Chapter 5: Conditional Expectations

Exercise 5.1 : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and a non-negative random variable X defined on Ω. Prove that $\{\mathbb{E}[X \mid \mathcal{G}] > 0\}$ is the smallest $\mathcal{G}$-measurable set containing $\{X > 0\}$. Here, the inclusion is defined up to a set of zero measure.

Exercise 5.2 : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Given a sequence $(X_n)_{n\geqslant 1}$ of non-negative random variables and a sequence $(\mathcal{F}_n)_{n\geqslant 1}$ of sub-σ-algebras of $\mathcal{F}$. Suppose that $\mathbb{E}[X_n \mid \mathcal{F}_n]$ converges in probability to 0.

(1) Prove that X_n also converges in probability to 0.

(2) Does the converse hold? If yes, prove it, otherwise, find a counter-example.

Exercise 5.3 : Given random variables X and Y defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$. We say that X and Y are independent knowing $\mathcal{G}$ if for all non-negative measurable functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, we have,

$$\mathbb{E}[f(X)g(Y) \mid \mathcal{G}] = \mathbb{E}[f(X) \mid \mathcal{G}]\ \mathbb{E}[g(Y) \mid \mathcal{G}].$$

(1) What does it mean if $\mathcal{G} = \{\varnothing, \Omega\}$? And if $\mathcal{G} = \mathcal{F}$?

(2) Show that the above definition is equivalent to any of the two following properties.

   (a) For any non-negative $\mathcal{G}$-measurable random variable Z and any non-negative measurable functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, we have,

   $$\mathbb{E}[f(X)g(Y)Z] = \mathbb{E}[f(X)Z\ \mathbb{E}[g(Y) \mid \mathcal{G}]].$$

   (b) For any non-negative measurable function $g : \mathbb{R} \longrightarrow \mathbb{R}$, we have,

   $$\mathbb{E}[g(Y) \mid \mathcal{G} \vee \sigma(X)] = \mathbb{E}[g(Y) \mid \mathcal{G}].$$

Exercise 5.4 : Below we want to extend some properties of the expectation to the conditional expectation.

(1) Prove that for any non-negative random variable X, the following equality holds almost surely,

$$\mathbb{E}[X \mid \mathcal{G}] = \int_0^\infty \mathbb{P}(X > t \mid \mathcal{G})\, \mathrm{d}t.$$

(2) (Markov's inequality) For any random variable $X \in L^p$ and $a > 0$, the following inequality holds almost surely,

$$\mathbb{P}(|X| \geqslant a \mid \mathcal{G}) \leqslant a^{-p}\, \mathbb{E}[|X|^p \mid \mathcal{G}].$$

(3) How to extend the Chebyshev's inequality and the Hölder's inequality?

Exercise 5.5 : Given a square-integrable random variable X and two sub-σ-algebras $\mathcal{G}_1 \subset \mathcal{G}_2$. First, explain using the properties of the conditional expectation why the following inequality needs to hold,

$$\mathbb{E}[(X - \mathbb{E}[X \,|\, \mathcal{G}_2])^2] \leqslant \mathbb{E}[(X - \mathbb{E}[X \,|\, \mathcal{G}_1])^2],$$

before giving a proof.

Exercise 5.6 : Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F})_{n \geqslant 0}$ be a sequence consisting of sub-σ-algebras of $\mathcal{F}$. Assume that $\mathcal{F}_0 = \mathcal{F}$ and that the sequence $(\mathcal{F})_{n \geqslant 0}$ is non-increasing. Consider $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and prove the following statements.

(1) The random variables $\mathbb{E}[X \,|\, \mathcal{F}_n] - \mathbb{E}[X \,|\, \mathcal{F}_{n+1}]$ are orthogonal in L^2.

(2) The following series of random variables converges in L^2,

$$\sum_{n \geqslant 0} (\mathbb{E}[X \,|\, \mathcal{F}_n] - \mathbb{E}[X \,|\, \mathcal{F}_{n+1}]).$$

(3) Let $\mathcal{F}_\infty = \bigcap_{n \geqslant 0} \mathcal{F}_n$. Show that $\mathbb{E}[X \,|\, \mathcal{F}_n]$ converges to $\mathbb{E}[X \,|\, \mathcal{F}_\infty]$ in L^2.

Exercise 5.7 : Given two non-negative random variables X and Y and assume $\mathbb{E}[X \,|\, Y] = Y$ and $\mathbb{E}[Y \,|\, X] = X$.

(1) Show that if X and Y are both in L^2, then $X = Y$ a.s.

(2) Show that for any non-negative random variable Z and a non-negative real number a, we have,

$$\mathbb{E}[Z \,|\, Z \wedge a] \wedge a = Z \wedge a, \quad \text{a.s.}.$$

(3) Show that for any $a \geqslant 0$, the couple $(X \wedge a, Y \wedge a)$ satisfies the same assumption. Deduce $X = Y$ a.s.

Exercise 5.8 : Let (X, Y) be a random variable with values in $\mathbb{R}^n \times \mathbb{R}^m$. Assume that its density function (with respect to the Lebesgue measure) exists and satisfies $p(x, y) > 0$.

(1) Given a Borel function $h : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $h(X, Y)$ is integrable. Determine $\mathbb{E}[h(X, Y) \,|\, Y]$.

(2) Suppose $n = m = 1$. Assume that Y is a Gamma distribution with parameters $(2, \lambda)$ (see Exercise 3.14 for the definition and properties) and that the conditional expectation of X knowing Y is the uniform distribution on $[0, Y]$. Show that X and $Y - X$ are independent and they both follow the exponential distribution of parameter λ.

Exercise 5.9 : Given two positive real numbers a, b and a random variable (X, Y) with values in $\mathbb{Z}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$ satisfying,

$$\mathbb{P}(X = n, Y \leqslant t) = b \int_0^t \frac{(ay)^n}{n!} e^{-(a+b)y} \, \mathrm{d}y.$$

(1) Given a measurable function $h : \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}$ such that $h(Y)$ is integrable. Determine $\mathbb{E}[h(Y) \,|\, X]$.

(2) Determine $\mathbb{E}[\frac{Y}{X+1}]$.

(3) Determine $\mathbb{E}[\mathbb{1}_{X=n} \,|\, Y]$.

(4) Determine $\mathbb{E}[X \,|\, Y]$.

Exercise 5.10 : Let $X_1, \ldots, X_n$ be i.i.d. random variables with exponential distribution of parameter λ. We denote $T = X_1 + \cdots + X_n$. Determine $\mathbb{E}[h(X_1) \,|\, T]$ for all Borel function $h : \mathbb{R} \longrightarrow \mathbb{R}$. How to interprete the result when $h = Id$?