Exercise 2.1 : Describe the sample space in different random experiments.

(1) Toss a coin with two sides denoted head and tail, and stop when tail appeaers for the first time.

(2) In a set of cards with four colors and 13 numbers, randomly pick up five cards (the order does not matter).

(3) In the unit disk, random trajectories that start from the origin and stop when they hit the boundary of the disk.

We assume that we have a fair coin in (1), and the cards are picked up randomly and uniformly in (2), describe the corresponding probability measure.

Exercise 2.2 (Bertrand's Paradox) **:** The French mathematician Joseph Bertrand (1822-1990) asked a question in his book "Probability Computations" (Calculs des probabilités) published in 1889: given a equilateral triangle and its circumscribed circle, what is the probability that a *randomly chosen* chord on the circle is longer than a side of the equilateral triangle? At the same time, he suggested three methods to compute, while obtaining three different results.

(1) The two endpoints of the chord are chosen uniformly at random on the circle.

(2) Choose a point inside the disk uniformly at random then draw the chord going through this point such that the midpoint of the chord is the randomly chosen point.

(3) Choose a point uniformly at random on the circle, draw the radius going through this point, choose a point uniformly at random on the radius and draw the chord which is perpendicular to the radius and going through this point.

In the above three different random experiments, give

(a) the probability space;

(b) the random variable giving the chord length and its density function;

(c) the probability that an chord randomly chosen as such is longer than a side of the equilateral triangle.

習題 2.1 ： 請描述不同機率試驗中的樣本空間。

(1) 投擲一顆正反兩面硬幣，直到出現反面為止。

(2) 在一副四種花色、十三個數字的撲克牌組中，抽出五張牌（不考慮順序）。

(3) 在單為圓盤中，從原點出發、且在碰到圓盤邊界便停止的隨機運動軌跡。

在 (1) 中，假設硬幣為公正，及在 (2) 中，假設撲克牌為均勻抽排，描述所相對應的機率測度。

習題 2.2 【Bertrand 悖論】**：** 法國數學家 Joseph Bertrand (1822-1900) 在 1889 出版的「機率計算」（Calcul des probabilités）一書之中，提出了下列問題：給定一個正三角形及其外接圓，隨機取圓上一弦，試問此弦長度大於正三角形的邊長為何？Bertrand 同時提出了三種計算方式，卻得到三個不同答案。

(1) 弦的兩個端點為圓上均勻隨機選擇的兩個點。

(2) 在圓內均勻隨機選擇一點，畫出過此點的弦，使得弦的中點為此隨機點。

(3) 事先在圓上均勻隨機選擇一點，畫出過此點及圓心的半徑，在此半徑上均勻隨機選擇一點，並畫出過此點且與此半徑垂直的弦。

請給出在以上三種不同狀況（隨機試驗）下的

(a) 機率空間；

(b) 描述弦長的隨機變數及其密度函數；

(c) 弦長大於正三角形邊長的機率。

Exercise 2.3 : A probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is called atomless (無原子的) if for any $A \in \mathcal{A}$ with $\mathbb{P}(A) > 0$, we can find $B \in \mathcal{A}$ such that $B \subseteq A$ and $0 < \mathbb{P}(B) < \mathbb{P}(A)$. Otherwise, we say that $(\Omega, \mathcal{A}, \mathbb{P})$ is atomic (有原子的).

(1) Suppose that the measurable space $(\Omega, \mathcal{A})$ is given by $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Prove that $(\Omega, \mathcal{A}, \mathbb{P})$ is an atomless probability space if and only if $\mathbb{P}$ is an atomless probability measure. (We recall Definition 1.1.12.)

(2) Give respectively an example of an atomless probability space and an atomic probability space.

Given an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and fix $A \in \mathcal{A}$ with $\mathbb{P}(A) > 0$.

(3) Show that for any $\varepsilon > 0$, we can find $B \in \mathcal{A}$ and $B \subseteq A$ with $0 < \mathbb{P}(B) < \varepsilon$.

(4) Show that if $0 < a < \mathbb{P}(A)$, then there exists $B \in \mathcal{A}$ and $B \subseteq A$ with $\mathbb{P}(B) = a$.

Exercise 2.4 : Fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Take $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ satisfying the following conditions,

(a) $\varnothing \in \mathcal{C}$;

(b) if $A \in \mathcal{C}$, then $A^c \in \mathcal{C}$;

(c) if $A, B \in \mathcal{C}$, then $A \cap B \in \mathcal{C}$.

We call such a $\mathcal{C}$ an algebra of sets (集合代數). Additionally, we also assume that $\mathcal{A} := \sigma(\mathcal{C})$.

(1) Prove that for any $B \in \mathcal{A}$, we have $\inf\{\mathbb{P}(A \Delta B) : A \in \mathcal{C}\} = 0$.

(2) Given a bounded random variable X and $\varepsilon > 0$, show that there exists a simple function $Y = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k}$, where $A_k \in \mathcal{C}$, such that $\mathbb{P}(|X - Y| > \varepsilon) < \varepsilon$.

