Chapter 3: Independence of Random Variables

Exercise 3.1 (Question 3.1.2) **:** Given n events $A_1, \dots, A_n \in \mathcal{A}$. Are $A_1, \dots, A_n$ independent events if only one of the following conditions holds?

- $\mathbb{P}(A_1 \cap \dots \cap A_n) = \mathbb{P}(A_1) \dots \mathbb{P}(A_n)$;
- for any pair $1 \leqslant i < j \leqslant n$, we have, $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\,\mathbb{P}(A_j)$.

Exercise 3.2 (Question 3.1.15) **:** Construct two random variables X and Y such that $\mathrm{Cov}(X, Y) = 0$ without X and Y being independent. In Exercise 3.20, we will see a specific condition under which $\mathrm{Cov}(X, Y) = 0$ implies the independence between X and Y.

Exercise 3.3 : Let X and Y be two independent standard normal random variables. Show that $\frac{X+Y}{\sqrt{2}}$ are $\frac{X-Y}{\sqrt{2}}$ also independent with standard normal distribution.

Exercise 3.4 : Let X and Y be bounded random variables. Show that X and Y are independent if and only if

$$\forall k, \ell \in \mathbb{N}, \qquad \mathbb{E}\left[X^k Y^\ell\right] = \mathbb{E}\left[X^k\right] \mathbb{E}\left[Y^\ell\right].$$

Exercise 3.5 : Let U be an exponential random variable with parameter 1 and V be a uniform random variable on $[0, 1]$. Assume that these two random variables are independent. Prove that $X = \sqrt{U}\cos(2\pi V)$ and $Y = \sqrt{U}\sin(2\pi V)$ are also independent.

Exercise 3.6 : Let X and Y be random variables with normal distribution with $\mathrm{Cov}(X, Y) = 0$. Are they independent? If we add the condition "the random variable $aX + bY$ is also a normal distribution for all $a, b \in \mathbb{R}$", please answer to the same question.

Exercise 3.7 (Question 3.1.17) **:** Let $X_1, \dots, X_n$ be real-valued random variables. The following properties are equivalent.

(i) $X_1, \dots, X_n$ are independent random variables.

(ii) For any $a_1, \dots, a_n \in \mathbb{R}$, we have $\mathbb{P}(X_1 \leqslant a_1, \dots, X_n \leqslant a_n) = \prod_{i=1}^n \mathbb{P}(X_i \leqslant a_i)$.

(iii) If $f_1, \dots, f_n$ are continuous and compactly supported (緊緻支撐) functions from $\mathbb{R}$ to $\mathbb{R}_{\geqslant 0}$, then

$$\mathbb{E}\left[\prod_{i=1}^n f_i(X_i)\right] = \prod_{i=1}^n \mathbb{E}\left[f_i(X_i)\right].$$

(iv) The characteristic function of X writes,

$$\Phi_X(\xi_1, \dots, \xi_n) = \prod_{i=1}^n \Phi_{X_i}(\xi_i).$$

Exercise 3.8 (Question 3.1.19) **:** Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be independent σ-algebras and $n_0 = 0 < n_1 < \cdots < n_p = n$. Then, the following σ-algebras are independent,

$$\begin{aligned}
\mathcal{D}_1 &= \mathcal{B}_1 \vee \cdots \vee \mathcal{B}_{n_1} \overset{\text{(def)}}{=} \sigma(\mathcal{B}_1, \ldots, \mathcal{B}_{n_1}),\\
\mathcal{D}_2 &= \mathcal{B}_{n_1+1} \vee \cdots \vee \mathcal{B}_{n_2},\\
&\vdots\\
\mathcal{D}_p &= \mathcal{B}_{n_{p-1}+1} \vee \cdots \vee \mathcal{B}_{n_p}.
\end{aligned}$$

Exercise 3.9 (Question 3.2.5) **:** Explain why it is necessary to assume that $(A_n)_{n\geqslant 1}$ is an independent sequence of events in (2) of Lemma 3.2.4.

Exercise 3.10 : Let $\alpha > 0$ and a sequence of independent random variables $(Z_n)_{n\geqslant 1}$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that for all $n \geqslant 1$, the distribution of Z_n is the following Bernoulli distribution,

$$\mathbb{P}(Z_n = 1) = \frac{1}{n^\alpha} \quad \text{and} \quad \mathbb{P}(Z_n = 0) = 1 - \frac{1}{n^\alpha},$$

Show that Z_n converges to 0 in L^1 but we have,

$$\text{a.s.,} \quad \limsup_{n\to\infty} Z_n = \begin{cases} 1 & \text{if } \alpha \leqslant 1, \\ 0 & \text{if } \alpha > 1. \end{cases}$$

Exercise 3.11 (Proposition 3.3.2) **:** If $(X_k)_{1\leqslant k\leqslant n}$ is a sequence of i.i.d. random variables where each term follows the Poisson distribution of parameter λ, then $X_1 + \cdots + X_n$ is a Poisson distribution of parameter $n\lambda$.

Exercise 3.12 (Proposition 3.3.3) **:** If $(X_k)_{1\leqslant k\leqslant n}$ is a sequence of independent random variables such that for all $1 \leqslant k \leqslant n$, X_k has the Gaussian distribution of parameter $(0, \sigma_k^2)$, then $X_1 + \cdots + X_n$ has the Gaussian distribution of parameter $(0, \sigma_1^2 + \cdots + \sigma_n^2)$.

