

Exercise 7.1 : Let E be a countable state space, (G, \mathcal{G}) be a measurable space and $(Z_n)_{n \geq 1}$ be an i.i.d. sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in (G, \mathcal{G}) . Let $\varphi : E \times G \rightarrow E$ be a measurable function. The random process $(X_n)_{n \geq 0}$ is defined as follows. Fix $x \in E$ and define $X_0 = x \in E$. Then, define $X_{n+1} = \varphi(X_n, Z_{n+1})$ for all $n \geq 0$. Show that $(X_n)_{n \geq 0}$ is a Markov chain and find its transition probability function.

Exercise 7.2 : Let $X = (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, (X_n)_{n \geq 0}, (\mathbb{P}_x)_{x \in E})$ be the canonical Markov chain on a countable set E with transition matrix Q . Take $F \subset E$, let $T = \inf\{n \geq 0 : X_n \in F\}$ and set $Y_n = X_{n \wedge T}$.

- (1) Show that $Y = (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, (Y_n)_{n \geq 0}, (\mathbb{P}_x)_{x \in E})$ is a Markov chain and find its transition matrix. ◦
- (2) Given a non-negative function $f : E \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\forall x \in F^c, \quad f(x) \geq Qf(x).$$

Show that for all $x \in E$, the stochastic process $(f(Y_n))_{n \geq 0}$ is a non-negative supermartingale for \mathbb{P}_x .

Exercise 7.3 : Let E be a countable state space and \mathcal{H} be the vector space consisting of the bounded real functions on E . Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a Markov chain $(X_n)_{n \geq 0}$ with transition matrix $Q = (Q(i, j))_{(i, j) \in E \times E}$ is defined and denote its canonical filtration (正則濾鏈) by $(\mathcal{F}_n)_{n \geq 0}$. Show that there exists a linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ such that for all $f \in \mathcal{H}$, the sequence $(M_n^f)_{n \geq 0}$ defined below is a martingale with respect to $(\mathcal{F}_n)_{n \geq 0}$,

$$\begin{aligned} M_0^f &= f(X_0) \\ M_n^f &= f(X_n) - \sum_{i=0}^{n-1} (Af)(X_i), \quad \forall n \geq 1. \end{aligned}$$

Exercise 7.4 (Symmetric random walk) : Let $(\xi_n)_{n \geq 0}$ be an i.i.d. sequence of random variables with values in \mathbb{Z} . Write μ for their common distribution. The one-dimensional random walk is defined by $S_0 = 0$ and $S_n = \sum_{k=1}^n \xi_k$ for all $n \geq 1$. Moreover, let $\tilde{S}_n = \sup_{0 \leq k \leq n} S_k$ for all $n \geq 0$. Define the stopping time $T_a = \inf\{n \geq 0 : S_n \geq a\}$ for all non-negative integer $a \geq 0$.

- (1) Suppose that $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$. Given integers $0 \leq b \leq a$ and $n \geq 0$.
 - (a) Show that T_a is a stopping time which is almost surely finite for all $a \geq 0$.
 - (b) Show that $(T_{a+1} - T_a)_{a \geq 0}$ is an i.i.d. sequence.
 - (c) Show that $\mathbb{P}(\tilde{S}_n \geq a, S_n \leq b) = \mathbb{P}(S_n \geq 2a - b)$.
 - (d) Deduce that $\mathbb{P}(\tilde{S}_n \geq a) = 2 \mathbb{P}(S_n \geq a) - \mathbb{P}(S_n = a) = \mathbb{P}(|S_n| \geq a) - \mathbb{P}(S_n = a)$.
- (2) Suppose that μ is only a symmetric distribution and $\mu \neq \delta_0$. Show that $\mathbb{P}(\tilde{S}_n \geq a) \leq 2 \mathbb{P}(S_n \geq a)$.

