9 Brownian Motion

9.1 Limit of Random Walks

In physics, we interprete the Brownian motion as the movement of a tiny particle under the shocks of the other particles. The trajectory of such a particle is quite irregular and highly unpredictable; moreover, the direction of the movement can change drastically at any time. From the point of view of mathematics, we can modelize this using the trajectory of a particle following a random walk on $\mathbb{Z}^d$ after a proper scaling and time-change.

More precisely speaking, let us consider a random walk $(S_n)_{n\geqslant 0}$ on $\mathbb{Z}^d$ started from 0, defined as below,

$$S_n = Y_1 + \cdots + Y_n, \qquad \forall n \geqslant 0,$$

where $(Y_n)_{n\geqslant 1}$ is an i.i.d. sequence of random variables with values in $\mathbb{Z}^d$ with distribution μ, Besides, suppose that μ satisfies the following properties.

(1) Centered and in L^2,

$$\sum_{x\in\mathbb{Z}^d} x\mu(x) = 0 \qquad \text{and} \qquad \sum_{x\in\mathbb{Z}^d} |x|^2\mu(x) < \infty.$$

(2) Isotropic: there exists $\sigma > 0$ such that for all $1 \leqslant i, j \leqslant d$, we have

$$\sum_{x\in\mathbb{Z}^d} x_i x_j \mu(x) = \sigma^2 \delta_{i,j}.$$

Theorem 4.5.4 tells us that the rescaled random walk $\frac{1}{\sqrt{n}}S_n$ converges in distribution to the d-dimensional Gaussian variable with covariance matrix given by $\sigma^2\mathrm{Id}$. It is not hard to check that the symmetric simple random walk on $\mathbb{Z}^d$ satisfies the above properties.

We are interested in the *global* behavior of this random walk both in time and in space, that is the behavior of the trajectory $n \mapsto S_n$ for large time. Hence, we introduce the following scaling,

$$S_t^{(n)} = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}, \qquad \forall t \geqslant 0, \quad \forall n \geqslant 1.$$

Alternatively speaking, the new time-scale t corresponds to the original random walk at time $\lfloor nt \rfloor$; besides, due to the central limit theorem, we know that $S_{\lfloor nt \rfloor}$ is of order $\sqrt{n}$ when n is large, explaining the division by $\sqrt{n}$ for a potentially non trivial limit.

Proposition 9.1.1 : *For any positive integer $p \geqslant 1$ and real numbers $0 = t_0 < t_1 < \cdots < t_p$, the following convergence holds,*

$$S_{t_1}^{(n)}, \ldots, S_{t_p}^{(n)} \xrightarrow{\mathcal{L}} (U_{t_1}, \ldots, U_{t_p}), \tag{9.1}$$

where the limiting distribution is characterized by the following properties,

(1) the sequence $(U_{t_j} - U_{t_{j-1}})_{1\leqslant j\leqslant p}$ of random variables is independent ($U_0 = 0$);

(2) for any $1 \leqslant j \leqslant p$, $U_{t_j} - U_{t_{j-1}}$ is a centered Gaussian vector whose covariance matrix writes $\sigma^2(t_j - t_{j-1})\mathrm{Id}$.

Remark 9.1.2 :

(1) If we define

$$p_a(x) = \frac{1}{(2\pi a)^{d/2}} \exp\Big(-\frac{|x|^2}{2a}\Big), \qquad x \in \mathbb{R}^d,$$

then the density function of $U_{t_j} - U_{t_{j-1}}$ is $p_{\sigma^2(t_j - t_{j-1})}(x)$. Therefore, the density function of the limiting distribution $(U_{t_1}, \ldots, U_{t_p})$ given above can be written as

$$f(y_1, \ldots, y_p) = \prod_{j=1}^{p} p_{\sigma^2(t_j - t_{j-1})}(y_j - y_{j-1}).$$

(2) The convergence in Eq. (9.1) is the finite-dimensional convergence in distribution of the random process $(S_t^{(n)})_{t \geqslant 0}$, which does not imply the convergence in distribution of the whole process. For instance, the below convergence cannot be seen as a consequence of the finite-dimension convergence in distribution,

$$\sup_{t \geqslant 0} S_t^{(n)} = \sup_{k \in \frac{1}{n}\mathbb{Z}_{\geqslant 0}} S_k^{(n)} \xrightarrow[n\to\infty]{?} \sup_{t \geqslant 0} U_t.$$

The main reason is that the regularity on the limiting path (the infinite-dimensional distribution) cannot be obtained from the knowledge of all the finite-dimensional distributions. Later in Section 9.2 we will construct the Brownian motion as a continuous path, and the Donsker's theorem (not discussed in this lecture) allows us to make sense of the convergence of random walk paths to the Brownian motion under a proper topological space.

