

Chapter 4: Convergence of Random Variables

Exercise 4.1 : Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $(X_n)_{n \geq 1}$ be a sequence of real random variables and X be a real random variable. Assume that under the probability measure $\mathbb{P}$, X_n converges in probability to X . If $\mathbb{Q}$ is another probability measure on $(\Omega, \mathcal{A})$ which is absolutely continuous with respect to $\mathbb{P}$, prove that X_n also converges in probability to X with respect to $\mathbb{Q}$.

Exercise 4.2 : Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Assume that Ω is a countable set and that $\mathcal{A} = \mathcal{P}(\Omega)$. Prove that for any sequence of random variables with values in a metric space (E, d) , “almost sure convergence” and “convergence in probability” are equivalent.

Exercise 4.3 : Let $(X_n)_{n \geq 1}$ be a sequence of independent random variables satisfying,

$$\begin{aligned} \mathbb{P}_{X_1} &= \delta_0, \\ \forall n \geq 2, \quad \mathbb{P}_{X_n} &= \frac{1}{2n \ln(n+1)}(\delta_n + \delta_{-n}) + \left(1 - \frac{1}{n \ln(n+1)}\right)\delta_0. \end{aligned}$$

Let $Y_n = \frac{1}{n}(X_1 + \dots + X_n)$.

- (1) Prove Y_n converges in probability to 0.
- (2) Prove that Y_n diverges almost surely.

Exercise 4.4 : Fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a real number $p \geq 1$. Given random variables $(X_n)_{n \geq 1}$ and X defined on $(\Omega, \mathcal{A}, \mathbb{P})$. We want to discuss the relations between convergence in probability, almost sure convergence, and convergence in L^p . In particular, you need to find out whether each of the following statements is true or false: if it is true, explain in a simple way; if it is false, find out a counterexample.

- (1) If $X_n \xrightarrow{\text{a.s.}} X$, then $X_n \xrightarrow{(\mathbb{P})} X$.
- (2) If $X_n \xrightarrow{(\mathbb{P})} X$, then $X_n \xrightarrow{\text{a.s.}} X$.
- (3) If $X_n \xrightarrow{(\mathbb{P})} X$, then there exists a subsequence $(n_k)_{k \geq 1}$ such that $X_{n_k} \xrightarrow{\text{a.s.}} X$.
- (4) If $X_n \xrightarrow{(\mathbb{P})} X$, then $X_n \xrightarrow{L^p} X$.
- (5) If $X_n \xrightarrow{L^p} X$, then $X_n \xrightarrow{(\mathbb{P})} X$.
- (6) If $X_n \xrightarrow{\text{a.s.}} X$, then $X_n \xrightarrow{L^p} X$.
- (7) If $X_n \xrightarrow{L^p} X$, then $X_n \xrightarrow{\text{a.s.}} X$.

Exercise 4.5 : Fix a positive integer $n \geq 1$ and let $(X_k)_{k \geq 1}$ be an i.i.d. sequence of uniform random variables with values in $\{1, \dots, n\}$. Let

$$T_n := \inf\{m \geq 1 : \{X_1, \dots, X_m\} = \{1, \dots, n\}\}.$$

- (1) For all $1 \leq k \leq n$, let $\tau_k^n := \inf\{m \geq 1 : |\{X_1, \dots, X_m\}| = k\}$. Prove that the random variables $(\tau_k^n - \tau_{k-1}^n)_{2 \leq k \leq n}$ are independent and find their respective distribution.
- (2) Deduce that $T_n/(n \ln n)$ converges in probability to 1.

Exercise 4.6 (Question 4.2.3) : Given an i.i.d. sequence of random variables $(X_n)_{n \geq 1}$ and define σ -algebras $\mathcal{B}_k$ and $\mathcal{B}_\infty$ as in Definition 4.2.1. Show the following properties.

- (1) If Y is a real random variable that is $\mathcal{B}_\infty$ -measurable, then it is almost surely a constant.
- (2) If $\frac{1}{n}(X_1 + \dots + X_n)$ converges almost surely, deduce from the previous question that its limit must almost surely be a constant.

Exercise 4.7 : Let $(X_n)_{n \geq 1}$ be an i.i.d. sequence of random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and suppose that there exists a random variable Y such that

$$\frac{1}{n}(X_1 + \dots + X_n) \xrightarrow{\text{a.s.}} Y.$$

- (1) Prove that $\frac{X_n}{n} \xrightarrow{\text{a.s.}} 0$.
- (2) Prove that for any random variable Z defined on $(\Omega, \mathcal{A}, \mathbb{P})$, the following properties are equivalent,

$$Z \in \mathcal{L}^1(\Omega, \mathcal{A}, \mathbb{P}) \quad \Longleftrightarrow \quad \sum_{n \geq 1} \mathbb{P}(|Z| \geq n) < \infty.$$

- (3) Deduce that $X_1 \in \mathcal{L}^1(\Omega, \mathcal{A}, \mathbb{P})$.