習題 7.1 : 令 E 為可數狀態空間， (G, \mathcal{G}) 為可測空間，且 $(Z_n)_{n \geq 1}$ 為定義在機率空間 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的 i.i.d. 且取值在 (G, \mathcal{G}) 中的隨機變數序列。令 $\varphi : E \times G \rightarrow E$ 為可測函數。隨機過程 $(X_n)_{n \geq 0}$ 定義如下：選定 $x \in E$ 並定義 $X_0 = x \in E$ ，對於所有 $n \geq 0$ ，定義 $X_{n+1} = \varphi(X_n, Z_{n+1})$ 。證明 $(X_n)_{n \geq 0}$ 是個馬可夫鏈並給出他的轉移機率函數。

習題 7.2 : 令 $X = (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, (X_n)_{n \geq 0}, (\mathbb{P}_x)_{x \in E})$ 為在可數集合 E 上，轉移矩陣為 Q 的正則馬可夫鍊。考慮 $F \subset E$ 及 $T = \inf\{n \geq 0 : X_n \in F\}$ 且設 $Y_n = X_{n \wedge T}$ 。

- (1) 證明 $Y = (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, (Y_n)_{n \geq 0}, (\mathbb{P}_x)_{x \in E})$ 是個馬可夫鍊，求其轉移矩陣。
- (2) 考慮非負函數 $f : E \rightarrow \mathbb{R}_{\geq 0}$ 使得

$$\forall x \in F^c, \quad f(x) \geq Qf(x).$$

證明對於所有 $x \in E$ ，隨機過程 $(f(Y_n))_{n \geq 0}$ 對於 \mathbb{P}_x 是個非負上鞅。

習題 7.3 : 令 E 為可數的狀態空間， \mathcal{H} 為由在 E 上有界實函數構成的向量空間。考慮定義在機率空間 $(\Omega, \mathcal{F}, \mathbb{P})$ 上，轉移矩陣為 $Q = (Q(i, j))_{(i, j) \in E \times E}$ 的馬可夫鏈 $(X_n)_{n \geq 0}$ ，並將其生成的正則濾鏈 (canonical filtration) 記作 $(\mathcal{F}_n)_{n \geq 0}$ 。證明存在線性運算子 $A : \mathcal{H} \rightarrow \mathcal{H}$ 使得對於所有 $f \in \mathcal{H}$ ，下列序列 $(M_n^f)_{n \geq 0}$ 對 $(\mathcal{F}_n)_{n \geq 0}$ 是個鞅：

$$\begin{aligned} M_0^f &= f(X_0) \\ M_n^f &= f(X_n) - \sum_{i=0}^{n-1} (Af)(X_i), \quad \forall n \geq 1. \end{aligned}$$

習題 7.4 【對稱隨機漫步】 : 令 ξ_i 為取值在 \mathbb{Z} 上的 i.i.d. 隨機變數，並將他們的共同分佈記作 μ 。定義一維的隨機漫步： $S_0 = 0$ 以及對於所有 $n \geq 1$ ， $S_n = \sum_{k=1}^n \xi_k$ 。此外，對於所有 $n \geq 0$ ，令 $\tilde{S}_n = \sup_{0 \leq k \leq n} S_k$ 。對於所有非負整數 $a \geq 0$ ，定義停止時間 $T_a = \inf\{n \geq 0 : S_n \geq a\}$ 。

- (1) 假設 $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$ 。給定整數 $0 \leq b \leq a$ 以及 $n \geq 0$ 。
 - (a) 證明對於所有 $a \geq 0$ ， T_a 為 a.s. 有限的停止時間。
 - (b) 證明 $(T_{a+1} - T_a)_{a \geq 0}$ 是個 i.i.d. 序列。
 - (c) 證明 $\mathbb{P}(\tilde{S}_n \geq a, S_n \leq b) = \mathbb{P}(S_n \geq 2a - b)$ 。
 - (d) 推得 $\mathbb{P}(\tilde{S}_n \geq a) = 2 \mathbb{P}(S_n \geq a) - \mathbb{P}(S_n = a) = \mathbb{P}(|S_n| \geq a) - \mathbb{P}(S_n = a)$ 。
- (2) 假設 μ 只是個對稱的分佈且 $\mu \neq \delta_0$ ，證明 $\mathbb{P}(\tilde{S}_n \geq a) \leq 2 \mathbb{P}(S_n \geq a)$ 。

Exercise 7.5 : Let $(X_n)_{n \geq 0}$ be a Markov chain defined on the discrete space E with transition matrix Q . Show that there does not exist a non-empty set $F \subsetneq E$ such that

$$\forall x \in F, \forall y \in F^c, \quad Q(x, y) = 0,$$

if and only if $(X_n)_{n \geq 0}$ is irreducible.