Exercise 2.5 : Let X, Y and Z be real random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

(1) Assume that X and Y are equal almost everywhere (with respect to the probability measure $\mathbb{P}$). Prove that X and Y have the same distribution. Does the converse hold?

(2) Assume that X and Y have the same distribution.
   (1) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Borel function. Prove that $f(X)$ and $f(Y)$ have the same distribution.
   (2) Prove that XZ and YZ does not necessarily have the same distribution.

Exercise 2.6 : Let $X \sim \text{Unif}([0, 1])$ be a uniform random variable on $[0, 1]$.

(1) Let $Y = -\ln(1 - X)$ and find the distribution of Y.

(2) Let $Z = \tan(\pi X - \frac{\pi}{2})$ and find the distribution of Z.

習題 2.3 ： 在機率空間 $(\Omega, \mathcal{A}, \mathbb{P})$ 中，若對於任意 $A \in \mathcal{A}$ 滿足 $\mathbb{P}(A) > 0$，我們可以找到 $B \in \mathcal{A}$，使得 $B \subseteq A$ 且 $0 < \mathbb{P}(B) < \mathbb{P}(A)$，則我們說 $(\Omega, \mathcal{A}, \mathbb{P})$ 是無原子的 (atomless)；反之，則說 $(\Omega, \mathcal{A}, \mathbb{P})$ 是有原子的 (atomic)。

(1) 這裡我們假設可測空間 $(\Omega, \mathcal{A})$ 為 $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$。證明若且唯若 $\mathbb{P}$ 是無原子機率測度，則 $(\Omega, \mathcal{A}, \mathbb{P})$ 也是無原子機率空間。（可以參考定義 1.1.12）

(2) 分別給出無原子機率空間及有原子機率空間的例子。

給定無原子機率空間 $(\Omega, \mathcal{A}, \mathbb{P})$，並固定 $A \in \mathcal{A}$ 滿足 $\mathbb{P}(A) > 0$。

(3) 證明對於所有 $\varepsilon > 0$，我們可以找到 $B \in \mathcal{A}$ 且 $B \subseteq A$ 使得 $0 < \mathbb{P}(B) < \varepsilon$。

(4) 證明若 $0 < a < \mathbb{P}(A)$，則存在 $B \in \mathcal{A}$ 且 $B \subseteq A$ 使得 $\mathbb{P}(B) = a$。

習題 2.4 ： 固定機率空間 $(\Omega, \mathcal{A}, \mathbb{P})$。考慮 $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ 滿足下列條件：

(a) $\varnothing \in \mathcal{C}$；

(b) 若 $A \in \mathcal{C}$，則 $A^c \in \mathcal{C}$；

(c) 若 $A, B \in \mathcal{C}$，則 $A \cap B \in \mathcal{C}$。

我們稱這樣的 $\mathcal{C}$ 為集合代數 (algebra of sets)。此外，我們也假設 $\mathcal{A} := \sigma(\mathcal{C})$。

(1) 證明對於任何 $B \in \mathcal{A}$，我們有 $\inf\{\mathbb{P}(A \Delta B) : A \in \mathcal{C}\} = 0$。

(2) 給定有界的隨機變數 X 及 $\varepsilon > 0$，證明存在簡單函數 $Y = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k}$，其中 $A_k \in \mathcal{C}$，使得 $\mathbb{P}(|X - Y| > \varepsilon) < \varepsilon$。

習題 2.5 ： 令 X, Y 及 Z 為定義在機率空間 $(\Omega, \mathcal{A}, \mathbb{P})$ 上的實隨機變數。

(1) 假設 X 與 Y （相對於機率測度 $\mathbb{P}$）幾乎處處相等，證明 X 與 Y 有相同的分佈。試問其逆命題是否成立？

(2) 假設 X 與 Y 有相同的分佈。
   (a) 令 $f : \mathbb{R} \longrightarrow \mathbb{R}$ 為伯雷爾函數，證明 $f(X)$ 與 $f(Y)$ 有著相同的分佈。
   (b) 證明 XZ 與 YZ 未必有相同的分佈。

習題 2.6 ： 令 $X \sim \text{Unif}([0, 1])$ 為 $[0, 1]$ 上的均勻隨機變數。

(1) 令 $Y = -\ln(1 - X)$ 並求 Y 的分佈。

(2) 令 $Z = \tan(\pi X - \frac{\pi}{2})$ 並求 Z 的分佈。

Exercise 2.7 (Question 2.1.21) **:** Construct two bi-dimensional real random variables $X = (X_1, X_2)$ and $X' = (X_1', X_2')$ such that for $j = 1, 2$, X_j and X_j' have the same marginal distribution, where as the distributions of X and X' are not equal.

Exercise 2.8 : We are given a bidimensional real random variable (X, Y) on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Use the method mentioned in Remark 2.1.16 to find the distribution of the following random variables.

