

**Exercise 6.1** : Let  $(X_n)_{n \geq 1}$  be an i.i.d. sequence of random variables defined on  $(\Omega, \mathcal{A}, \mathbb{P})$  with distribution given by  $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$ . Let  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  and  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$  for all  $n \geq 1$ . Set

$$\forall n \geq 0, \quad S_n = \sum_{i=1}^n X_i.$$

- (1) Show that  $(S_n)_{n \geq 0}$  is a  $(\mathcal{F}_n)_{n \geq 0}$ -adapted martingale.
- (2) Show that  $(S_n^2 - n)_{n \geq 0}$  is a  $(\mathcal{F}_n)_{n \geq 0}$ -adapted martingale.
- (3) Show that  $(S_n^3 - 3nS_n)_{n \geq 0}$  is a  $(\mathcal{F}_n)_{n \geq 0}$ -adapted martingale.
- (4) Let  $P$  be a polynomial in two variables. Prove that  $(P(S_n, n))_{n \geq 0}$  is a  $(\mathcal{F}_n)_{n \geq 0}$ -adapted martingale if and only if for all  $s \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}$ ,

$$P(s+1, n+1) + P(s-1, n+1) = 2P(s, n).$$

- (5) Given  $\lambda \in \mathbb{R}$ . Find  $\xi \in \mathbb{R}$  such that  $(\exp(\lambda S_n - \xi n))_{n \geq 0}$  is a  $(\mathcal{F}_n)_{n \geq 0}$ -adapted martingale.

**Exercise 6.2** : Find examples with following properties.

- (1) An unbounded martingale in  $L^1$ .
- (2) An unbounded martingale in  $L^1$  which converges almost surely.
- (3) A martingale converging almost surely to  $+\infty$ .
- (4) A submartingale  $(X_n)$  such that  $(X_n^2)$  is a supermartingale.

**Exercise 6.3** : Let  $T$  be a stopping time for the filtration  $(\mathcal{F}_n)_{n \geq 0}$ . Suppose that there exist  $\varepsilon > 0$  and a positive integer  $N \geq 1$  such that

$$\mathbb{P}(T \leq n + N | \mathcal{F}_n) > \varepsilon, \quad \text{a.s.,} \quad \forall n \geq 0.$$

Show that  $T$  is almost surely finite and  $\mathbb{E}[T] < \infty$ .

**Exercise 6.4** (Question 6.2.7) : Check that  $\mathcal{F}_T$  is a  $\sigma$ -algebra and that  $\mathcal{F}_T = \mathcal{F}_n$  for a constant stopping time  $T = n$ .

**習題 6.1** : 令  $(X_n)_{n \geq 1}$  為 i.i.d. 且定義在  $(\Omega, \mathcal{A}, \mathbb{P})$  上的隨機變數，分佈滿足  $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$ 。令  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  以及對於所有  $n \geq 1$ ， $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ 。設

$$\forall n \geq 0, \quad S_n = \sum_{i=1}^n X_i.$$

- (1) 證明  $(S_n)_{n \geq 0}$  是個適應濾鏈  $(\mathcal{F}_n)_{n \geq 0}$  的鞅。
- (2) 證明  $(S_n^2 - n)_{n \geq 0}$  是個適應濾鏈  $(\mathcal{F}_n)_{n \geq 0}$  的鞅。
- (3) 證明  $(S_n^3 - 3nS_n)_{n \geq 0}$  是個適應濾鏈  $(\mathcal{F}_n)_{n \geq 0}$  的鞅。
- (4) 令  $P$  為雙變數多項式。證明若且唯若  $(P(S_n, n))_{n \geq 0}$  是個適應濾鏈  $(\mathcal{F}_n)_{n \geq 0}$  的鞅，則對於所有  $s \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}$ ，我們有

$$P(s+1, n+1) + P(s-1, n+1) = 2P(s, n).$$

- (5) 給定  $\lambda \in \mathbb{R}$ 。求  $\xi \in \mathbb{R}$  使得  $(\exp(\lambda S_n - \xi n))_{n \geq 0}$  是個適應濾鏈  $(\mathcal{F}_n)_{n \geq 0}$  的鞅。