Exercise 7.6 (Question 7.3.4) : Check the relation $U = (I - Q)^{-1}$ holds where U is the potential kernel and Q is the transition matrix of a given Markov chain.

Exercise 7.7 : Let $(X_n)_{n \geq 0}$ be a Markov chain defined on the discrete space E with transition matrix Q . Denote by U its potential kernel and assume that $U(x, y) < \infty$ for all $x, y \in E$. Let $f : E \rightarrow \mathbb{R}_{\geq 0}$ be a non-negative function on E .

(1) Show that

$$\mathbb{E}_x \left[\sum_{n \geq 0} f(X_n) \sum_{p \geq n} f(X_p) \right] = U(f \cdot Uf)(x).$$

(2) Rewrite the following expression in terms of U ,

$$\mathbb{E}_x \left[\left(\sum_{n \geq 0} f(X_n) \right)^2 \right].$$

(3) Find an expression for $\mathbb{E}_x[N_y^2]$ using $U(x, y)$ and $U(y, y)$.

Exercise 7.8 : We write E for a finite or countable state space, Q for a transition matrix and $(X_n)_{n \geq 0}$ for the corresponding Markov chain. Given $x \in E$, define $N_x = \sum_{n \geq 0} \mathbb{1}_{X_n=x}$ to be the number of times that the Markov chain returns to x .

- (1) Does there exist a Markov chain departing from x such that the set $V_x = \{X_n : n \geq 0\}$ of the vertices it visits is not a deterministic set, that is not almost surely a constant?
- (2) If $x, y \in E$, is the following statement true? “ y is recurrent and there exists n such that $Q^n(x, y) > 0$, then we have $N_y = \infty$, \mathbb{P}_x -a.s.”
- (3) Given $x, y \in E$. Show that if $\mathbb{E}_x[N_y] = \infty$, then y is recurrent. Does the converse hold?
- (4) Given $x, y \in E$. Is it possible to have $0 < \mathbb{E}_x[N_y] < \infty$ with y being a recurrent state?
- (5) Given $x, y \in E$. If $\mathbb{E}_x[N_y] = \infty$, what are the possible values of $\mathbb{E}_y[N_x]$?
- (6) Suppose that the set $V_x = \{y \in E : \exists n \text{ s.t. } Q^n(x, y) > 0\}$ is finite for all $x \in E$, show that there exists a recurrent state.
- (7) Suppose that there exists $x_0 \in E$ such that $\sum Q^n(x_0, x) > 0$ and $\mathbb{P}_x(H_{x_0} < \infty) = 1$ for all $x \in E \setminus \{x_0\}$. Does there exist a recurrent state?

習題 7.5 : 令 $(X_n)_{n \geq 0}$ 是個取值在離散空間 E 中，轉移矩陣為 Q 的馬可夫鏈。證明若且唯若 $(X_n)_{n \geq 0}$ 為不可約，則不存在非空子集合 $F \subsetneq E$ 使得

$$\forall x \in F, \forall y \in F^c, \quad Q(x, y) = 0.$$

習題 7.6 【問題 7.3.4】 : 固定馬可夫鏈，將其位勢核記作 U ，轉移矩陣記作 Q ，驗證下列關係： $U = (I - Q)^{-1}$ 。

習題 7.7 : 令 $(X_n)_{n \geq 0}$ 是個取值在離散空間 E 中，轉移矩陣為 Q 的馬可夫鏈。我們將其位勢核記作 U 並假設對於所有 $x, y \in E$ ，我們有 $U(x, y) < \infty$ 。令 $f : E \rightarrow \mathbb{R}_{\geq 0}$ 為在 E 上的非負函數。