Proof : By Lévy's continuity theorem (Theorem 4.4.15), we only need to prove that the following convergence holds for all $\xi_1, \ldots, \xi_p \in \mathbb{R}^d$,

$$\mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \xi_j \cdot S_{t_j}^{(n)}\Big)\Big] \xrightarrow[n\to\infty]{} \mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \xi_j \cdot U_{t_j}\Big)\Big].$$

In other words, we need the following convergence to hold for all $\eta_1, \ldots, \eta_p \in \mathbb{R}^d$,

$$\mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \eta_j \cdot (S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})\Big)\Big] \xrightarrow[n\to\infty]{} \mathbb{E}\Big[\exp\Big(\mathrm{i}\sum_{j=1}^{p} \eta_j \cdot (U_{t_j} - U_{t_{j-1}})\Big)\Big]. \tag{9.2}$$

Let us start with the computation of the above term on the left. First, we have

$$S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} = \frac{1}{\sqrt{n}} \sum_{k=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j \rfloor} Y_k,$$

that is, the sequence $(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})_{1 \leqslant j \leqslant p}$ of random variables is independent. Additionally, we have, for a fixed j,

$$S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)} \overset{(\mathrm{d})}{=} \frac{1}{\sqrt{n}} S_{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor} = \sqrt{\frac{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor}{n}} \frac{S_{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor}}{\sqrt{\lfloor nt_j \rfloor - \lfloor nt_{j-1} \rfloor}}.$$

The central limit theorem states that the above random variable converges in distribution to $\sqrt{t_j - t_{j-1}}N$, where N is a Gaussian vector with covariance matrix $\sigma^2\mathrm{Id}$. Hence, the below convergence holds for all $1 \leqslant j \leqslant p$,

$$\mathbb{E}\left[\exp\left(\mathrm{i}\,\eta_j \cdot (S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})\right)\right] \xrightarrow[n\to\infty]{} \mathbb{E}[\exp(\mathrm{i}\sqrt{t_j - t_{j-1}}\eta_j \cdot N)] = \exp\left(-\frac{\sigma^2|\eta_j|^2(t_j - t_{j-1})}{2}\right).$$

The last equality in the above formula is the Laplace transform of the Gaussian distribution, computed in Lemma 2.4.14. Finally, using the independence of $(S_{t_j}^{(n)} - S_{t_{j-1}}^{(n)})_{1\leqslant j\leqslant p}$, we can get Eq. (9.2) and conclude the proof. □

Definition 9.1.3 : Given a sequence $(B_t)_{t\geqslant 0}$ of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$. We say that $(B_t)_{t\geqslant 0}$ is a d-dimensional *standard Brownian motion* (標準布朗運動) started from 0 if the following properties are satisfied.

(BM1) $B_0 = 0$ a.s. and for all positive integer $p \geqslant 1$ and real numbers $0 = t_0 < t_1 < \cdots < t_p$, the sequence $(B_{t_j} - B_{t_{j-1}})_{1\leqslant j\leqslant p}$ of random variables is independent and for all $1 \leqslant j \leqslant p$, the Gaussian vector $B_{t_j} - B_{t_{j-1}}$ is centered with covariance matrix $(t_j - t_{j-1})\mathrm{Id}$.

(BM2) For all $\omega \in \Omega$, the function $t \mapsto B_t(\omega)$ is continuous.

Remark 9.1.4 :

(1) The most of the time, we will say Brownian motion instead of standard Brownian motion.

