = \sum_{n=1}^{\infty} 2^{-n} \sup_{0\leqslant t\leqslant n} (|w(t)-w'(t)| \wedge 1).$$

我們將由 d 定義的布雷爾 σ 代數記作 $\mathcal{B}$。

一方面，我們注意到，由於座標函數 $w \mapsto w(t)$ 為由 $(\mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d), d)$ 映射至 $\mathbb{R}^d$ 的連續函數，所以會對伯雷爾 σ 代數 $\mathcal{B}$ 可測，因此我們會有 $\mathcal{C} \subseteq \mathcal{B}$。

另一方面，賦距空間 $(\mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d), d)$ 是可分 (separable) 的，因此此空間中任意開集皆可以寫作開球的可數聯集。因此若要證明 $\mathcal{B} \subseteq \mathcal{C}$，我們只需要證明任意 $\mathcal{B}$ 的開球皆會在 $\mathcal{C}$ 中，換句話說也就是對任意給定的 $w_0 \in \mathcal{C}(\mathbb{R}_{\geqslant 0},\mathbb{R}^d)$，函數 $w \mapsto d(w_0,w)$ 是 $\mathcal{C}$ 可測的。若我們將定義距離 d 的 sup 重新寫作：

$$\sup_{t\in[0,n]} (|w(t)-w_0(t)| \wedge 1) = \sup_{t\in[0,n]\cap\mathbb{Q}} (|w(t)-w_0(t)| \wedge 1)$$

則顯然我們有 $\mathcal{C}$ 可測的性質。 □

Definition 9.2.5 : Let $(B_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ started from 0. The d-dimensional *Wiener's measure* (Wiener 測度) is a probability measure on $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$, defined as the image measure of $\mathbb{P}$ under the following map

$$\begin{array}{rccc} \Phi: & \Omega & \longrightarrow & \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d) \\ & \omega & \mapsto & (B_t(\omega))_{t\geqslant 0} \end{array},$$

denoted $\mathbb{P}_0 := \Phi_* \mathbb{P} = \mathbb{P} \circ \Phi^{-1}$.

Remark 9.2.6 :

(1) By Lemma 7.2.2, we know that Φ is measurable, since the composition of Φ with any coordinate function $w \mapsto w(t)$ is $\omega \mapsto B_t(\omega)$ which is measurable.

(2) $\mathbb{P}_0$ is well-defined because it does not depend on the choice of the standard Brownian motion B. If we are given $0 = t_0 < t_1 < \cdots < t_p$, then for any Borel sets $A_0, A_1, \ldots, A_p$ of $\mathbb{R}^d$, we get from Eq. (9.3) that

$$\begin{aligned} & \mathbb{P}_0(\{w \in \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d) : w(t_0) \in A_0, \ldots, w(t_p) \in A_p\}) \\ & = \mathbb{P}(B_{t_0} \in A_0, \ldots, B_{t_p} \in A_p) \\ & = \mathbb{1}_{A_0}(0) \int_{A_1 \times \cdots \times A_p} \prod_{j=1}^{p} p_{t_j - t_{j-1}}(y_{t_j} - y_{t_{j-1}}) \prod_{j=1}^{p} \mathrm{d}y_j. \end{aligned} \tag{9.4}$$

Clearly, for any standard Brownian motion, the same formula above holds. Then the monotone class lemma states that a probability measure on $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ is characterized by the cylindrical events, which are events of the following type,

$$\{w \in \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d) : w(t_0) \in A_0, \ldots, w(t_p) \in A_p\}.$$

As a consequence, $\mathbb{P}_0$ is unique and is independent of the choice of the standard Brownian motion B. In other words, all the standard Brownian motions have the same distribution, which is Wiener's measure.

(3) Wiener's measure for $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ is like the Lebesgue measure for $[0, 1]$.