(1) 證明

$$\mathbb{E}_x \left[\sum_{n \geq 0} f(X_n) \sum_{p \geq n} f(X_p) \right] = U(f \cdot Uf)(x).$$

(2) 將下列式子用 U 來表示：

$$\mathbb{E}_x \left[\left(\sum_{n \geq 0} f(X_n) \right)^2 \right].$$

(3) 將 $\mathbb{E}_x[N_y^2]$ 以 $U(x, y)$ 及 $U(y, y)$ 來表示。

習題 7.8 : 我們將有限或可數的狀態空間記作 E ，轉移矩陣記作 Q ，馬可夫鏈記作 $(X_n)_{n \geq 0}$ 。給定 $x \in E$ ，我們定義 $N_x = \sum_{n \geq 0} \mathbb{1}_{X_n=x}$ 為馬可夫鏈回到 x 的次數。

- (1) 是否存在馬可夫鏈使得從 x 出發，他拜訪的集合 $V_x = \{X_n : n \geq 0\}$ 不會是個固定的集合，也就是說不會 a.s. 為常數？
- (2) 若 $x, y \in E$ ，下列敘述是否為真？「 y 為重現狀態，且存在 n 使得 $Q^n(x, y) > 0$ ，則 \mathbb{P}_x -a.s.，我們有 $N_y = \infty$ 。」
- (3) 給定 $x, y \in E$ 。證明若 $\mathbb{E}_x[N_y] = \infty$ ，則 y 為重現狀態。逆命題是否也成立？
- (4) 給定 $x, y \in E$ ，是否有可能 $0 < \mathbb{E}_x[N_y] < \infty$ 且 y 為重現狀態呢？
- (5) 給定 $x, y \in E$ ，若 $\mathbb{E}_x[N_y] = \infty$ ，哪些是 $\mathbb{E}_y[N_x]$ 可以取的值？
- (6) 假設對於所有 $x \in E$ ，集合 $V_x = \{y \in E : \exists n \text{ s.t. } Q^n(x, y) > 0\}$ 是有限的。證明存在重現狀態。
- (7) 假設存在 $x_0 \in E$ 使得對於所有 $x \in E \setminus \{x_0\}$ ，我們有 $\sum Q^n(x_0, x) > 0$ 以及 $\mathbb{P}_x(H_{x_0} < \infty) = 1$ 。試問，此馬可夫鏈是否有重現狀態？

Exercise 7.9 : Consider the random walk on \mathbb{Z} defined by

$$Y_n = Y_0 + \sum_{i=1}^n \xi_i,$$

where ξ_i are i.i.d. random variables taking values in \mathbb{Z} with distribution μ . Assume that all the ξ_i are independent of Y_0 and write $m = \mathbb{E}[\xi_1]$ for the expectation of μ .

- (1) Show that all the states $x \in \mathbb{Z}$ have the same recurrence type, that is either all of them are recurrent or all of them are transient.

Suppose that $\mathbb{E}[|\xi_1|] < \infty$ and define $m = \mathbb{E}[\xi_1]$.

- (2) If $m \neq 0$, show that all the states are transient.
- (3) If $m = 0$, we want to show that all the states are recurrent. By contradiction, assume that $Y_0 = 0$ is transient.
- (a) Show that there exists $C > 0$ such that for all $n \geq 1$, we have

$$\sum_{|x| \leq n} U(0, x) \leq Cn.$$

- (b) Given $\varepsilon > 0$. Show that $\frac{Y_n}{n}$ converges almost surely to 0 and that there exists a large enough N such that for all $n \geq p \geq N$, we have

$$\sum_{|x| \leq \varepsilon n} Q_p(0, x) \geq \sum_{|x| \leq \varepsilon p} Q_p(0, x) > \frac{1}{2}.$$

- (c) Conclude by contradiction.
- (4) We keep considering the case $m = 0$. Show that the Markov chain is irreducible if and only if the subgroup generated by $\{y \in \mathbb{Z} : \mu(y) > 0\}$ is the whole \mathbb{Z} .