(1) Assume that (X, Y) has distribution
$$\lambda\mu e^{-\lambda x - \mu y} \mathbb{1}_{\mathbb{R}_+^2}(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$
Determine the distribution of the random variable $U = \min(X, Y)$.

(2) Assume that (X, Y) has distribution
$$\frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\,\mathrm{d}x\,\mathrm{d}y.$$
Determine the distribution of the random variable $\frac{X}{Y}$.

Exercise 2.9 : Let X be a random variable with values in $[0, 1)$ such that its distribution $\mathbb{P}_X$ is the Lebesgue measure on $[0, 1)$. We define the sequence $(X_n)_{n\geqslant 1}$ of random variables as follows,
$$\begin{aligned} X_1 &:= \lfloor 2X \rfloor, \\ X_{n+1} &:= \lfloor 2^{n+1} X - \sum_{k=1}^{n} 2^{n+1-k} X_k \rfloor, \qquad \forall n \geqslant 1. \end{aligned}$$

(1) Check that almost surely all the X_n's take values in the set $\{0, 1\}$. In other words, show that
$$\mathbb{P}\left(\forall n \geqslant 1, X_n \in \{0, 1\}\right) = 1.$$
(Hint: show that the sequence $(X_n)_{n\geqslant 1}$ is the dyadic expansion (二進位展開) of X.)

(2) Describe the σ-algebra generated by X_1, which we denote by $\sigma(X_1)$. (Write down all the elements in this σ-algebra.)

(3) Describe the σ algebra generated by X_1 and X_2, which we denote by $\sigma(X_1, X_2)$. Is this the same as $\sigma(X_2)$, the σ-algebra generated only by X_2? You may want to write down explicitly the elements of these two σ-algebras.

(4) Fix a positive integer $n \geqslant 1$ and find the σ-algebra generated by $X_1, \ldots, X_n$ without writing down all of its elements. How many elements does this σ-algebra contain?

Exercise 2.10 : Compute the expectation and the variance of the probability distributions in Section 2.3.

習題 2.7 【問題 2.1.21】**：** 試著構造兩個二維的實隨機變數 $X = (X_1, X_2)$ 及 $X' = (X_1', X_2')$ 使得對於 $j = 1, 2$，X_j 及 X_j' 有著相同的邊緣分佈，但 X 及 X' 的分佈卻不相同。

習題 2.8 ： 給定機率空間 $(\Omega, \mathcal{A}, \mathbb{P})$ 以及定義在此空間上的二維實隨機變數 (X, Y)。使用註解 2.1.16 中提到的方式，來求隨機變數的分佈。

(1) 假設 (X, Y) 的分佈為
$$\lambda\mu e^{-\lambda x - \mu y} \mathbb{1}_{\mathbb{R}_+^2}(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$
求隨機變數 $U = \min(X, Y)$ 的分佈。

(2) 假設 (X, Y) 的分佈為
$$\frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\,\mathrm{d}x\,\mathrm{d}y.$$
求隨機變數 $\frac{X}{Y}$ 的分佈。

習題 2.9 ： 令 X 為取值在 $[0, 1)$ 上的隨機變數，其分佈 $\mathbb{P}_X$ 為在 $[0, 1)$ 上的勒貝格測度。我們以遞迴方式來定義隨機變數序列 $(X_n)_{n\geqslant 1}$：
$$\begin{aligned} X_1 &:= \lfloor 2X \rfloor, \\ X_{n+1} &:= \lfloor 2^{n+1} X - \sum_{k=1}^{n} 2^{n+1-k} X_k \rfloor, \qquad \forall n \geqslant 1. \end{aligned}$$

(1) 驗證幾乎必然所有的 X_n 取值皆在集合 $\{0, 1\}$ 中，也就是說
$$\mathbb{P}\left(\forall n \geqslant 1, X_n \in \{0, 1\}\right) = 1.$$
（提示：證明序列 $(X_n)_{n\geqslant 1}$ 為 X 的二進位展開 (dyadic expansion)。）

(2) 請描述由 X_1 生成的 σ 代數 $\sigma(X_1)$。（寫出 σ 代數中的所有元素）

(3) 請描述由 X_1 及 X_2 生成的 σ 代數 $\sigma(X_1, X_2)$。這與只由 X_2 生成的 σ 代數 $\sigma(X_2)$ 相同嗎？你可以嘗試把這兩個 σ 代數中的元素都寫下來。

(4) 固定正整數 $n \geqslant 1$，請找出由 $X_1, \ldots, X_n$ 生成的 σ 代數（不用寫出裡面所有的元素）。請問此 σ 代數中有多少個元素？

習題 2.10 ： 求第 2.3 節中不同隨機分佈的期望值及變異數。

Exercise 2.11 : Given $p \in (0,1)$ and if $X \sim \text{Geo}(p)$, then show that for any integers $m, n \geqslant 0$ we have,

$$\mathbb{P}(X \geqslant n) = \mathbb{P}(X \geqslant m+n \,|\, X \geqslant m).$$

We call it the memoryless property (無記憶性質) of the geometric distribution.