(2) In the below Section 9.2, we will construct a standard Brownian motion, giving its existence. In consequence, the statement in Proposition 9.1.1 can be reformulated as follows. For any $0 \leqslant t_1 < \cdots < t_p$, the following convergence in distribution holds,

$$(S_{t_1}^{(n)}, \ldots, S_{t_p}^{(n)}) \xrightarrow{(d)} (\sigma B_{t_1}, \ldots, \sigma B_{t_p}).$$

This tells us that a random walk convergence to a Brownian motion after a proper space-time scaling. From the point of view of physics, this can be interpreted as the macroscopic random trajectory arised from the microscopic shocks from particles.

(3) As we have seen above, the distribution of the random vector $(B_{t_1}, \ldots, B_{t_p})$ is characterized by

$$\mathbb{P}\left((B_{t_1}, \ldots, B_{t_p}) \in A\right) = \int_A \prod_{j=1}^{p} p_{t_j - t_{j-1}}(y_{t_j} - y_{t_{j-1}}) \prod_{j=1}^{p} \mathrm{d}y_j, \qquad \forall A \in \mathcal{B}((\mathbb{R}^d)^p) = \mathcal{B}(\mathbb{R}^d)^{\otimes p}. \tag{9.3}$$

9.2 Construction of Brownian Motion

In this section, we want to construct a Brownian motion as a random continuous function. More precisely, we need to construct an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we can define a random variable with values in $\mathcal{C}([0,1], \mathbb{R})$. Then, since the Brownian motion we want to define is a random process taking values in $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$, we need to glue i.i.d. copies of the random process constructed above together, then repeat the same operation on all the d coordinates to obtain the Brownian motion we looked for.

9.2.1 Lévy's Construction

The construction we present below is from Lévy. The idea is to start with an appropriate increasing sets $\mathcal{D}_n$ of $[0,1]$ such that $\cup \mathcal{D}_n$ is dense in $[0,1]$, then construct a random process with property (BM1) satisfied on $\mathcal{D}_n$. Next, we extend this random process by linear interpolation to points in $[0,1] \backslash \mathcal{D}_n$. In the end, we show that, when n tends to infinity, such a construction converges uniformly on $[0,1]$, so the limit satisfies the property (BM2) as well.

Before describing Lévy's construction in details, we start with the following lemma, used as the induction step in the construction, and is also the key idea of this method.

Lemma 9.2.1 : *Given S, T and Z three random varibles with normal distributions $\mathcal{N}(0,s)$, $\mathcal{N}(0,t)$ and $\mathcal{N}(0,1)$ and suppose that S, $T-S$ and Z are independent. Define*

$$U = \frac{S+T}{2} + \frac{\sqrt{t-s}}{2} Z.$$

Then, S, $U-S$ and $T-U$ are independent random variables with normal distributions $\mathcal{N}(0,s)$, $\mathcal{N}(0,u-s)$ and $\mathcal{N}(0,t-u)$, where $u = \frac{s+t}{2}$.

Proof : By definition, the covariance matrix of $(S, T-S, Z)$ writes

$$K_{(S,T-S,Z)} = \mathrm{Diag}(s, t-s, 1).$$

Besides, if we define the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{t-s}}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{t-s}}{2} \end{pmatrix},$$

then we have $(S, U-S, T-U)^T = A(S, T-S, Z)^T$. By Exercise 2.21, we can write

$$K_{(S,U-S,T-U)} = AK_{(S,T-S,Z)}A^T = \mathrm{Diag}(s, \tfrac{t-s}{2}, \tfrac{t-s}{2}).$$

Since $(S, U-S, T-U)$ is a Gaussian vector with diagonal covariance matrix, Proposition 3.4.1 tells us that its components are independent. □

Theorem 9.2.2 : *A standard Brownian motion exists, that is, we can find an appropriate probability space and a random process $(B_t)_{t \geqslant 0}$ such that the two properties in Definition 9.1.3 are satisfied.*

Proof : To begin with, let us define the following dyadic sets

$$\forall n \geqslant 0, \quad \mathcal{D}_n = \{\tfrac{k}{2^n} : 0 \leqslant k \leqslant 2^n\}.$$

Next, we will construct, for each n, a random continuous function on $[0,1]$ such that its marginal distributions satisfy the property required by Definition 9.1.3. And in the end, we will show that such random continuous functions will converge uniformly to a random continuous function, which is the Brownian motion we are looking for.