Exercise 7.10 : Let us consider the branching process defined in (4) of Example 7.1.11.

- (1) Explain why when $\mu = \delta_1$, all the states are recurrent. In what follows, we will assume $\mu \neq \delta_1$.
- (2) Show that 0 is a recurrent state.
- (3) We show that 0 is the *only* recurrent state in two steps.
- (a) Suppose that $\mu(0) > 0$, show that $U(x, 0) > 0$ and $U(0, x) = 0$ for any $x \geq 1$ and conclude.
- (b) Suppose that $\mu(0) = 0$, for any $x \geq 1$, find $y \geq 1$ such that $U(x, y) > 0$ and $U(y, x) = 0$. Conclude.
- (4) Show that the Markov chain defined by the branching process satisfies one of the two following properties.
- There exists $N \geq 0$ such that $X_n = 0$ for all $n \geq N$.
 - We have $X_n \rightarrow +\infty$ when $n \rightarrow \infty$.

習題 7.9 : 考慮在 \mathbb{Z} 上的隨機漫步定義做

$$Y_n = Y_0 + \sum_{i=1}^n \xi_i,$$

其中 ξ_i 為取值在 \mathbb{Z} 中的 i.i.d. 隨機變數，並將他們的共同分佈記作 μ ，假設所有的 ξ_i 皆與 Y_0 獨立，並記 $m = \mathbb{E}[\xi_1]$ 為 μ 的期望值。

- (1) 證明所有的狀態 $x \in \mathbb{Z}$ 皆有同樣的重現性；也就是說，全部皆為重現狀態，或全部皆為暫現狀態。

假設 $\mathbb{E}[|\xi_1|] < \infty$ 並定義 $m = \mathbb{E}[\xi_1]$ 。

- (2) 若 $m \neq 0$ ，證明所有的狀態皆為暫現狀態。
- (3) 若 $m = 0$ ，我們想用反證法證明所有的狀態皆為重現狀態。假設 $Y_0 = 0$ 是個暫現狀態。
- (a) 證明存在 $C > 0$ 使得對於所有 $n \geq 1$ ，我們有

$$\sum_{|x| \leq n} U(0, x) \leq Cn.$$

- (b) 給定 $\varepsilon > 0$ 。證明 $\frac{Y_n}{n}$ 會 a.s. 收斂至 0，並且證明存在 N 使得對於所有 $n \geq p \geq N$ 有

$$\sum_{|x| \leq \varepsilon n} Q_p(0, x) \geq \sum_{|x| \leq \varepsilon p} Q_p(0, x) > \frac{1}{2}.$$

- (c) 以反證法總結。
- (4) 繼續考慮 $m = 0$ 的情況，證明若且唯若 $\{y \in \mathbb{Z} : \mu(y) > 0\}$ 生成的子群為整個 \mathbb{Z} ，則馬可夫鏈是不可約的。

習題 7.10 : 我們考慮範例 7.1.11 中 (4) 所定義的分支過程。

- (1) 解釋為什麼在 $\mu = \delta_1$ 時，所有的狀態皆為重現狀態。後續我們將假設 $\mu \neq \delta_1$ 。
- (2) 證明 0 是個重現狀態。
- (3) 下面我們分兩步驟證明 0 是唯一的重現狀態。
- (a) 假設 $\mu(0) > 0$ ，對於任意 $x \geq 1$ ，證明 $U(x, 0) > 0$ 及 $U(0, x) = 0$ 並總結。
- (b) 假設 $\mu(0) = 0$ ，對於任意 $x \geq 1$ ，構造 $y \geq 1$ 滿足 $U(x, y) > 0$ 及 $U(y, x) = 0$ ，總結。
- (4) 證明由分之過程定義出來的馬可夫鏈滿足下列兩行為之一：
- 存在 $N \geq 0$ 使得對於所有 $n \geq N$ ，我們有 $X_n = 0$ ；
 - 當 $n \rightarrow \infty$ ，我們有 $X_n \rightarrow +\infty$ 。

Exercise 7.11 (Wright-Fischer model) : Fix a positive integer $N \geq 1$ and consider the following model for the evolution of the gene pool in the population of a species. Assume that there are N haploid (單倍體) individuals in the population and that the allele (等位基因) is either A or a. From the generation n to the following generation $n + 1$, we make the following assumptions.