Exercise 2.12 : For any $0 \leqslant k \leqslant n$, show that we hvae

$$\lim_{\substack{N,K\to\infty \\ K/N\to p}} f_X(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

where f_X is the mass function of $X \sim \text{Hypergeo}(N,K,n)$. Please give an explanation to this result.

Exercise 2.13 : Consider a sequence $(p_n)_{n\geqslant 1}$ of nonnegative real numbers and a sequence $(X_n)_{n\geqslant 1}$, where X_n is a random variable following the binomial distribution $\text{Bin}(n, p_n)$ for all $n \geqslant 1$. Suppose that we have $np_n \to \lambda$ when n tends to infinity. Then, show that

$$\forall k \in \mathbb{N}_0, \qquad \lim_{n\to\infty} \mathbb{P}(X_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

Exercise 2.14 : Let X be a random variable with distribution $\mathcal{N}(\mu, \sigma^2)$. Prove the following inequality in two different ways,

$$\forall t > 0, \quad \mathbb{P}(X \geqslant \mu + t) \leqslant \max\Big(\frac{1}{\sqrt{2\pi}}\frac{\sigma}{t}, 1\Big) \exp\big(-\frac{t^2}{2\sigma^2}\big).$$

(1) Use integration by parts.

(2) Use the moment generating function $\mathbb{E}[e^{sX}]$ of X。

Exercise 2.15 : For any $u > 0$, the density function of the *Cauchy distribution* (柯西分佈) writes,

$$c_u(x) = \frac{1}{\pi}\frac{u}{u^2 + x^2}, \quad x \in \mathbb{R}.$$

(1) Check that for all $u > 0$, c_u defines a probability distribution.

(2) Prove that its expectation does not exist.

(3) Prove that $c_u * c_v = c_{u+v}$.

Exercise 2.16 : Let $Z \sim \mathcal{N}(0,1)$ and $X := Z^2$. Show that $X \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$.

習題 2.11 ： 給定任意 $p \in (0,1)$，若 $X \sim \text{Geo}(p)$，則對於所有整數 $m, n \geqslant 0$，證明

$$\mathbb{P}(X \geqslant n) = \mathbb{P}(X \geqslant m+n \,|\, X \geqslant m).$$

我們稱此為幾何分佈的無記憶性質 (memoryless property)。

習題 2.12 ： 對任何 $0 \leqslant k \leqslant n$，證明

$$\lim_{\substack{N,K\to\infty \\ K/N\to p}} f_X(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

其中 f_X 是 $X \sim \text{Hypergeo}(N,K,n)$ 的質量函數。請詮釋此結果。

習題 2.13 ： 考慮非負實數序列 $(p_n)_{n\geqslant 1}$ 及隨機變數序列 $(X_n)_{n\geqslant 1}$ 使得對於所有 $n \geqslant 1$，隨機變數 X_n 滿足二項式分佈 $\text{Bin}(n, p_n)$。假設當 n 趨近無窮大時，我們有 $np_n \to \lambda$，證明

$$\forall k \in \mathbb{N}_0, \qquad \lim_{n\to\infty} \mathbb{P}(X_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

習題 2.14 ： 令 X 為分佈為 $\mathcal{N}(\mu, \sigma^2)$ 的隨機變數。利用兩種方法證明下列不等式

$$\forall t > 0, \quad \mathbb{P}(X \geqslant \mu + t) \leqslant \max\Big(\frac{1}{\sqrt{2\pi}}\frac{\sigma}{t}, 1\Big) \exp\big(-\frac{t^2}{2\sigma^2}\big).$$

(1) 使用分部積分法。

(2) 使用 X 的動差生成函數 $\mathbb{E}[e^{sX}]$。

習題 2.15 ： 對於任意 $u > 0$，柯西分佈 (Cauchy distribution) 的密度函數寫作

$$c_u(x) = \frac{1}{\pi}\frac{u}{u^2 + x^2}, \quad x \in \mathbb{R}.$$

(1) 驗證對於所有 $u > 0$，c_u 定義一個機率分佈。

(2) 證明其期望值不存在。

(3) 證明 $c_u * c_v = c_{u+v}$。

習題 2.16 ： 令 $Z \sim \mathcal{N}(0,1)$ 及 $X := Z^2$，證明 $X \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$。

Exercise 2.17 : Let $X \sim \mathcal{N}(0,1)$ be a random variable with standard normal distribution.

(1) Find its moment generating function $M(t) := \mathbb{E}[e^{tX}]$.

(2) Prove by induction that for any nonnegative positive number $k \geqslant 0$, there exists a polynomial P_k satisfying

(a) $M^{(k)}(t) = P_k(t)e^{t^2/2}$;

(b) $\deg(P_k) = k$;

(c) P_k only contains odd degree terms when k is odd; P_k only contains even degree terms when k is even;

(d) $P_{k+1}(t) = P_k'(t) + tP_k(t)$.