Write $\mathcal{D} = \bigcup_{n\geqslant 0} \mathcal{D}_n$ and consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we can define an i.i.d. sequence $(Z_d)_{d\in\mathcal{D}}$ of standard normal random varibales. We construct by induction the sequence $(B_d)_{d\in\mathcal{D}}$ of random processes satisfying

(a) for any $s < t$ in $\mathcal{D}_n$, the distribution of the random variable $B_t - B_s$ is $\mathcal{N}(0, t-s)$ and is independent of B_s;

(b) for any non-negative integer $n \geqslant 0$, the random process $(B_d)_{d\in\mathcal{D}_n}$ and $(Z_t)_{t\in\mathcal{D}\setminus D_n}$ are independent.

Our construction goes as follows.

(1) Define $B_0 = 0$ and $B_1 = Z_1$. The properties (a) and (b) clearly hold.

(2) Given $n \geqslant 1$ and assume that B_d for $d \in \mathcal{D}_{n-1}$ is well-defined and that the properties (a) and (b) are satified for any $d \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$. Next, we define

$$B_d = \frac{1}{2}(B_{d-2^{-n}} + B_{d+2^{-n}}) + \frac{Z_d}{2^{(n+1)/2}}.$$

We need to check that in the induction step (2), the properties (a) and (b) are still satisfied, which is a direct consequence of Lemma 9.2.1.

Now that we have defined a random process on dyadic points in $\mathcal{D}$, we interpolate its value outside of $\mathcal{D}$ linearly. To be more precise, let us define

$$F_0(t) = \begin{cases} 0 & t = 0, \\ Z_1 & t = 1, \end{cases}$$

and interpolate $F_0(t)$, for $t \in (0,1)$, linearly between $t = 0$ and $t = 1$. Next, we define

$$\forall n \geqslant 1, \quad F_n(t) = \begin{cases} 2^{-(n+1)/2} Z_t & t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}, \\ 0 & t \in \mathcal{D}_{n-1}, \end{cases}$$

and define $F_n(t)$, for $t \notin \mathcal{D}_n$, to be the linear interpolation between the two closent points from $\mathcal{D}_n$. All the above functions are continuous on $[0,1]$ and for all $n \geqslant 0$ and $d \in \mathcal{D}_n$, we have

$$B_d = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d).$$

We want to show that $\sum_{n\geqslant 0} F_n$ converges uniformly on $[0,1]$, meaning that we need to estimate the tail of Z_d. Exercise 2.14 states that for any $c > 1$ and large enough n, the following inequality holds,

$$\mathbb{P}(|Z_d| \geqslant c\sqrt{n}) \leqslant \exp\big(-\frac{c^2 n}{2}\big).$$

This implies that

$$\sum_{n\geqslant 0}\mathbb{P}(\exists d\in\mathcal{D}_n : |Z_d|\geqslant c\sqrt{n})\leqslant\sum_{n\geqslant 0}\sum_{d\in\mathcal{D}_n}\mathbb{P}(|Z_d|\geqslant c\sqrt{n})\leqslant\sum_{n\geqslant 0}(2^n+1)\exp\big(-\frac{c^2 n}{2}\big),$$

where the right side converge as long as $c>\sqrt{2\ln 2}$. Borel–Cantelli lemma then states that, there exists a random variable $N<\infty$ a.s. such that for all $n\geqslant N$ and $d\in\mathcal{D}_n$, we have $|Z_d|<c\sqrt{n}$, that is, we have for all $n\geqslant N$,

$$\|F_n\|_\infty < c\sqrt{n}2^{-(n+1)/2},$$

and in consequence, $\sum_{n\geqslant 0}F_n$ indeed converges uniformly on $[0,1]$ to a (random) continuous function, denoted B.