- The population size stays constant, meaning that there are N individuals in each generation.
- Each gene from the new generation is chosen uniformly at random from the old generation.

Thus, this model is a Markov chain defined on $E = \{0, \dots, N\}$ with transition matrix

$$Q(i, j) = \binom{N}{j} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j}, \quad \forall i, j \in E.$$

- (1) Which states in E are recurrent? Which are transient?
- (2) Given $k \in E$, show that $(X_n)_{n \geq 0}$ is a martingale under \mathbb{P}_k . Prove that the limit $X_\infty = \lim_{n \rightarrow \infty} X_n$ exists almost surely. Find the distribution of X_∞ under \mathbb{P}_k .

Exercise 7.12 (Question 7.4.6) : Does there exist an invariant measure which is not reversible?

Exercise 7.13 : Does any general Markov chain have a non-zero invariant measure? How about in the case where it is irreducible? If a non-zero invariant measure exists, does this imply the uniqueness?

Exercise 7.14 : Given a Markov chain living on a finite state space. We know that if it is irreducible, then its invariant measure exists and is unique from Theorem 7.4.10 and Theorem 7.4.12. Please find the statement of Perron-Frobenius Theorem and compare to the Markov chain mentioned above.

Exercise 7.15 (Question 7.4.9) : Given $p \in (0, 1)$ and consider the random walk on \mathbb{Z} defined by the following transition matrix,

$$\begin{aligned} Q(i, i+1) &= p, \\ Q(i, i-1) &= 1-p. \end{aligned}$$

In the case where $p \neq \frac{1}{2}$, construct two different invariant measures (two measures defined up to a multiplicative constant are considered as the same).

Exercise 7.16 (Kolmogorov's condition) : Consider a Markov chain on E with transition matrix Q . Suppose that it is irreducible. Prove that there exists a reversible measure (but not necessarily a probability measure) if and only if the two following conditions are satisfied.

- For all $(x, y) \in E^2$, if $Q(x, y) > 0$, then we also have $Q(y, x) > 0$.
- For all $x_0, x_1, \dots, x_n = x_0$, if $\prod_{i=1}^n Q(x_i, x_{i-1}) > 0$, then we have

$$\prod_{i=1}^n \frac{Q(x_{i-1}, x_i)}{Q(x_i, x_{i-1})} = 1.$$

習題 7.11 【Wright-Fischer 模型】：固定正整數 $N \geq 1$ ，我們考慮下列描述族群基因庫演變的模型。假設族群中有 N 個單倍體 (haploid) 個體，以及等位基因 (allele) 有兩個可能：A 或 a。從世代 n 到下一個世代 $n + 1$ 時，我們做下列假設：

- 族群大小不變，也就是每個世代皆有 N 個個體。
- 新世代每個個體的基因，為均勻隨機從舊世代選取。

因此，這個模型是個在 $E = \{0, \dots, N\}$ 上的馬可夫鏈，其轉移矩陣寫作

$$Q(i, j) = \binom{N}{j} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j}, \quad \forall i, j \in E.$$

- (1) E 中的狀態哪些是重現狀態，哪些是暫現狀態？
- (2) 給定 $k \in E$ ，證明 $(X_n)_{n \geq 0}$ 在 \mathbb{P}_k 之下是個鞅。證明極限 $X_\infty = \lim_{n \rightarrow \infty} X_n$ a.s. 存在。求 X_∞ 在 \mathbb{P}_k 之下的分佈。

習題 7.12 【問題 7.4.6】：是否存在不可逆的不變測度？

習題 7.13：一般的馬可夫鏈是否都有非零的不變測度？如果他是不可約的情況下？若非零的不變測度存在，是否有唯一性？

習題 7.14：給定一個定義在有限集合的狀態空間上的馬可夫鏈，我們知道如果他是不可約的，則定理 7.4.10 及定理 7.4.12 給出不變測度的存在性及唯一性。請去尋找 Perron-Frobenius 定理的敘述，並且拿來和這樣的馬可夫鏈做比較。