(3) Prove that when k is odd, we have $\mathbb{E}[X^k] = 0$. (Use two different approaches: one using the moment generating function and the other one using integration.)

(4) Compute $\mathbb{E}[X^k]$ for all non-negative integer $k \geqslant 0$.

(5) Use a change of variables and properties of the Γ function to find the same result as in the previous question.

Exercise 2.18 (Estimates on Gaussian integrals) **:** Let f be the density function of the standard normal distribution, that is,

$$\forall x \in \mathbb{R}, \qquad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

We write F for the distribution function of f, i.e.,

$$\forall x \in \mathbb{R}, \qquad F(x) = \int_{-\infty}^{x} f(y)\,\mathrm{d}y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,\mathrm{d}y.$$

(1) Show that we have the following asymptotic behavior when $x \to +\infty$,

$$1 - F(x) \sim \frac{1}{x}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

(Hint: differentiate the above formula on both the left and the right hand sides.)

(2) Prove the following inequality for $x > 1$,

$$\Big(\frac{1}{x} - \frac{1}{x^3}\Big) f(x) < 1 - F(x) < \frac{f(x)}{x}.$$

(Hint: differentiate again the above formula on both the left and the right hand sides.)

(3) More generally, show that we have the following asymptotic formula for any integer $k \geqslant 0$,

$$\begin{aligned} 1 - F(x) &= f(x)\Big(\sum_{j=0}^{k} (-1)^j \frac{(2j-1)!!}{x^{2j+1}} + o\Big(\frac{1}{x^{2k+1}}\Big)\Big) \\ &= f(x)\Big(\sum_{j=0}^{k} \Big(-\frac{1}{2}\Big)^j \binom{2j}{j} j! + o\Big(\frac{1}{x^{2k+1}}\Big)\Big). \end{aligned}$$

習題 2.17 ： 令 $X \sim \mathcal{N}(0,1)$ 為標準常態分佈的隨機變數。

(1) 求他的動差生成函數 $M(t) := \mathbb{E}[e^{tX}]$。

(2) 使用數學歸納法，證明對於所有非負整數 $k \geqslant 0$，存在多項式 P_k 滿足下列條件：

(a) $M^{(k)}(t) = P_k(t)e^{t^2/2}$；

(b) $\deg(P_k) = k$；

(c) 當 k 為奇數時，P_k 只包含奇數次方的項；當 k 為偶數時，P_k 只包含偶數次方的項；

(d) $P_{k+1}(t) = P_k'(t) + tP_k(t)$。

(3) 證明當 k 為奇數時，$\mathbb{E}[X^k] = 0$。（請提出兩種方法：一種使用動差生成函數，一種使用積分。）

(4) 對所有非負整數 $k \geqslant 0$，計算 $\mathbb{E}[X^k]$。

(5) 使用變數變換及 Γ 函數的性質，得到與上題相同的結果。

習題 2.18 【高斯積分的估計】 **：** 令 f 為標準常態分佈的密度函數，也就是說

$$\forall x \in \mathbb{R}, \qquad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

我們把 f 的分佈函數記作 F，也就是說

$$\forall x \in \mathbb{R}, \qquad F(x) = \int_{-\infty}^{x} f(y)\,\mathrm{d}y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,\mathrm{d}y.$$

(1) 證明當 $x \to +\infty$ 時，我們有漸進行為

$$1 - F(x) \sim \frac{1}{x}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

（提示：考慮左右兩側的微分。）

(2) 當 $x > 1$ 時，證明下列不等式

$$\Big(\frac{1}{x} - \frac{1}{x^3}\Big) f(x) < 1 - F(x) < \frac{f(x)}{x}.$$

（提示：再次考慮左右兩側的微分。）

(3) 更一般來說，證明對於所有整數 $k \geqslant 0$，我們有漸進式

$$\begin{aligned} 1 - F(x) &= f(x)\Big(\sum_{j=0}^{k} (-1)^j \frac{(2j-1)!!}{x^{2j+1}} + o\Big(\frac{1}{x^{2k+1}}\Big)\Big) \\ &= f(x)\Big(\sum_{j=0}^{k} \Big(-\frac{1}{2}\Big)^j \binom{2j}{j} j! + o\Big(\frac{1}{x^{2k+1}}\Big)\Big). \end{aligned}$$

(4) Show that for all $a > 0$, we have the following limit when $x \to +\infty$,

$$\frac{1 - F\left(x + \frac{a}{x}\right)}{1 - F(x)} \to e^{-a}.$$

(5) Show that for all $b > 0$, we have the following limit when $x \to +\infty$,

$$\frac{1 - F(x + b)}{1 - F(x)} \to 0.$$

Exercise 2.19 : On the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we define the random variable $(X_1, \ldots, X_n)$ with values in $\mathbb{R}^n$ and law

$$\mathbb{1}_{[0,1]^n}(x_1, \ldots, x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