(4) If $x \in \mathbb{R}^d$, we can write $\mathbb{P}_x(\mathrm{d}w)$ for the image measure of $\mathbb{P}_0(\mathrm{d}w)$ under the translation $w \mapsto x + w$, which is the law of the standard Brownian motion started from x.

Below we describe how to construct the *canonical Brownian motion* (正則布朗運動). Consider the sample space $\Omega = \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ equipped with the σ-algebra $\mathcal{C}$ and the probability measure $\mathbb{P}_0$. We define

$$B_t(w) = w(t), \qquad \forall w \in \Omega, \quad \forall t \geqslant 0. \tag{9.5}$$

We can check that the random process $(B_t)_{t\geqslant 0}$ defined on the probability space $(\Omega, \mathcal{C}, \mathbb{P}_0)$ is a standard Brownian motion started at 0: the property (BM2) clearly holds and the property (BM1) can be obtained by the computations in Eq. (9.4). Similarly, under the probability measure $\mathbb{P}_x$, the random process $(B_t)_{t\geqslant 0}$ defined in Eq. (9.5) is a standard Brownian motion started from x.

定義 9.2.5 ： 令 $(B_t)_{t\geqslant 0}$ 為定義在機率空間 $(\Omega, \mathcal{F}, \mathbb{P})$ 上 d 維的標準布朗運動（出發點為 0）。我們將 d 維的<u>*Wiener* 測度</u> (Wiener's measure) 定義為 $\mathbb{P}$ 在下列函數下的影像測度

$$\begin{array}{rccc} \Phi: & \Omega & \longrightarrow & \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d) \\ & \omega & \mapsto & (B_t(\omega))_{t\geqslant 0} \end{array},$$

這會是個在 $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ 上的機率測度，且我們將他記作 $\mathbb{P}_0 := \Phi_* \mathbb{P} = \mathbb{P} \circ \Phi^{-1}$。

註解 9.2.6 ：

(1) 從引理 7.2.2 我們知道 Φ 是可測的，因為 Φ 與任意座標函數 $w \mapsto w(t)$ 的合成為 $\omega \mapsto B_t(\omega)$ 是個可測函數。

(2) $\mathbb{P}_0$ 是定義良好的，因為他並不取決於標準布朗運動 B 的選擇。若給定 $0 = t_0 < t_1 < \cdots < t_p$，則對於任意 $\mathbb{R}^d$ 的布雷爾集合 $A_0, A_1, \ldots, A_p$，從式 (9.3) 我們得到

$$\begin{aligned} & \mathbb{P}_0(\{w \in \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d) : w(t_0) \in A_0, \ldots, w(t_p) \in A_p\}) \\ & = \mathbb{P}(B_{t_0} \in A_0, \ldots, B_{t_p} \in A_p) \\ & = \mathbb{1}_{A_0}(0) \int_{A_1 \times \cdots \times A_p} \prod_{j=1}^{p} p_{t_j - t_{j-1}}(y_{t_j} - y_{t_{j-1}}) \prod_{j=1}^{p} \mathrm{d}y_j. \end{aligned} \tag{9.4}$$

顯然，對任意標準布朗運動，上式的結果皆相同。接著使用單調類引理，也就是說，在 $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ 上的機率測度是被他在圓柱事件上的機率所決定，也就是說下列的事件集合

$$\{w \in \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d) : w(t_0) \in A_0, \ldots, w(t_p) \in A_p\}.$$

因此這告訴我們，$\mathbb{P}_0$ 有唯一性，且不取決於我們考慮的標準布朗運動 B，換句話說，所有的標準布朗運動皆有相同的分佈，也就是 Wiener 測度。

(3) Wiener 測度之於 $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ 如同勒貝格測度之於 $[0, 1]$。

(4) 若 $x \in \mathbb{R}^d$，我們可以將 $\mathbb{P}_x(\mathrm{d}w)$ 記作 $\mathbb{P}_0(\mathrm{d}w)$ 在位移 $w \mapsto x + w$ 之下的影像測度，這會是由 x 出發的標準布朗運動的分佈。