習題 7.15 【問題 7.4.9】：給定 $p \in (0, 1)$ 並考慮由下列轉移矩陣定義，在 \mathbb{Z} 上的隨機漫步：

$$\begin{aligned} Q(i, i+1) &= p, \\ Q(i, i-1) &= 1-p. \end{aligned}$$

在 $p \neq \frac{1}{2}$ 的情況下，構造兩個不同的不變測度（若只相差常數倍，我們仍將他們視為相同）。

習題 7.16 【Kolmogorov 條件】：考慮轉移矩陣為 Q ，在可數集合 E 上的馬可夫鏈，並假設他是不可約的。證明若且唯若下列兩條件成立，則存在可逆測度（但未必是機率測度）。

- 對於所有 $(x, y) \in E^2$ ，若 $Q(x, y) > 0$ ，則我們也有 $Q(y, x) > 0$ 。
- 對於所有的 $x_0, x_1, \dots, x_n = x_0$ ，若 $\prod_{i=1}^n Q(x_i, x_{i-1}) > 0$ ，則我們有

$$\prod_{i=1}^n \frac{Q(x_{i-1}, x_i)}{Q(x_i, x_{i-1})} = 1.$$

Exercise 7.17 (Ehrenfest model) : Given an integer $N \geq 1$ and consider a Markov chain defined on $\{0, 1, \dots, N\}$ with transition matrix given by

$$\begin{aligned} Q(j, j+1) &= \frac{N-j}{N} & \text{if } 0 \leq j \leq N-1, \\ Q(j, j-1) &= \frac{j}{N} & \text{if } 1 \leq j \leq N. \end{aligned}$$

This model describes a system consisting of a container on the left and another one on the right, connected by a tube with N particles inside. At each step, a particle is picked up uniformly at random then transported through the tube to the other container. The Markov chain $(X_k)_{k \geq 0}$ counts the number of particles in the left container during this evolution.

- (1) Please find at least one invariant measure of this Markov chain. Is there uniqueness?
- (2) Show that there exists a unique invariant probability measure denoted by π . Find π .

Define the following entropy (熵) with respect to the Markov chain at time $k \geq 0$,

$$\forall k \geq 0, \quad H(X_k) = - \sum_{i=0}^N \mathbb{P}(X_k = i) \ln \left(\frac{\mathbb{P}(X_k = i)}{\pi_i} \right).$$

- (3) Using Jensen's inequality, show that this entropy increases in time and find its limit.
- (4) Please describe its relation with the second law of thermodynamics.

習題 7.17 【Ehrenfest 模型】：給定整數 $N \geq 1$ 並考慮由下列轉移矩陣定義，在 $\{0, 1, \dots, N\}$ 上的馬可夫鏈：

$$\begin{aligned} Q(j, j+1) &= \frac{N-j}{N} & \text{若 } 0 \leq j \leq N-1, \\ Q(j, j-1) &= \frac{j}{N} & \text{若 } 1 \leq j \leq N. \end{aligned}$$

此模型描述一個透過連通管連接的左右兩個空間，且含有 N 個粒子的系統，在每個步驟中，均勻隨機選擇一顆粒子，將他透過連通管送到另一個空間中。馬可夫鏈 $(X_k)_{k \geq 0}$ 紀錄左邊空間中粒子個數的變化。

- (1) 請找出此馬可夫鏈至少一個不變測度，是否有唯一性？
- (2) 證明存在唯一的不變機率測度 π ，計算 π 。

將不變測度記作 π ，並對在時間 $k \geq 0$ 的馬可夫鏈定義下列熵 (entropy)：

$$\forall k \geq 0, \quad H(X_k) = - \sum_{i=0}^N \mathbb{P}(X_k = i) \ln \left(\frac{\mathbb{P}(X_k = i)}{\pi_i} \right).$$

- (3) 利用琴生不等式，證明此熵隨著時間而遞增，並求其極限。
- (4) 請描述與熱力學第二定律的關係。