(1) For any permutation (排列) $\sigma \in \mathcal{S}_n$，let $A_\sigma = \{X_{\sigma_1} < \ldots < X_{\sigma_n}\}$. Construct the random variable $(Y_1, \ldots, Y_n)$ on $(\Omega, \mathcal{A}, \mathbb{P})$ using $(A_\sigma)_{\sigma \in \mathcal{S}_n}$, such that the following statement holds almost surely,

$$Y_1 < \ldots < Y_n \qquad \text{and} \qquad \{Y_1, \ldots, Y_n\} = \{X_1, \ldots, X_n\}.$$

(2) Determine the law of the two vectors of random variables $(Y_1, \ldots, Y_n)$ and $(\frac{Y_1}{Y_2}, \ldots, \frac{Y_{n-1}}{Y_n})$.

Exercise 2.20 (Question 2.4.9) **:** Prove that when X is a d-dimensional real random variable, its covariance matrix K_X is a positive semi-definite (半正定) symmetric matrix. In other words, prove that for all $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$, we have $\lambda K_X \lambda^T \geqslant 0$.

Exercise 2.21 (Question 2.4.10) **:** If A is a matrix of size $n \times d$ and X is a d-dimensional real random variable, define $Y = AX$ and prove that $K_Y = AK_XA^T$.

Exercise 2.22 : Consider the following density function (with respect to the Lebesgue measure),

$$p_0(y) = \frac{1}{\sqrt{2\pi y^2}} \exp\Big(-\frac{\ln(y)^2}{2}\Big), \quad y > 0.$$

Let Y be a random variable whose density is given by p_0.

(1) Let $X \sim \mathcal{N}(0, 1)$. Find a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y \overset{(\mathrm{d})}{=} f(X)$.

(2) Compute the n-th moment of Y for any non-negative integer $n \geqslant 0$.

(3) Let

$$q(y) = \sin(2\pi \ln(y)) p_0(y).$$

Compute $\int_{\mathbb{R}_{>0}} y^n q(y)\,\mathrm{d}y$ for all non-negative integer $n \geqslant 0$.

(4) For all $a \in \mathbb{R}$, let

$$p_a(y) = p_0(y) + aq(y).$$

Under which condition, p_a is the density function of a probability distribution?

(4) 證明對於所有 $a > 0$，在 $x \to +\infty$ 時，我們有下列極限

$$\frac{1 - F\left(x + \frac{a}{x}\right)}{1 - F(x)} \to e^{-a}.$$

(5) 證明對於所有 $b > 0$，在 $x \to +\infty$ 時，我們有下列極限

$$\frac{1 - F(x + b)}{1 - F(x)} \to 0.$$

習題 2.19 ： 在機率空間 $(\Omega, \mathcal{A}, \mathbb{P})$ 上，我們給定 n 維實隨機變數 $(X_1, \ldots, X_n)$ 使得其分佈寫作

$$\mathbb{1}_{[0,1]^n}(x_1, \ldots, x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

(1) 對於所有排列 (permutation) $\sigma \in \mathcal{S}_n$，令 $A_\sigma = \{X_{\sigma_1} < \ldots < X_{\sigma_n}\}$。利用 $(A_\sigma)_{\sigma \in \mathcal{S}_n}$ 構造在 $(\Omega, \mathcal{A}, \mathbb{P})$ 上的隨機變數 $(Y_1, \ldots, Y_n)$，使得下列敘述幾乎必然為真：

$$Y_1 < \ldots < Y_n \qquad \text{且} \qquad \{Y_1, \ldots, Y_n\} = \{X_1, \ldots, X_n\}.$$

(2) 給出下列兩隨機變數向量的分佈：$(Y_1, \ldots, Y_n)$ 以及 $(\frac{Y_1}{Y_2}, \ldots, \frac{Y_{n-1}}{Y_n})$。

習題 2.20 【問題 2.4.9】**：** 試證明當 X 是個 d 維的實隨機變數時，其共變異數矩陣 K_X 是個半正定 (positive semi-definite) 對稱矩陣，換句話說，證明對於所有的 $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$，我們有 $\lambda K_X \lambda^T \geqslant 0$。

習題 2.21 【問題 2.4.10】**：** 若 A 是個 $n \times d$ 維的矩陣，且 X 為 d 維實隨機變數，定義 $Y = AX$，試證明 $K_Y = AK_XA^T$。

習題 2.22 ： 考慮密度函數（對於勒貝格測度）如下的機率分佈：

$$p_0(y) = \frac{1}{\sqrt{2\pi y^2}} \exp\Big(-\frac{\ln(y)^2}{2}\Big), \quad y > 0.$$

令 Y 是個密度函數由 p_0 給定的隨機變數

(1) 令 $X \sim \mathcal{N}(0, 1)$。找出可測函數 $f : \mathbb{R} \to \mathbb{R}$ 使得 $Y \overset{(\mathrm{d})}{=} f(X)$。

(2) 對於所有非負整數 $n \geqslant 0$，計算 Y 的 n 階動差。

(3) 令

$$q(y) = \sin(2\pi \ln(y)) p_0(y).$$

對於所有非負整數 $n \geqslant 0$，計算 $\int_{\mathbb{R}_{>0}} y^n q(y)\,\mathrm{d}y$。

(4) 對於所有 $a \in \mathbb{R}$，令

$$p_a(y) = p_0(y) + aq(y).$$

(5) Which conclusion can we deduce from the above questions?