Exercise 3.13 : We use the notations from Exercise 2.15. Let $u > 0$. Assume that $X_1, \ldots, X_n$ are i.i.d. Cauchy random variables with density function c_u, show that $\frac{1}{n}(X_1 + \cdots + X_n)$ has the same density function c_u.

Exercise 3.14 (Gamma distribution) **:** Let $(X_n)_{n\geqslant 1}$ be an i.i.d. sequence of exponential random variables of parameter $\lambda > 0$. Given two parameters $k, \theta > 0$, if the random variable X has its values in $\mathbb{R}_{\geqslant 0}$ and the density function,

$$\gamma_{k,\theta}(x) = \frac{x^{k-1}e^{-x/\theta}}{\theta^k \Gamma(k)} \mathbb{1}_{x\geqslant 0}, \quad \forall x \in \mathbb{R}.$$

we call it the *Gamma distribution* (伽瑪分佈) of parameter (k, θ), denoted $X \sim \Gamma(k, \theta)$.

(1) Given $k, \theta > 0$, compute the expectation and the variance of $\Gamma(k, \theta)$.

(2) Show that $X_1 + \cdots + X_n \sim \Gamma(n, \lambda^{-1})$.

Exercise 3.15 : Let $X_1, \ldots, X_n$ be i.i.d. standard Gaussian distributions. Show that the density function of $\chi_n^2 = X_1^2 + \ldots X_n^2$ is

$$\frac{x^{n/2-1}e^{-x/2}}{2^{n/2}\Gamma(n/2)}\mathbb{1}_{x>0}.$$

It is called the χ^2 *distribution* (χ^2 分佈) with n degrees of freedom. What is the relation of this distribution with the Gamma distribution (Exercise 3.14)?

Exercise 3.16 : Let $X_1, \ldots, X_n$ be random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that they are independent and have exponential distributions of parameters $c_1, \ldots, c_n$. Let $Y = \min_{1\leqslant k\leqslant n} X_k$. Prove the following statements.

(1) Y is a random variable with exponential distribution of parameter $c_1 + \cdots + c_n$.

(2) There exists a $\mathcal{A}$-measurable function $N : \Omega \longrightarrow \{1, \ldots, n\}$ such that,

$$\mathbb{P}\text{-a.s.}, \qquad X_N = Y \quad \text{and} \quad X_N < \min\{X_k : k \in \{1, \ldots, n\}\backslash\{N\}\}.$$

(3) The random variable N satisfies: for all $k \in \{1, \ldots, n\}$, we have,

$$\mathbb{P}(N = k) = \frac{c_k}{c_1 + \cdots + c_n}.$$

(4) N and Y are independent random variables.

Exercise 3.17 (Poisson process) **:** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(X_n)_{n\geqslant 1}$ be a sequence of i.i.d. random variables with exponential distribution of parameter 1. Let $T_0 = 0$ and for all $n \geqslant 1$, let

$$T_n = X_1 + \cdots + X_n.$$

For all $t \geqslant 0$, let

$$N_t = \max\{n \geqslant 0 : T_n \leqslant t\}.$$

(1) Suppose $n \geqslant 1$, determine the distribution of the n-tuple $(T_1, \ldots, T_n)$.

(2) For any $t > 0$, determine the distribution of N_t.

(3) Given $n \geqslant 1$ and $t > 0$, we define a new probability measure $\mathbb{Q}^{n,t}$ on Ω,

$$\forall A \in \mathcal{A}, \qquad \mathbb{Q}^{n,t}(A) = \frac{\mathbb{P}(A \cap \{N_t = n\})}{\mathbb{P}(N_t = n)}.$$

Determine the distribution of the n-tuple $(T_1, \ldots, T_n)$ under the probability measure $\mathbb{Q}^{n,t}$.

Exercise 3.18 : We are given n independent random variables $X_1, \ldots, X_n$ on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that they are all follow the uniform distribution on $[0, 1]$. Define $m = \min_{1\leqslant i\leqslant n} X_i$ and $M = \max_{1\leqslant i\leqslant n} X_i$. Determine the density functions of m and M, compute their expectations and variances.

Exercise 3.19 : Let $X = (X_1, \dots, X_d)$ be a d-dimensional real-valuded random variable. We want to show that the following three conditions are equivalent. Moreover, when one of the conditions is satisfied, we say that X has a multivariate normal distribution (多元常態分佈).

(a) There exist a d-dimensional real-valued random variable $Z = (Z_1, \dots, Z_d)$ such that the comonents are i.i.d. standard normal distributions, a square matrix A of size $d \times d$ and a vector $B \in \mathbb{R}^d$ such that $X \overset{\text{(d)}}{=} AZ + B$.

(b) For any $\alpha \in \mathbb{R}^d$, the random variable $\alpha^T X$ also has a normal distribution.

(c) There exist a semi-definite symmetric matrix Σ of size $d \times d$ and a vector $B \in \mathbb{R}^d$ such that the characteristic function of X writes,

$$\Phi_X(\xi) = \mathbb{E}\left[e^{\mathrm{i}\,\xi \cdot X}\right] = \exp\left(\mathrm{i}\,\xi^T B - \tfrac{1}{2}\xi^T \Sigma \xi\right).$$

Exercise 3.20 : Let (X, Y) be a pair of random variables with *bivariate normal distribution* (二元常態分佈) . Prove that X and Y are independent if and only if $\mathrm{Cov}(X, Y) = 0$.