We can easily check that $(B_t)_{t\in[0,1]}$ satisfies the property (BM1) on the dense set $\mathcal{D}$, and by interpolation, this property is also satisfied on $[0,1]$. Also, by construction, the property (BM2) clearly holds. In conclusion, we have constructed a one-dimensional Brownian motion on the time interval $[0,1]$. If want to extend the time interval to $\mathbb{R}_{\geqslant 0}$, we may consider an i.i.d. sequence of Brownian motion constructed above $(B^{(k)})_{k\geqslant 0}$, where each random variable $B^{(k)}$ is a random function in $C([0,1],\mathbb{R})$. We may glue them together to obtain a Brownian motion defined on the time interval $\mathbb{R}_{\geqslant 0}$. To define a Brownian motion in dimension d, we take d i.i.d. components and we are done with the construction.
$\square$

Remark 9.2.3 : There is a more algebraic perspective of this construction.

(1) We may consider a countably infinite family $(Z_n)_{n\geqslant 0}$ of i.i.d. $\mathcal{N}(0,1)$ random variables on a probability space $L^2(\Omega,\mathcal{F},\mathbb{P})$. The probability space $L^2(\Omega,\mathcal{F},\mathbb{P})$ is a Hilbert space and the random variables $(Z_n)_{n\geqslant 0}$ form an orthonormal family on it.

Moreover, $L^2([0,1],\mathcal{B}([0,1]),\lambda)$ is also a Hilbert space. which has an orthonormal basis given by the family of functions given by $h_0\equiv 1$, and

$$\forall n\geqslant 0,\quad \forall k=0,\dots,2^n-1,\quad h_n^k=2^{n/2}\mathbb{1}_{[2k/2^{n+1},(2k+1)/2^{n+1})}-2^{n/2}\mathbb{1}_{[(2k+1)/2^{n+1},(2k+2)/2^{n+1})}.$$

Since both Hilbert spaces $L^2(\Omega,\mathcal{F},\mathbb{P})$ and $L^2([0,1],\mathcal{B}([0,1]),\lambda)$ are of infinite dimension, we can define a linear isometry between $L^2([0,1],\mathcal{B}([0,1]),\lambda)$ and a subspace of $L^2(\Omega,\mathcal{F},\mathbb{P})$. First, we reorder the elements of $(Z_n)_{n\geqslant 0}$ into $Z_0,(Z_n^k)_{n\geqslant 0,0\leqslant k\leqslant 2^n-1}$ and we set $\Phi(h_0)=Z_0$, and $\Phi(h_n^k)=Z_n^k$ for all $n\geqslant 0$ and $0\leqslant k\leqslant 2^n-1$. This extends naturally Φ on the whole space by setting

$$\Phi(f)=\Big(\int_0^1 f(t)h_0(t)\,\mathrm{d}t\Big)Z_0+\sum_{n\geqslant 0}\sum_{0\leqslant k\leqslant 2^n-1}\Big(\int_0^1 f(t)h_n^k(t)\,\mathrm{d}t\Big)Z_n^k,$$

for all $f\in L^2([0,1],\mathcal{B}([0,1]),\lambda)$. We can also easily check that

$$\mathbb{E}[\Phi(f)\Phi(g)]=\|fg\|_2\,,\qquad \forall f,g\in L^2([0,1],\mathcal{B}([0,1]),\lambda).$$

Then, we define $B_t=\Phi(\mathbb{1}_{[0,t]})$ for $t\in[0,1]$. This definition allows us to check (BM1) easily for the random process $B:=(B_t)_{t\geqslant 0}$. We may also check that the random process B agrees with the one given by Lévy constructed in Theorem 9.2.2, giving us the continuity property (BM2). Alternatively, we may also apply Exercise 9.4 to regularize the trajectories of B, giving the continuity as an easy consequence.

(2) The very first construction of Brownian motion was given by Norbert Wiener in 1923 in "Differential

Space”. It writes

$$B_t = Z_0 t + \sqrt{2}\sum_{n\geqslant 1} Z_n \frac{\sin(\pi n t)}{\pi n}, \qquad \forall t \in [0,1].$$

If we adapt the linear isometry perspective described above, this construction corresponds to another choice of the orthonormal basis on $L^2([0,1],\mathcal{B}([0,1]),\lambda)$, which is given by trigonometric functions,

$$\{1\} \cup \{t \mapsto \sqrt{2}\cos(\pi n t)\}_{n\geqslant 1}.$$

9.2.2 Wiener’s Process

Let $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ be the space of continuous functions from $\mathbb{R}_{\geqslant 0}$ to $\mathbb{R}^d$ equipped with the σ-algebra $\mathcal{C}$ which is the smallest σ-algebra making all the coordinate functions $w \mapsto w(t)$, for all $t \geqslant 0$, measurable.