**習題 6.2** : 請找出符合敘述的例子。

- (1) 在  $L^1$  中無界的鞅。
- (2) 會 a.s. 收斂但在  $L^1$  中無界的鞅。
- (3) 會 a.s. 收斂至  $+\infty$  的鞅。
- (4)  $(X_n)$  是個下鞅但  $(X_n^2)$  是個上鞅。

**習題 6.3** : 令  $T$  是個對於濾鏈  $(\mathcal{F}_n)_{n \geq 0}$  的停止時間。假設存在  $\varepsilon > 0$  及正整數  $N \geq 1$  使得

$$\mathbb{P}(T \leq n + N | \mathcal{F}_n) > \varepsilon, \quad \text{a.s.,} \quad \forall n \geq 0.$$

證明  $T$  殆必有限且  $\mathbb{E}[T] < \infty$ 。

**習題 6.4** 【問題 6.2.7】 : 檢查  $\mathcal{F}_T$  為  $\sigma$  代數且當  $T = n$  為常數停止時間時， $\mathcal{F}_T = \mathcal{F}_n$ 。

**Exercise 6.5** (Question 6.2.11) : Consider an i.i.d. sequence  $(Y_n)_{n \geq 1}$  of random variables with distribution  $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$ . Define  $X_0 = 0$  and for all  $n \geq 1$ , define  $X_n = Y_1 + \dots + Y_n$ , that is to say  $(X_n)_{n \geq 0}$  is a random walk on  $\mathbb{Z}$  started at 0 with symmetric distribution. Let

$$T = \inf\{n \geq 0 : X_n = 1\}.$$

- (1) Apply Proposition 4.2.4 to show that  $T < \infty$  a.s.
- (2) Show that  $\mathbb{E}[X_T] = 1$  and  $\mathbb{E}[X_0] = 0$ .
- (3) Explain.

**Exercise 6.6** (Question 6.3.2) : Given an adapted process  $(X_n)_{n \geq 0}$ , then for all  $k \geq 1$ ,  $S_k(X)$  and  $T_k(X)$  are random variables with values in  $\mathbb{N}_0 \cup \{+\infty\}$ . Check the following points:

- (1) For all  $k \geq 1$ ,  $S_k(X)$  and  $T_k(X)$  are both stopping times.
- (2)  $N_n([a, b], X)$  is  $\mathcal{F}_n$ -measurable.

**Exercise 6.7** : Given a non-negative supermartingale  $(X_n)_{n \geq 0}$ . Show that  $X_n$  converges almost surely to a limit that we denote by  $X$ , satisfying  $\mathbb{E}[X] \leq \mathbb{E}[X_0]$ .

**Exercise 6.8** : Let  $(X_n)_{n \geq 1}$  be a sequence of independent random variables. Define

$$\begin{aligned} \forall n \geq 1, \quad \mathcal{F}_n &= \sigma(X_1, \dots, X_n), & \mathcal{F}_\infty &= \sigma\left(\bigcup_{n \geq 1} \mathcal{F}_n\right), \\ \forall n \geq 1, \quad \mathcal{F}^n &= \sigma(X_n, X_{n+1}, \dots), & \mathcal{F}^\infty &= \bigcap_{n \geq 1} \mathcal{F}^n. \end{aligned}$$

Given  $A \in \mathcal{F}^\infty$  and define  $M_n = \mathbb{E}[\mathbb{1}_A | \mathcal{F}_n]$  for all  $n \geq 1$ . Prove that  $\mathbb{P}(A) = 0$  or 1 using  $(M_n)_{n \geq 1}$ .