Exercise 2.23 : Given a real sequence $(\mu_k)_{k\geqslant 0}$ satisfying,

$$\limsup_{k\to\infty} \frac{\mu_{2k}^{1/2k}}{2k} = r < \infty.$$

(1) Assume that there exists a distribution function F such that its moments are given by $(\mu_k)_{k\geqslant 0}$. For all $k \geqslant 0$, let $\nu_k = \int |x|^k \,\mathrm{d}F(x)$. Prove that for all $k \geqslant 0$, we have $\nu_{2k+1}^2 \leqslant \mu_{2k}\mu_{2k+2}$ and deduce,

$$\limsup_{k\to\infty} \frac{\nu_k^{1/k}}{k} = r.$$

(2) Let X be a random variable with distribution function F. Write Ψ_X for its characteristic function. Prove that for all $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$, we have,

$$\left| e^{\mathrm{i}\,tx} - \sum_{k=0}^{n-1} \frac{(\mathrm{i}\,tx)^k}{k!} \right| \leqslant \frac{|tx|^n}{n!}.$$

Then, deduce,

$$\forall \xi \in \mathbb{R}, \qquad \left| \Psi_X(\xi+t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \Psi_X^{(k)}(\xi) \right| \leqslant \frac{|t|^n}{n!}\nu_n.$$

(3) Given $\xi \in \mathbb{R}$, how to choose t so that the following equality holds?

$$\Psi_X(\xi+t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Psi_X^{(k)}(\xi).$$

(4) Explain what we have shown here. See Exercise 2.22 and conclude.

Exercise 2.24 (Methods of moments) **:** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

(1) Consider a finite set I and measurable events $A_i \in \mathcal{A}$ for $i \in I$. Show that the following inequality holds,

$$\mathbb{P}\Big(\bigcup_{i\in I} A_i\Big) \leqslant \sum_{i\in I} \mathbb{P}(A_i).$$

(2) Let X be a random variable in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ with expectation $\mu := \mathbb{E}[X] < \infty$. Prove that

$$\mathbb{P}(X \leqslant \mu) > 0 \quad \text{and} \quad \mathbb{P}(X \geqslant \mu) > 0.$$

(3) Let X be a non-constant random variable in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ with non-negative integer values. Show that

$$\mathbb{P}(X = 0) \leqslant \frac{\mathrm{Var}(X)}{\mathbb{E}[X^2]}.$$

什麼情況下，p_a 會是個機率密度函數？

(5) 從上述問題，我們可以得到什麼結論？

習題 2.23 ： 給定滿足下列條件的實數序列 $(\mu_k)_{k\geqslant 0}$：

$$\limsup_{k\to\infty} \frac{\mu_{2k}^{1/2k}}{2k} = r < \infty.$$

(1) 假設存在分佈函數 F 使得其動差由 $(\mu_k)_{k\geqslant 0}$ 給出。對於所有 $k \geqslant 0$，令 $\nu_k = \int |x|^k \,\mathrm{d}F(x)$。證明對於所有 $k \geqslant 0$，我們有 $\nu_{2k+1}^2 \leqslant \mu_{2k}\mu_{2k+2}$，因此由此可以推得

$$\limsup_{k\to\infty} \frac{\nu_k^{1/k}}{k} = r.$$

(2) 令 X 為分佈函數為 F 的隨機變數，我們將其特徵函數記作 Ψ_X。證明對於所有 $x \in \mathbb{R}$，$t \in \mathbb{R}$ 以及 $n \in \mathbb{N}$，我們有

$$\left| e^{\mathrm{i}\,tx} - \sum_{k=0}^{n-1} \frac{(\mathrm{i}\,tx)^k}{k!} \right| \leqslant \frac{|tx|^n}{n!}.$$

進而推得

$$\forall \xi \in \mathbb{R}, \qquad \left| \Psi_X(\xi+t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \Psi_X^{(k)}(\xi) \right| \leqslant \frac{|t|^n}{n!}\nu_n.$$

(3) 給定 $\xi \in \mathbb{R}$，該如何選擇 t，我們才有下列的等式呢？

$$\Psi_X(\xi+t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Psi_X^{(k)}(\xi).$$

(4) 解釋我們在這裡證明出了什麼，並參考習題 2.22 以做總結。

Exercise 2.25 : For any subset A of an additive group $(G, +)$, if for all $a, b \in A$, $a + b \notin A$, then we say that A is a sum-free set (無和集合). Given a set of non-zero integers $B = \{b_1, \dots b_n\}$, we want to prove that there exists a subset $A \subseteq B$ with $|A| > \frac{n}{3}$ such that A is a sum-free set using the methods of moments introduced in Exercise 2.24.