Lemma 9.2.4 : *The σ-algebra $\mathcal{C}$ conincides with the Borel σ-algebra when $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ is equipped with the topology of the uniform convergence on all compact sets.*

Proof : First, let us note that the topology of the uniform convergence on all compact sets is metrizable by the following distance,

$$d(w,w') = \sum_{n=1}^{\infty} 2^{-n} \sup_{0\leqslant t\leqslant n} (|w(t) - w'(t)| \wedge 1).$$

We write $\mathcal{B}$ for the Borel σ-algebra given by d.

On one hand, the coordinate functions $w \mapsto w(t)$ are clearly continuous from $(\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d), d)$ to $\mathbb{R}^d$, so they are also $\mathcal{B}$-measurable, giving $\mathcal{C} \subseteq \mathcal{B}$.

On the other hand, the metric space $(\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d), d)$ is separable (可分), so any open set of the space can be written as a countable union of open balls. In consequence, in order to show $\mathcal{B} \subseteq \mathcal{C}$, it is sufficient to prove that any open ball of $\mathcal{B}$ is in $\mathcal{C}$. That is, for given any $w_0 \in \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$, we need to show that the function $w \mapsto d(w_0, w)$ is $\mathcal{C}$-measurable. If we rewrite sup in the definition of the distance d as follows,

$$\sup_{t\in[0,n]} (|w(t) - w_0(t)| \wedge 1) = \sup_{t\in[0,n]\cap\mathbb{Q}} (|w(t) - w_0(t)| \wedge 1)$$

then clearly we obtain the $\mathcal{C}$-measurability. □

Definition 9.2.5 : Let $(B_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ started from 0. The d-dimensional *Wiener’s measure* (Wiener 测度) is a probability measure on $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$, defined as the image measure of $\mathbb{P}$ under the following map

$$\begin{array}{cccc} \Phi: & \Omega & \longrightarrow & \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d) \\ & \omega & \mapsto & (B_t(\omega))_{t\geqslant 0} \end{array},$$

denoted $\mathbb{P}_0 := \Phi_* \mathbb{P} = \mathbb{P} \circ \Phi^{-1}$.

Remark 9.2.6 :

(1) By Lemma 7.2.2, we know that Φ is measurable, since the composition of Φ with any coordinate function $w \mapsto w(t)$ is $\omega \mapsto B_t(\omega)$ which is measurable.

(2) $\mathbb{P}_0$ is well-defined because it does not depend on the choice of the standard Brownian motion B. If we are given $0 = t_0 < t_1 < \cdots < t_p$, then for any Borel sets $A_0, A_1, \ldots, A_p$ of $\mathbb{R}^d$, we get from Eq. (9.3) that

$$\begin{aligned} &\mathbb{P}_0(\{w \in \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d) : w(t_0) \in A_0, \ldots, w(t_p) \in A_p\}) \\ &= \mathbb{P}(B_{t_0} \in A_0, \ldots, B_{t_p} \in A_p) \\ &= \mathbb{1}_{A_0}(0) \int_{A_1 \times \cdots \times A_p} \prod_{j=1}^{p} p_{t_j - t_{j-1}}(y_{t_j} - y_{t_{j-1}}) \prod_{j=1}^{p} \mathrm{d}y_j. \end{aligned} \tag{9.4}$$

Clearly, for any standard Brownian motion, the same formula above holds. Then the monotone class lemma states that a probability measure on $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ is characterized by the cylindrical events, which are events of the following type,

$$\{w \in \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d) : w(t_0) \in A_0, \ldots, w(t_p) \in A_p\}.$$

As a consequence, $\mathbb{P}_0$ is unique and is independent of the choice of the standard Brownian motion B. In other words, all the standard Brownian motions have the same distribution, which is Wiener's measure.

(3) Wiener's measure for $\mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ is like the Lebesgue measure for $[0, 1]$.