我們以下列方式來構造<u>正則布朗運動</u> (canonical Brownian motion)。考慮樣本空間 $\Omega = \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ 且在上面配置 σ 代數 $\mathcal{C}$ 以及機率測度 $\mathbb{P}_0$。我們定義

$$B_t(w) = w(t), \qquad \forall w \in \Omega, \quad \forall t \geqslant 0. \tag{9.5}$$

我們可以檢查，定義在機率空間 $(\Omega, \mathcal{C}, \mathbb{P}_0)$ 上的隨機過程 $(B_t)_{t\geqslant 0}$ 是個由 0 出發的標準布朗運動：性質 (BM2) 顯然成立；性質 (BM1) 則是由式 (9.4) 中的計算所給出。相同的，在機率測度 $\mathbb{P}_x$ 之下，定義在式 (9.5) 中隨機過程 $(B_t)_{t\geqslant 0}$ 會是個由 x 出發的標準布朗運動。

9.2.3 Regularity of the Trajectory

Since the Brownian motion is a continuous function, it is uniformly continuous on all compact sets. We can consider the segment $[0,1]$ and discuss the regularity of the Brownian motion on this segment. We introduce first the notion of (global) *modulus of continuity*. A (random) continuous function φ is called a *modulus of continuity* (連續性尺度) if it satisfies

$$\lim_{h\downarrow 0}\varphi(h)=0 \qquad \text{and} \qquad \limsup_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1-h}\frac{|B_{t+h}-B_t|}{\varphi(h)}\leqslant 1.$$

The below theorem says that the Brownian motion has a non random modulus of continuity.

Theorem 9.2.7 : *There exists a constant $C>0$ such that almost surely, the following inequality holds*

$$|B_{t+h}-B_t|\leqslant C\sqrt{h\ln(1/h)},$$

for all small enough h and all $0\leqslant t\leqslant 1-h$.

Remark 9.2.8 :

(1) This theorem tells us indirectly that the trajectory of the Brownian motion is γ–Hölder continuous for all $\gamma<\frac{1}{2}$.

(2) This theorem gives the best order for the global modulus of continuity in the following sense,

$$\limsup_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1-h}\frac{|B_{t+h}-B_t|}{\sqrt{2h\ln(1/h)}}=1,$$

which is a result due to Lévy, called Lévy's (global) modulus of continuity theorem. We will not prove this in this lecture.

(3) In Exercise 9.10 we will see a result on the local modulus of continuity.

Proof : We keep the notations introduced during the construction of the Brownian motion in Theorem 9.2.2.

First note that each F_n is a piecewise linear function, so piecewise differentiable with derivative well-defined almost everywhere. Using a similar computation, we know that for any $c>\sqrt{2\ln 2}$, there exists a (random) positive integer N such that

$$\forall n\geqslant N, \qquad \|F_n'\|_\infty\leqslant\frac{2\,\|F_n\|_\infty}{2^{-n+1}}\leqslant c\sqrt{n}2^{(n-1)/2}.$$

Next, for any $t, t+h\in[0,1]$, consider positive integers $l>N$ and use the mean-value theorem (均值定理) to obtain

$$|B_{t+h}-B_t|\leqslant\sum_{n\geqslant 0}|F_n(t+h)-F_n(t)|$$

第三小節 軌跡的規律性

由於布朗運動是個連續函數，他在每個緊緻區間上會是均勻連續的。我們可以考慮線段 $[0,1]$ 並探討布朗運動在此線段上的規律性。首先，我們定義（全域）連續性尺度的概念：若（隨機）連續函數 φ 滿足

$$\lim_{h\downarrow 0}\varphi(h)=0 \qquad \text{以及} \qquad \limsup_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1-h}\frac{|B_{t+h}-B_t|}{\varphi(h)}\leqslant 1,$$