Exercise 4.8 : Consider an i.i.d. sequence $(X_n)_{n \geq 1}$ of real-valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution

$$\mathbb{P}(X_1 = +1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}.$$

- (1) Prove that $\mathbb{E}[e^{uX_1}] \leq e^{u^2/2}$ for all $u \in \mathbb{R}$.
- (2) Let $a_1, \dots, a_n$ be real numbers satisfying $\sum_{k=1}^n a_k^2 = 1$ and let $S = \sum_{k=1}^n a_k X_k$. Prove that $\mathbb{E}[e^{uS}] \leq e^{u^2/2}$ for all $u \in \mathbb{R}$. Deduce that

$$\forall t \geq 0, \quad \mathbb{P}(|S| \geq t) \leq 2 e^{-t^2/2}.$$

(Hint: consider two cases $S \geq 0$ and $S \leq 0$ and look at the moment generating function.)

- (3) Using the previous question, prove that for any real numbers $a_1, \dots, a_n$ and any positive real

number $p > 0$, we have

$$\|S\|_p := \mathbb{E} \left[\left| \sum_{k=1}^n a_k X_k \right|^p \right]^{1/p} \leq C_p \left(\sum_{k=1}^n a_k^2 \right)^{1/2}.$$

(Hint: use the Fubini's theorem to rewrite the expectation.)

- (4) For any $n \geq 1$, set $S_n = X_1 + \cdots + X_n$. Prove that we have the following convergence for any $\alpha > 1$,

$$\frac{S_n}{\sqrt{n(\ln n)^\alpha}} \xrightarrow{\text{a.s.}} 0.$$

(Hint: use the result in (2).)

Exercise 4.9 : Let $(X_n)_{n \geq 0}$ be an i.i.d. sequence of non-negative random variables.

- (1) Except for a particular case (please describe what it is), prove that $\sum_{n \geq 0} X_n = \infty$ a.s.
(2) Let X be a non-negative random variable. Prove that for all $\alpha > 0$, we have the following equivalent relation,

$$\mathbb{E}[X] < \infty \quad \Leftrightarrow \quad \sum_{n \geq 0} \mathbb{P}(X \geq \alpha n) < \infty.$$

- (3) Deduce the following dichotomy,

$$\text{a.s.,} \quad \limsup_{n \rightarrow \infty} \frac{X_n}{n} = \begin{cases} 0 & \text{if } \mathbb{E}[X_1] < \infty, \\ \infty & \text{if } \mathbb{E}[X_1] = \infty. \end{cases}$$

- (4) (Law of large numbers in the non-integrable case) Let (Y_n) be an i.i.d. sequence of random variables. For all $n \geq 0$, let $S_n = Y_1 + \cdots + Y_n$. Prove that if Y_1 is not integrable, then $(\frac{S_n}{n})_{n \geq 1}$ diverges almost surely.

Exercise 4.10 (Question 4.4.5) : Let $(X_n)_{n \geq 1}$ and X be random variables with values in $\mathbb{Z}^d$. Prove that X_n converges in distribution to X if and only if,

$$\forall x \in \mathbb{Z}^d, \quad \mathbb{P}(X_n = x) \longrightarrow \mathbb{P}(X = x).$$

Exercise 4.11 (Question 4.4.6) : Assume that for all $n \geq 1$, the random variable X_n has a density, denoted $\mathbb{P}_{X_n}(dx) = p_n(x) dx$. Suppose

- (1) $p_n(x) \longrightarrow p(x)$, dx -a.s.,
(2) there exists a non-negative function q such that $\int_{\mathbb{R}^d} q(x) dx < \infty$ and

$$\forall n, \quad p_n(x) \leq q(x), \quad dx\text{-a.s.}$$

Prove that p is the density function of a probability measure on $\mathbb{R}^d$ and that X_n converges in distribution to $p(x) dx$.

Exercise 4.12 (Question 4.4.7) : If $(X_n)_{n \geq 1}$ converges in distribution to X , is it true that for any $B \in \mathcal{B}(\mathbb{R}^d)$, we also have the following convergence?

$$\mathbb{P}(X_n \in B) \longrightarrow \mathbb{P}(X \in B).$$

Exercise 4.13 (Question 4.4.9) : Consider real-valued random variables $(X_n)_{n \geq 1}$ and X and we write $(F_{X_n})_{n \geq 1}$ and F_X for their distributions. Then, the sequence of random variables $(X_n)_{n \geq 1}$ converges in distribution to X if and only if for all the points of continuity x of F_X , the distribution function $F_{X_n}(x)$ converges to $F(x)$.

Exercise 4.14 (Question 4.4.14) : Given a sequence $(\mu_n)_{n \geq 1}$ of probability measures on $\mathbb{R}^d$. We say that $(\mu_n)_{n \geq 1}$ is a tight (緊密) sequence if for all $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq \mathbb{R}^d$ such that

$$\mu_n(K_\varepsilon) \geq 1 - \varepsilon.$$

Given a measure μ on $\mathbb{R}^d$ and prove that the following properties are equivalent.

- (1) μ_n converges weakly to μ .
- (2) μ_n converges vaguely to μ and $(\mu_n)_{n \geq 1}$ is tight.