(4) If $x \in \mathbb{R}^d$, we can write $\mathbb{P}_x(\mathrm{d}w)$ for the image measure of $\mathbb{P}_0(\mathrm{d}w)$ under the translation $w \mapsto x + w$, which is the law of the standard Brownian motion started from x.

Below we describe how to construct the *canonical Brownian motion* (正則布朗運動). Consider the sample space $\Omega = \mathcal{C}(\mathbb{R}_{\geqslant 0}, \mathbb{R}^d)$ equipped with the σ-algebra $\mathcal{C}$ and the probability measure $\mathbb{P}_0$. We define

$$B_t(w) = w(t), \qquad \forall w \in \Omega, \quad \forall t \geqslant 0. \tag{9.5}$$

We can check that the random process $(B_t)_{t \geqslant 0}$ defined on the probability space $(\Omega, \mathcal{C}, \mathbb{P}_0)$ is a standard Brownian motion started at 0: the property (BM2) clearly holds and the property (BM1) can be obtained by the computations in Eq. (9.4). Similarly, under the probability measure $\mathbb{P}_x$, the random process $(B_t)_{t \geqslant 0}$ defined in Eq. (9.5) is a standard Brownian motion started from x.

9.2.3 Regularity of the Trajectory

Since the Brownian motion is a continuous function, it is uniformly continuous on all compact sets. We can consider the segment $[0, 1]$ and discuss the regularity of the Brownian motion on this segment. We introduce first the notion of (global) *modulus of continuity*. A (random) continuous function φ is called a *modulus of continuity* (連續性尺度) if it satisfies

$$\lim_{h \downarrow 0} \varphi(h) = 0 \qquad \text{and} \qquad \limsup_{h \downarrow 0} \sup_{0 \leqslant t \leqslant 1-h} \frac{|B_{t+h} - B_t|}{\varphi(h)} \leqslant 1.$$

The below theorem says that the Brownian motion has a non random modulus of continuity.

Theorem 9.2.7 : *There exists a constant $C > 0$ such that almost surely, the following inequality holds*

$$|B_{t+h} - B_t| \leqslant C\sqrt{h\ln(1/h)},$$

for all small enough h and all $0 \leqslant t \leqslant 1 - h$.

Remark 9.2.8 :

(1) This theorem tells us indirectly that the trajectory of the Brownian motion is γ–Hölder continuous for all $\gamma < \frac{1}{2}$.

(2) This theorem gives the best order for the global modulus of continuity in the following sense,

$$\limsup_{h\downarrow 0} \sup_{0\leqslant t\leqslant 1-h} \frac{|B_{t+h} - B_t|}{\sqrt{2h\ln(1/h)}} = 1,$$

which is a result due to Lévy, called Lévy's (global) modulus of continuity theorem. We will not prove this in this lecture.

(3) In Exercise 9.10 we will see a result on the local modulus of continuity.

Proof : We keep the notations introduced during the construction of the Brownian motion in Theorem 9.2.2.

First note that each F_n is a piecewise linear function, so piecewise differentiable with derivative well-defined almost everywhere. Using a similar computation, we know that for any $c > \sqrt{2\ln 2}$, there exists a (random) positive integer N such that

$$\forall n \geqslant N, \qquad \|F_n'\|_\infty \leqslant \frac{2\|F_n\|_\infty}{2^{-n+1}} \leqslant c\sqrt{n}2^{(n-1)/2}.$$

Next, for any $t, t+h \in [0,1]$, consider positive integers $l > N$ and use the mean-value theorem (均值定理) to obtain

$$\begin{aligned} |B_{t+h} - B_t| &\leqslant \sum_{n\geqslant 0} |F_n(t+h) - F_n(t)| \\ &\leqslant \sum_{n=0}^{l} h\|F_n'\|_\infty + \sum_{n=l+1}^{\infty} 2\|F_n\|_\infty \\ &\leqslant \sum_{n=0}^{N-1} h\|F_n'\|_\infty + \sum_{n=N}^{l} ch\sqrt{n}2^{(n-1)/2} + \sum_{n=l+1}^{\infty} 2\|F_n\|_\infty . \end{aligned}$$