則稱 φ 為連續性尺度 (modulus of continuity)。

下面定理告訴我們，布朗運動有個非隨機的連續性尺度。

定理 9.2.7 ： 存在常數 $C>0$ 使得幾乎必然，對於所有夠小的 h 以及所有 $0\leqslant t\leqslant 1-h$，我們有不等式

$$|B_{t+h}-B_t|\leqslant C\sqrt{h\ln(1/h)}.$$

註解 9.2.8 ：

(1) 此定理間接告訴我們，對於所有的 $\gamma<\frac{1}{2}$，布朗運動的軌跡是 γ–Hölder 連續的。

(2) 此定理給出的全域連續性尺度數量級是最好的數量級，因為 Lévy 的（全域）連續性尺度定理告訴我們，下列等式幾乎必然成立：

$$\limsup_{h\downarrow 0}\sup_{0\leqslant t\leqslant 1-h}\frac{|B_{t+h}-B_t|}{\sqrt{2h\ln(1/h)}}=1.$$

我們在這門課程中不會證明此結果。

(3) 關於區域連續性尺度的結果，請參考習題 9.10 。

證明：我們使用在定理 9.2.2 中構造布朗運動時所引進的記號。

首先注意到，每個 F_n 皆是片段線性函數，因此片段可微，微分幾乎處處存在；根據相仿的計算，我們知道對任意 $c>\sqrt{2\ln 2}$，存在（隨機）正整數 N 使得

$$\forall n\geqslant N, \qquad \|F_n'\|_\infty\leqslant\frac{2\,\|F_n\|_\infty}{2^{-n+1}}\leqslant c\sqrt{n}2^{(n-1)/2}.$$

接著，對於任意的 $t, t+h\in[0,1]$，考慮正整數 $l>N$ 並使用均值定理 (mean-value theorem)：

$$\begin{aligned}|B_{t+h}-B_t|&\leqslant\sum_{n\geqslant 0}|F_n(t+h)-F_n(t)|\\&\leqslant\sum_{n=0}^{l}h\,\|F_n'\|_\infty+\sum_{n=l+1}^{\infty}2\,\|F_n\|_\infty\end{aligned}$$

$$\leqslant \sum_{n=0}^{l} h \|F_n'\|_\infty + \sum_{n=l+1}^{\infty} 2\|F_n\|_\infty$$
$$\leqslant \sum_{n=0}^{N-1} h \|F_n'\|_\infty + \sum_{n=N}^{l} ch\sqrt{n}2^{(n-1)/2} + \sum_{n=l+1}^{\infty} 2\|F_n\|_\infty .$$

In the above formula, we can take a small enough h so that the first term is bounded by $\sqrt{h\ln(1/h)}$; then we take $l > N$ so that $2^{-l} < h \leqslant 2^{-l+1}$, telling us the second and the thrid terms are both bounded by a constant times $\sqrt{h\ln(1/h)}$. The proof is complete. □

Theorem 9.2.9 : *Almost surely, the trajectory of the Brownian motion is not Lipschitz continuous at any point, hence not differentiable at any point.*

Proof : Fix a constant $M < \infty$ and let, for all $n \geqslant 1$,

$$A_n = \{\omega : \exists s \in [0,1], |t-s| \leqslant \tfrac{3}{n} \Rightarrow |B_t - B_s| \leqslant M|t-s|\}.$$

For all $0 \leqslant k \leqslant n-3$, define

$$Y_{k,n} = \max\left\{\left|B_{(k+j+1)/n} - B_{(k+j)/n}\right| : j = 0,1,2\right\},$$
$$C_n = \{\exists k : Y_{k,n} \leqslant \tfrac{5M}{n}\}.$$