**Exercise 6.9** : Let  $(Y_n)_{n \geq 0}$  be an i.i.d. sequence of non-negative random variables satisfying  $\mathbb{E}[Y_n] = 1$ ,  $\mathbb{P}(Y_n = 1) < 1$ .

- (1) Show that  $X_n = \prod_{0 \leq m \leq n} Y_m$  defines a martingale.
- (2) Show that  $X_n$  converges to 0 almost surely.
- (3) Show that  $\frac{1}{n} \ln X_n$  converges almost surely to  $c < 0$ .

**習題 6.5** 【問題 6.2.11】 : 考慮  $(Y_n)_{n \geq 1}$  為 i.i.d. 隨機變數序列，使得  $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$ 。定義  $X_0 = 0$  以及對於所有  $n \geq 1$ ， $X_n = Y_1 + \dots + Y_n$ ，也就是說  $(X_n)_{n \geq 0}$  是個從 0 出發，在  $\mathbb{Z}$  上且分佈對稱的隨機漫步。設

$$T = \inf\{n \geq 0 : X_n = 1\}.$$

- (1) 使用命題 4.2.4 來證明  $T < \infty$  a.s.。
- (2) 證明  $\mathbb{E}[X_T] = 1$  且  $\mathbb{E}[X_0] = 0$ 。
- (3) 解釋。

**習題 6.6** 【問題 6.3.2】 : 給定一個適應過程  $(X_n)_{n \geq 0}$ ，則對於所有  $k \geq 1$ ， $S_k(X)$  及  $T_k(X)$  為值域為  $\mathbb{N}_0 \cup \{+\infty\}$  的隨機變數。驗證下列兩點：

- (1) 對於所有  $k \geq 1$ ， $S_k(X)$  及  $T_k(X)$  為停止時間。
- (2)  $N_n([a, b], X)$  是  $\mathcal{F}_n$  可測的。

**習題 6.7** : 給定非負上鞅  $(X_n)_{n \geq 0}$ ，試證明  $X_n$  會 a.s. 收斂，將其極限記為  $X$ ，並證明其滿足  $\mathbb{E}[X] \leq \mathbb{E}[X_0]$ 。

**習題 6.8** : 令  $(X_n)_{n \geq 1}$  是個獨立的隨機變數序列。定義

$$\begin{aligned} \forall n \geq 1, \quad \mathcal{F}_n &= \sigma(X_1, \dots, X_n), & \mathcal{F}_\infty &= \sigma\left(\bigcup_{n \geq 1} \mathcal{F}_n\right), \\ \forall n \geq 1, \quad \mathcal{F}^n &= \sigma(X_n, X_{n+1}, \dots), & \mathcal{F}^\infty &= \bigcap_{n \geq 1} \mathcal{F}^n. \end{aligned}$$

給定  $A \in \mathcal{F}^\infty$ ，對所有  $n \geq 1$ ，定義  $M_n = \mathbb{E}[\mathbb{1}_A | \mathcal{F}_n]$ 。利用  $(M_n)_{n \geq 1}$  來證明  $\mathbb{P}(A) = 0$  或 1。

**習題 6.9** : 令  $(Y_n)_{n \geq 0}$  為 i.i.d. 非負隨機變數序列且滿足  $\mathbb{E}[Y_n] = 1$ ， $\mathbb{P}(Y_n = 1) < 1$ 。

- (1) 證明  $X_n = \prod_{0 \leq m \leq n} Y_m$  是個鞅。
- (2) 證明  $X_n$  會 a.s. 收斂至 0。
- (3) 證明  $\frac{1}{n} \ln X_n$  會 a.s. 收斂至  $c < 0$ 。

**Exercise 6.10** (Polya's urn) : At the initial time  $n = 0$ , consider an urn with  $a$  white balls and  $b = N - a$  black balls. We pick up a ball uniformly at random from the urn and put in two balls of the same color, resulting in the configuration at time  $n = 1$ . We repeat this procedure infinitely.

