Chapter 6: Discrete Time Martingales

Exercise 6.1 : Let $(X_n)_{n\geqslant 1}$ be an i.i.d. sequence of random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution given by $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. Let $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for all $n \geqslant 1$. Set

$$\forall n \geqslant 0, \quad S_n = \sum_{i=1}^{n} X_i.$$

(1) Show that $(S_n)_{n\geqslant 0}$ is a $(\mathcal{F}_n)_{n\geqslant 0}$-adapted martingale.

(2) Show that $(S_n^2 - n)_{n\geqslant 0}$ is a $(\mathcal{F}_n)_{n\geqslant 0}$-adapted martingale.

(3) Show that $(S_n^3 - 3nS_n)_{n\geqslant 0}$ is a $(\mathcal{F}_n)_{n\geqslant 0}$-adapted martingale.

(4) Let P be a polynomial in two variables. Prove that $(P(S_n, n))_{n\geqslant 0}$ is a $(\mathcal{F}_n)_{n\geqslant 0}$-adapted martingale if and only if for all $s \in \mathbb{Z}, n \in \mathbb{Z}_{\geqslant 0}$,

$$P(s+1, n+1) + P(s-1, n+1) = 2P(s, n).$$

(5) Given $\lambda \in \mathbb{R}$. Find $\xi \in \mathbb{R}$ such that $(\exp(\lambda S_n - \xi n))_{n\geqslant 0}$ is a $(\mathcal{F}_n)_{n\geqslant 0}$-adapted martingale.

Exercise 6.2 : Find examples with following properties.

(1) An unbounded martingale in L^1.

(2) An unbounded martingale in L^1 which converges almost surely.

(3) A martingale converging almost surely to $+\infty$.

(4) A submartingale (X_n) such that (X_n^2) is a supermartingale.

Exercise 6.3 : Let T be a stopping time for the filtration $(\mathcal{F}_n)_{n\geqslant 0}$. Suppose that there exist $\varepsilon > 0$ and a positive integer $N \geqslant 1$ such that

$$\mathbb{P}(T \leqslant n + N \,|\, \mathcal{F}_n) > \varepsilon, \quad \text{a.s.,} \quad \forall n \geqslant 0.$$

Show that T is almost surely finite and $\mathbb{E}[T] < \infty$.

Exercise 6.4 (Question 6.2.7) **:** Check that $\mathcal{F}_T$ is a σ-algebra and that $\mathcal{F}_T = \mathcal{F}_n$ for a constant stopping time $T = n$.

Exercise 6.5 (Question 6.2.11) **:** Consider an i.i.d. sequence $(Y_n)_{n\geqslant 1}$ of random variables with distribution $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$. Define $X_0 = 0$ and for all $n \geqslant 1$, define $X_n = Y_1 + \cdots + Y_n$, that is to say $(X_n)_{n\geqslant 0}$ is a random walk on $\mathbb{Z}$ started at 0 with symmetric distribution. Let

$$T = \inf\{n \geqslant 0 : X_n = 1\}.$$

(1) Apply Proposition 4.2.4 to show that $T < \infty$ a.s.

(2) Show that $\mathbb{E}[X_T] = 1$ and $\mathbb{E}[X_0] = 0$.

(3) Explain.

Exercise 6.6 (Question 6.3.2) **:** Given an adapted process $(X_n)_{n\geqslant 0}$, then for all $k \geqslant 1$, $S_k(X)$ and $T_k(X)$ are random variables with values in $\mathbb{N}_0 \cup \{+\infty\}$. Check the following points:

(1) For all $k \geqslant 1$, $S_k(X)$ and $T_k(X)$ are both stopping times.

(2) $N_n([a,b], X)$ is $\mathcal{F}_n$-measurable.

Exercise 6.7 : Given a non-negative supermartingale $(X_n)_{n\geqslant 0}$. Show that X_n converges almost surely to a limit that we denote by X, satisfying $\mathbb{E}[X] \leqslant \mathbb{E}[X_0]$.

Exercise 6.8 : Let $(X_n)_{n\geqslant 1}$ be a sequence of independent random variables. Define

$$\forall n \geqslant 1, \quad \mathcal{F}_n = \sigma(X_1, \ldots, X_n), \qquad \mathcal{F}_\infty = \sigma\Big(\bigcup_{n\geqslant 1} \mathcal{F}_n\Big),$$
$$\forall n \geqslant 1, \quad \mathcal{F}^n = \sigma(X_n, X_{n+1}, \ldots), \qquad \mathcal{F}^\infty = \bigcap_{n\geqslant 1} \mathcal{F}^n.$$

Given $A \in \mathcal{F}^\infty$ and define $M_n = \mathbb{E}[\mathbb{1}_A \,|\, \mathcal{F}_n]$ for all $n \geqslant 1$. Prove that $\mathbb{P}(A) = 0$ or 1 using $(M_n)_{n\geqslant 1}$.

Exercise 6.9 : Let $(Y_n)_{n\geqslant 0}$ be an i.i.d. sequence of non-negative random variables satisfying $\mathbb{E}[Y_n] = 1$, $\mathbb{P}(Y_n = 1) < 1$.

(1) Show that $X_n = \prod_{0\leqslant m\leqslant n} Y_m$ defines a martingale.

(2) Show that X_n converges to 0 almost surely.

(3) Show that $\frac{1}{n} \ln X_n$ converges almost surely to $c < 0$.

Exercise 6.10 (Polya's urn) **:** At the initial time $n = 0$, consider an urn with a white balls and $b = N - a$ black balls. We pick up a ball uniformly at random from the urn and put in two balls of the same color, resulting in the configuration at time $n = 1$. We repeat this procedure infinitely.

For all $n \geqslant 0$, we write Y_n for the number of white balls and $X_n = \frac{Y_n}{N+n}$ for the proportion of the white balls in the urn at time n and let $\mathcal{F}_n = \sigma(Y_0, Y_1, \ldots, Y_n)$.

(1) Find $\mathbb{P}(Y_{n+1} = Y_n + 1 \,|\, \mathcal{F}_n)$ and $\mathbb{P}(Y_{n+1} = Y_n \,|\, \mathcal{F}_n)$. Show that $(X_n)_{n\geqslant 0}$ is a martingale that converges almost surely. Denote its limit U and show that for all integers $k \geqslant 1$, we have the convergence of $\mathbb{E}[X_n^k]$ to $\mathbb{E}[U^k]$.

(2) Show that when $a = b = 1$, for all $n \geqslant 0$, Y_n is a random variable on $\{1, \ldots, n+1\}$. Find the distribution of U.

(3) Then consider the general case: for all $k \geqslant 1$ and $n \geqslant 0$, let

$$Z_n = \frac{Y_n(Y_n+1)\ldots(Y_n+k-1)}{(N+n)\cdots(N+n+k-1)}.$$

Show that $(Z_n)_{n\geqslant 0}$ is a $(\mathcal{F}_n)_{n\geqslant 0}$-adapted martingale and find $\mathbb{E}[U^k]$.

(4) Define the density function of the Beta distribution $\beta(a, b)$,

$$B(a,b)^{-1} u^{a-1}(1-u)^{b-1} \mathbb{1}_{[0,1]}(u),$$

where $B(a, b)$ is defined as follows and rewrites as,

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,\mathrm{d}u = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Show that $U \sim \beta(a, b)$.