From the definition, we can check that for any $\omega \in A_n$, pick up the corresponding $s = s(\omega) \in [0,1]$ and choose $0 \leqslant k \leqslant n-3$ so that $\frac{k}{n} \leqslant s < \frac{k+1}{n}$ (if $s \geqslant \frac{n-2}{n}$, we take $k = n-2$). The triangle inequality gives

$$|B_{(k+j+1)/n} - B_{(k+j)/n}| \leqslant |B_{(k+j+1)/n} - B_s| + |B_s - B_{(k+j)/n}| \leqslant \tfrac{5M}{n},$$

meaning that $\omega \in C_n$. Thus, we have $A_n \subseteq C_n$ and

$$\mathbb{P}(A_n) \leqslant \mathbb{P}(C_n) \leqslant n\ \mathbb{P}(|B_1| \leqslant \tfrac{5M}{\sqrt{n}})^3 \leqslant n(\tfrac{5M}{\sqrt{n}})^3 = n^{-1/2}(5M)^3.$$

Since $(A_n)_{n\geqslant 1}$ is a sequence of increasing events and the bound of the above probability tends to 0 when n goes to infinity, we know that $\mathbb{P}(A_n) = 0$ for all n. This implies $\mathbb{P}(A_n^c) = 1$ for all $n \geqslant 1$, that is, almost surely we have

$$\forall s \in [0,1], \qquad \limsup_{t\to s} \frac{|B_t - B_s|}{|t-s|} \geqslant M.$$

The above formula being true for all $M > 0$, we get the required property by the theorem. □

$$\leqslant \sum_{n=0}^{N-1} h \|F_n'\|_\infty + \sum_{n=N}^{l} ch\sqrt{n}2^{(n-1)/2} + \sum_{n=l+1}^{\infty} 2\|F_n\|_\infty .$$

在上式中，我們可以取夠小的 h 使得第一項會比 $\sqrt{h\ln(1/h)}$ 來得小；接著我們取 $l > N$ 使得 $2^{-l} < h \leqslant 2^{-l+1}$，這會告訴我們第二項和第三項皆會被常數乘上 $\sqrt{h\ln(1/h)}$ 所控制住，得證。□

定理 9.2.9 ： 幾乎必然，布朗運動的軌跡在任何點都不會是 Lipschitz 連續的，因此在任何點皆不可微分。

證明：固定常數 $M < \infty$，對於所有 $n \geqslant 1$，令

$$A_n = \{\omega : \exists s \in [0,1], |t-s| \leqslant \tfrac{3}{n} \Rightarrow |B_t - B_s| \leqslant M|t-s|\}.$$

對於所有 $0 \leqslant k \leqslant n-3$，定義

$$Y_{k,n} = \max\left\{\left|B_{(k+j+1)/n} - B_{(k+j)/n}\right| : j = 0,1,2\right\},$$
$$C_n = \{\exists k : Y_{k,n} \leqslant \tfrac{5M}{n}\}.$$

根據定義，我們可以驗證，對於任意 $\omega \in A_n$，選擇任意相對應的 $s = s(\omega) \in [0,1]$，再選擇 $0 \leqslant k \leqslant n-3$ 使得 $\frac{k}{n} \leqslant s < \frac{k+1}{n}$ （若 $s \geqslant \frac{n-2}{n}$，我們取 $k = n-2$） ，則根據三角不等式我們有

$$|B_{(k+j+1)/n} - B_{(k+j)/n}| \leqslant |B_{(k+j+1)/n} - B_s| + |B_s - B_{(k+j)/n}| \leqslant \tfrac{5M}{n},$$

也就是說 $\omega \in C_n$。因此我們得到 $A_n \subseteq C_n$。所以我們有

$$\mathbb{P}(A_n) \leqslant \mathbb{P}(C_n) \leqslant n\ \mathbb{P}(|B_1| \leqslant \tfrac{5M}{\sqrt{n}})^3 \leqslant n(\tfrac{5M}{\sqrt{n}})^3 = n^{-1/2}(5M)^3.$$

由於 $(A_n)_{n\geqslant 1}$ 是個遞增事件序列，且上式機率的上界在 n 趨近於無窮時會趨近於 0，因此我們得知對於所有的 n，我們皆有 $\mathbb{P}(A_n) = 0$。因此這告訴我們，對於所有 $n \geqslant 1$，我們有 $\mathbb{P}(A_n^c) = 1$，也就是說，下列性質幾乎必然成立：

$$\forall s \in [0,1], \qquad \limsup_{t\to s} \frac{|B_t - B_s|}{|t-s|} \geqslant M.$$

由於上式對於所有 $M > 0$ 成立，我們得到定理所描述的性質。□

Question 9.2.10: How to modify the proof of Theorem 9.2.9 to deduce the following properties?