For all  $n \geq 0$ , we write  $Y_n$  for the number of white balls and  $X_n = \frac{Y_n}{N+n}$  for the proportion of the white balls in the urn at time  $n$  and let  $\mathcal{F}_n = \sigma(Y_0, Y_1, \dots, Y_n)$ .

- (1) Find  $\mathbb{P}(Y_{n+1} = Y_n + 1 | \mathcal{F}_n)$  and  $\mathbb{P}(Y_{n+1} = Y_n | \mathcal{F}_n)$ . Show that  $(X_n)_{n \geq 0}$  is a martingale that converges almost surely. Denote its limit  $U$  and show that for all integers  $k \geq 1$ , we have the convergence of  $\mathbb{E}[X_n^k]$  to  $\mathbb{E}[U^k]$ .
- (2) Show that when  $a = b = 1$ , for all  $n \geq 0$ ,  $Y_n$  is a random variable on  $\{1, \dots, n+1\}$ . Find the distribution of  $U$ .
- (3) Then consider the general case: for all  $k \geq 1$  and  $n \geq 0$ , let

$$Z_n = \frac{Y_n(Y_n+1)\dots(Y_n+k-1)}{(N+n)\dots(N+n+k-1)}.$$

Show that  $(Z_n)_{n \geq 0}$  is a  $(\mathcal{F}_n)_{n \geq 0}$ -adapted martingale and find  $\mathbb{E}[U^k]$ .

- (4) Define the density function of the Beta distribution  $\beta(a, b)$ ,

$$B(a, b)^{-1} u^{a-1} (1-u)^{b-1} \mathbf{1}_{[0,1]}(u),$$

where  $B(a, b)$  is defined as follows and rewrites as,

$$B(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Show that  $U \sim \beta(a, b)$ .

**習題 6.10** 【Polya 球箱】：在初始時間  $n = 0$  時，考慮裝有  $a$  顆白球以及  $b = N - a$  顆黑球的箱子。我們均勻隨機從箱中抽取一顆球，然後放入兩顆相同顏色的球，得到的是箱子在時間  $n = 1$  的組態。我們重複此動作至任意時間。

對於  $n \geq 0$ ，我們記  $Y_n$  為在時間  $n$  時，箱子中白球的數量，以及  $X_n = \frac{Y_n}{N+n}$  為箱子中白球的比例，且令  $\mathcal{F}_n = \sigma(Y_0, Y_1, \dots, Y_n)$ 。

- (1) 求  $\mathbb{P}(Y_{n+1} = Y_n + 1 | \mathcal{F}_n)$  以及  $\mathbb{P}(Y_{n+1} = Y_n | \mathcal{F}_n)$ 。證明  $(X_n)_{n \geq 0}$  是個會 a.s. 收斂的鞅，將其極限記作  $U$ ，證明對於所有整數  $k \geq 1$ ，我們有  $\mathbb{E}[X_n^k]$  收斂至  $\mathbb{E}[U^k]$ 。
- (2) 證明當  $a = b = 1$  時，對於所有  $n \geq 0$ ， $Y_n$  是在  $\{1, \dots, n+1\}$  上的隨機變數。求  $U$  的分佈。
- (3) 接著考慮一般的情況：對於所有  $k \geq 1$  以及  $n \geq 0$ ，令

$$Z_n = \frac{Y_n(Y_n+1)\dots(Y_n+k-1)}{(N+n)\dots(N+n+k-1)}.$$

證明  $(Z_n)_{n \geq 0}$  是個適應  $(\mathcal{F}_n)_{n \geq 0}$  的鞅，並求  $\mathbb{E}[U^k]$ 。

- (4) 定義  $\beta(a, b)$  分佈的密度函數為

$$B(a, b)^{-1} u^{a-1} (1-u)^{b-1} \mathbf{1}_{[0,1]}(u),$$

其中  $B(a, b)$  定義及性質如下：

$$B(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

證明  $U \sim \beta(a, b)$ 。