Exercise 4.15 (Proposition 4.4.11) : Show that if $(X_n)_{n \geq 1}$ converges in distribution to X which is almost surely a constant, then (X_n) also converges in probability to X .

Exercise 4.16 : Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $(X_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$ be two sequences of real random variables, X and Y be two real random variables. Assume that $X_n \xrightarrow{(d)} X$ and $Y_n \xrightarrow{(d)} Y$.

- (1) Suppose that for all $n \geq 1$, the random variables X_n and Y_n are independent. Also suppose that X and Y are independent. Show that $(X_n, Y_n) \xrightarrow{(d)} (X, Y)$.
- (2) In general, does (X_n, Y_n) converge in distribution to (X, Y) ?
- (3) (Slutsky's lemma) If Y is almost surely a constant, show that (X_n, Y_n) converges in distribution to (X, Y) .

Exercise 4.17 : Let $(X_n)_{n \geq 1}$ be an i.i.d. sequence of random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and denote by μ their common distribution. For $n \geq 1$, let $M_n = \max(X_1, \dots, X_n)$.

- (1) If μ is the uniform distribution on $[0, 1]$, show that when $n \rightarrow \infty$, $(n(1 - M_n))_{n \geq 1}$ converges in distribution and find its limiting distribution.
- (2) If μ is the Cauchy distribution $\text{Cauchy}(0, 1)$, show that when $n \rightarrow \infty$, $\frac{n}{M_n}$ converges in distribution and find its limiting distribution. (Hint: we have $\arctan(x) = \frac{\pi}{2} - \frac{1}{x} + o(\frac{1}{x})$ when $x \rightarrow \infty$.)

Exercise 4.18 : Here we intend to use more elementary tools to show the central limit theorem and a result of large deviations (大偏差) in a simpler example. Consider an i.i.d. sequence of random variables $(X_n)_{n \geq 1}$ and assume that all the terms are Bernoulli distribution of parameter $\frac{1}{2}$ with ± 1 values, meaning that,

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}.$$

For all $n \geq 1$, let $S_n = X_1 + \cdots + X_n$.

- (1) Prove that when n is large, we have the following asymptotic formula,

$$\mathbb{P}(S_{2n} = 2k) \sim \frac{1}{\sqrt{\pi n}} \left(1 + \frac{k}{n}\right)^{-n-k-1/2} \left(1 - \frac{k}{n}\right)^{-n+k-1/2}.$$

- (2) If $2k/\sqrt{2n} \rightarrow x$, show that

$$\mathbb{P}(S_{2n} = 2k) \sim \frac{1}{\sqrt{\pi n}} e^{-x^2/2}.$$

- (3) Prove the central limit theorem: for any $a < b$, we have,

$$\mathbb{P}(a\sqrt{2n} \leq S_{2n} \leq b\sqrt{2n}) \xrightarrow{n \rightarrow \infty} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \mathbb{P}(a \leq \chi \leq b),$$

where χ is the distribution $\mathcal{N}(0, 1)$.

- (4) If $k/n \rightarrow x$, determine the asymptotic expansion of $\frac{1}{2n} \ln \mathbb{P}(S_{2n} = 2k)$ up to the order $o(1)$. Then give an inequality connecting $\mathbb{P}(S_{2n} = 2k)$ and $\mathbb{P}(S_{2n} = 2k + 2)$ for large enough n .
- (5) Prove the following result: for $a \in (0, 1)$,

$$\frac{1}{2n} \ln \mathbb{P}(S_{2n} \geq 2na) \xrightarrow{n \rightarrow \infty} -\gamma(a),$$

where $\gamma(a) = \frac{1}{2} [(1+a) \ln(1+a) + (1-a) \ln(1-a)]$. This result is called large deviations since this formula can be used to understand the asymptotic behavior of the tail probability which decays exponentially.

Exercise 4.19 : Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

- (1) If X is a square integrable random defined on $(\Omega, \mathcal{A}, \mathbb{P})$, prove that for any $a > 0$, we have,

$$\mathbb{E}[|X - \inf(X, a)|] \leq \sqrt{\mathbb{E}[X^2] \mathbb{P}(X \geq a)}.$$

Let $(X_n)_{n \geq 1}$ be an i.i.d. sequence of random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ with Poisson distribution of parameter 1. For all $n \geq 1$, let

$$S_n = \sum_{i=1}^n X_i \quad \text{and} \quad Y_n = \frac{1}{\sqrt{n}}(S_n - n).$$

- (2) For all $n \geq 1$, determine the distribution of S_n , compute $\mathbb{E}[Y_n^2]$ and prove that for any $a > 0$,

$$\mathbb{P}(Y_n^- \geq a) \leq \frac{1}{a^2}.$$

- (3) If Y is the distribution $\mathcal{N}(0, 1)$, prove that the sequence of random variables $(Y_n^-)_{n \geq 1}$ converges in distribution to Y^- .
- (4) Show that $\mathbb{E}[Y_n^-]$ converges to $\mathbb{E}[Y^-]$.

(5) Deduce the Stirling's formula from above,

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \quad \text{as } n \rightarrow \infty.$$