(1) For any function $f : [0,1) \longrightarrow \mathbb{R}$, we define its upper right derivative (上右微分) and lower right derivative (下右微分) as

$$\forall t \in [0,1), \qquad D^* f(t) = \limsup_{h \downarrow 0} \frac{f(t+h) - f(t)}{h},$$
$$D_* f(t) = \liminf_{h \downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

Show that for the Brownian motion, almost surely for all $t \in [0,1)$, we have

$$D^* B_t = +\infty \qquad \text{or} \qquad D_* B_t = -\infty.$$

(2) Show that for any $k \geqslant 3$, for all $\gamma > \frac{1}{2} + \frac{1}{k}$, the trajectory of the Brownian motion is almost surely not γ–Hölder continuous.

Question 9.2.11: Let us define the following random set

$$\mathcal{H}_\gamma(\omega) = \{t \geqslant 0 : s \mapsto B_s(\omega) \text{ is } \gamma\text{–Hölder continuous at } t\}.$$

(1) Show that $\mathbb{P}(\mathcal{H}_\gamma = [0,\infty)) = 1$ for all $\gamma < \frac{1}{2}$.
(2) Show that $\mathbb{P}(\mathcal{H}_\gamma = \varnothing) = 1$ for all $\gamma > \frac{1}{2}$.
(3) Show that $\mathbb{P}(t \in \mathcal{H}_{1/2}) = 0$ for all $t \geqslant 0$.
(4) Burgess Davis proved in 1983 that $\mathbb{P}(\mathcal{H}_{1/2} \neq \varnothing) = 1$. Please explain why it is not contradictory to (3)?

問題 9.2.10：如何修改定理 9.2.9 的證明，進而推得下列性質。

(1) 對於任意函數 $f : [0,1) \longrightarrow \mathbb{R}$，我們定義上右微分 (upper right derivative) 及下右微分 (lower right derivative)：

$$\forall t \in [0,1), \qquad D^* f(t) = \limsup_{h \downarrow 0} \frac{f(t+h) - f(t)}{h},$$
$$D_* f(t) = \liminf_{h \downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

試證對布朗運動來說，幾乎必然，對於所有 $t \in [0,1)$，我們有

$$D^* B_t = +\infty \qquad \text{或} \qquad D_* B_t = -\infty.$$

(2) 證明對於任意 $k \geqslant 3$，對所有 $\gamma > \frac{1}{2} + \frac{1}{k}$，布朗運動的軌跡幾乎必然不會是 γ–Hölder 連續的。

問題 9.2.11：我們可以定義下列隨機集合

$$\mathcal{H}_\gamma(\omega) = \{t \geqslant 0 : s \mapsto B_s(\omega) \text{ 在 } t \text{ 是 } \gamma\text{–Hölder 連續的}\}.$$

(1) 證明對於所有 $\gamma < \frac{1}{2}$，我們有 $\mathbb{P}(\mathcal{H}_\gamma = [0,\infty)) = 1$。
(2) 證明對於所有 $\gamma > \frac{1}{2}$，我們有 $\mathbb{P}(\mathcal{H}_\gamma = \varnothing) = 1$。
(3) 證明對於所有 $t \geqslant 0$，我們有 $\mathbb{P}(t \in \mathcal{H}_{1/2}) = 0$。
(4) Burgess Davis 在 1983 年證明了 $\mathbb{P}(\mathcal{H}_{1/2} \neq \varnothing) = 1$，請解釋為何與 (3) 沒有矛盾？