(1) For any positive integer k, prove that $C_k := \{k+1, \dots, 2k+1\}$ is sum-free in the cyclic ring $\mathbb{Z}/(3k+2)\mathbb{Z}$.

(2) Let $p = 3k+2$ be a prime number satisfying $p > 2\max\{|b_i| : 1 \leqslant i \leqslant n\}$. Let X be a uniform random variable on $\{1, \dots, p-1\}$. Consider n random variables, $0 \leqslant D_i < p$ such that $D_i \equiv X b_i$ (mod p). Prove that all the D_i's have the same distribution. Determine this distribution and prove that $\mathbb{P}(D_i \in C_k) > \frac{1}{3}$.

(3) Conclude.

Exercise 2.26 (Question 2.4.26) **:** Given a random variable X, prove that the three following statements are equivalent.

(1) X is a sub-Gaussian distribution.

(2) There exist $C, c > 0$ such that $\mathbb{E}[e^{tX}] \leqslant C\exp(ct^2)$.

(3) There exists $C > 0$ such that for all $k \geqslant 1$, we have $\mathbb{E}[|X|^k] \leqslant (Ck)^{k/2}$.

Exercise 2.27 (Question 2.4.27) **:** Let X be a real random variable and $k > 0$. If X is in L^k, prove that when $\lambda \longrightarrow \infty$, we have that

$$\lambda^k \, \mathbb{P}(|X| > \lambda) \longrightarrow 0.$$

習題 2.24 【動差方法】： 令 $(\Omega, \mathcal{A}, \mathbb{P})$ 為機率空間。

(1) 考慮有限集合 I 且對於所有 $i \in I$，取可測事件 $A_i \in \mathcal{A}$。證明下列不等式成立：

$$\mathbb{P}\Big(\bigcup_{i\in I} A_i\Big) \leqslant \sum_{i \in I} \mathbb{P}(A_i).$$

(2) 令 X 為在 $L^1(\Omega, \mathcal{A}, \mathbb{P})$ 中，且期望值為 $\mu := \mathbb{E}[X] < \infty$ 的隨機變數，證明

$$\mathbb{P}(X \leqslant \mu) > 0 \quad \text{且} \quad \mathbb{P}(X \geqslant \mu) > 0.$$

(3) 令 X 為在 $L^2(\Omega, \mathcal{A}, \mathbb{P})$ 中，非常數且取值為非負整數的隨機變數，證明

$$\mathbb{P}(X = 0) \leqslant \frac{\mathrm{Var}(X)}{\mathbb{E}[X^2]}.$$

習題 2.25 ： 對任意加法群 $(G, +)$ 的子集合 A，若對所有 $a, b \in A$，$a + b \notin A$，則我們說 A 是個無和集合 (sum-free set)。給定對任意一個由非零整數構成的集合 $B = \{b_1, \dots b_n\}$，我們想要利用在習題 2.24 引入的動差方法，證明存在 $A \subseteq B$，且 $|A| > \frac{n}{3}$，使得 A 是個無和集合。

(1) 對於任意正整數 k，證明 $C_k := \{k+1, \dots, 2k+1\}$ 在環群 $\mathbb{Z}/(3k+2)\mathbb{Z}$ 中是個無和集合。

(2) 令 $p = 3k+2$ 為滿足 $p > 2\max\{|b_i| : 1 \leqslant i \leqslant n\}$ 的質數。令 X 為在 $\{1, \dots, p-1\}$ 之上的均勻隨機變數，令 $0 \leqslant D_i < p$ 使得 $D_i \equiv X b_i$ (mod p) 等 n 個隨機變數。試證所有的 D_i 皆有相同的分佈。求此分佈並證明 $\mathbb{P}(D_i \in C_k) > \frac{1}{3}$。

(3) 總結。

習題 2.26 【問題 2.4.26】： 給定隨機變數 X，則下列三敘述等價：

(1) X 是個次高斯分佈；

(2) 存在 $C, c > 0$ 使得對於所有 $t \in \mathbb{R}$，我們有 $\mathbb{E}[e^{tX}] \leqslant C\exp(ct^2)$；

(3) 存在 $C > 0$ 使得對於所有 $k \geqslant 1$，我們有 $\mathbb{E}[|X|^k] \leqslant (Ck)^{k/2}$。

習題 2.27 【問題 2.4.27】： 令 X 為實隨機變數且 $k > 0$。若 X 在 L^k 中，證明當 $\lambda \longrightarrow \infty$ 時，我們有

$$\lambda^k \, \mathbb{P}(|X| > \lambda) \longrightarrow 0.$$