In the above formula, we can take a small enough h so that the first term is bounded by $\sqrt{h\ln(1/h)}$; then we take $l > N$ so that $2^{-l} < h \leqslant 2^{-l+1}$, telling us the second and the thrid terms are both bounded by a constant times $\sqrt{h\ln(1/h)}$. The proof is complete. □

Theorem 9.2.9 : *Almost surely, the trajectory of the Brownian motion is not Lipschitz continuous at any point, hence not differentiable at any point.*

Proof : Fix a constant $M < \infty$ and let, for all $n \geqslant 1$,

$$A_n = \{\omega : \exists s \in [0,1], |t-s| \leqslant \tfrac{3}{n} \Rightarrow |B_t - B_s| \leqslant M|t-s|\}.$$

For all $0 \leqslant k \leqslant n-3$, define

$$Y_{k,n} = \max\left\{\left|B_{(k+j+1)/n} - B_{(k+j)/n}\right| : j = 0,1,2\right\},$$
$$C_n = \{\exists k : Y_{k,n} \leqslant \tfrac{5M}{n}\}.$$

From the definition, we can check that for any $\omega \in A_n$, pick up the corresponding $s = s(\omega) \in [0,1]$ and choose $0 \leqslant k \leqslant n-3$ so that $\frac{k}{n} \leqslant s < \frac{k+1}{n}$ (if $s \geqslant \frac{n-2}{n}$, we take $k = n-2$). The triangle inequality gives

$$|B_{(k+j+1)/n} - B_{(k+j)/n}| \leqslant |B_{(k+j+1)/n} - B_s| + |B_s - B_{(k+j)/n}| \leqslant \tfrac{5M}{n},$$

meaning that $\omega \in C_n$. Thus, we have $A_n \subseteq C_n$ and

$$\mathbb{P}(A_n) \leqslant \mathbb{P}(C_n) \leqslant n\ \mathbb{P}(|B_1| \leqslant \tfrac{5M}{\sqrt{n}})^3 \leqslant n(\tfrac{5M}{\sqrt{n}})^3 = n^{-1/2}(5M)^3.$$

Since $(A_n)_{n\geqslant 1}$ is a sequence of increasing events and the bound of the above probability tends to 0 when n goes to infinity, we know that $\mathbb{P}(A_n) = 0$ for all n. This implies $\mathbb{P}(A_n^c) = 1$ for all $n \geqslant 1$, that is, almost surely we have

$$\forall s \in [0,1], \qquad \limsup_{t\to s} \frac{|B_t - B_s|}{|t-s|} \geqslant M.$$

The above formula being true for all $M > 0$, we get the required property by the theorem. □

Question 9.2.10: How to modify the proof of Theorem 9.2.9 to deduce the following properties?

(1) For any function $f : [0,1) \longrightarrow \mathbb{R}$, we define its upper right derivative (上右微分) and lower right derivative (下右微分) as

$$\forall t \in [0,1), \qquad D^* f(t) = \limsup_{h\downarrow 0} \frac{f(t+h) - f(t)}{h},$$
$$D_* f(t) = \liminf_{h\downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

Show that for the Brownian motion, almost surely for all $t \in [0,1)$, we have

$$D^* B_t = +\infty \qquad \text{or} \qquad D_* B_t = -\infty.$$

(2) Show that for any $k \geqslant 3$, for all $\gamma > \frac{1}{2} + \frac{1}{k}$, the trajectory of the Brownian motion is almost surely not γ–Hölder continuous.

Question 9.2.11: Let us define the following random set

$$\mathcal{H}_\gamma(\omega) = \{t \geqslant 0 : s \mapsto B_s(\omega) \text{ is } \gamma\text{–Hölder continuous at } t\}.$$

(1) Show that $\mathbb{P}(\mathcal{H}_\gamma = [0, \infty)) = 1$ for all $\gamma < \frac{1}{2}$.

(2) Show that $\mathbb{P}(\mathcal{H}_\gamma = \varnothing) = 1$ for all $\gamma > \frac{1}{2}$.

(3) Show that $\mathbb{P}(t \in \mathcal{H}_{1/2}) = 0$ for all $t \geqslant 0$.

(4) Burgess Davis proved in 1983 that $\mathbb{P}(\mathcal{H}_{1/2} \neq \varnothing) = 1$. Please explain why it is not contradictory to (3)?