Chapter 7: Discrete Time Markov Chains

Exercise 7.1 : Let E be a countable state space, $(G,\mathcal{G})$ be a measurable space and $(Z_n)_{n\geqslant 1}$ be an i.i.d. sequence of random variables defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ with values in $(G,\mathcal{G})$. Let $\varphi : E\times G \longrightarrow E$ be a measurable function. The random process $(X_n)_{n\geqslant 0}$ is defined as follows. Fix $x\in E$ and define $X_0 = x\in E$. Then, define $X_{n+1} = \varphi(X_n, Z_{n+1})$ for all $n\geqslant 0$. Show that $(X_n)_{n\geqslant 0}$ is a Markov chain and find its transition probability function.

Exercise 7.2 : Let $X = (\Omega,\mathcal{F},(\mathcal{F}_n)_{n\geqslant 0},(X_n)_{n\geqslant 0},(\mathbb{P}_x)_{x\in E})$ be the canonical Markov chain on a countable set E with transition matrix Q. Take $F\subset E$, let $T=\inf\{n\geqslant 0 : X_n\in F\}$ and set $Y_n = X_{n\wedge T}$.

(1) Show that $Y = (\Omega,\mathcal{F},(\mathcal{F}_n)_{n\geqslant 0},(Y_n)_{n\geqslant 0},(\mathbb{P}_x)_{x\in E})$ is a Markov chain and find its transition matrix. ∘

(2) Given a non-negative function $f : E\longrightarrow \mathbb{R}_{\geqslant 0}$ such that

$$\forall x\in F^c,\quad f(x)\geqslant Qf(x).$$

Show that for all $x\in E$, the stochastic process $(f(Y_n))_{n\geqslant 0}$ is a non-negative supermartingale for $\mathbb{P}_x$.

Exercise 7.3 : Let E be a countable state space and $\mathcal{H}$ be the vector space consisting of the bounded real functions on E. Consder a probability space $(\Omega,\mathcal{F},\mathbb{P})$ on which a Markov chain $(X_n)_{n\geqslant 0}$ with transition matrix $Q = (Q(i,j))_{(i,j)\in E\times E}$ is defined and denote its canonical filtration (正則濾鏈) by $(\mathcal{F}_n)_{n\geqslant 0}$. Show that there exists a linear operator $A:\mathcal{H}\longrightarrow\mathcal{H}$ such that for all $f\in\mathcal{H}$, the sequence $(M_n^f)_{n\geqslant 0}$ defined below is a martingale with respect to $(\mathcal{F}_n)_{n\geqslant 0}$,

$$\begin{aligned} M_0^f &= f(X_0)\\ M_n^f &= f(X_n) - \sum_{i=0}^{n-1}(Af)(X_i),\quad \forall n\geqslant 1. \end{aligned}$$

Exercise 7.4 (Symmetric random walk) **:** Let $(\xi_n)_{n\geqslant 0}$ be an i.i.d. sequence of random variables with values in $\mathbb{Z}$. Write μ for their commun distribution. The one-dimensional random walk is defined by $S_0 = 0$ and $S_n = \sum_{k=1}^n \xi_k$ for all $n\geqslant 1$. Moreover, let $\widetilde{S}_n = \sup_{0\leqslant k\leqslant n} S_k$ for all $n\geqslant 0$. Define the stopping time $T_a = \inf\{n\geqslant 0 : S_n\geqslant a\}$ for all non-negative integer $a\geqslant 0$.

(1) Suppose that $\mu = \frac{1}{2}(\delta_{-1}+\delta_1)$. Given integers $0\leqslant b\leqslant a$ and $n\geqslant 0$.
   (a) Show that T_a is a stopping time which is almost surely finite for all $a\geqslant 0$.
   (b) Show that $(T_{a+1}-T_a)_{a\geqslant 0}$ is an i.i.d. sequence.
   (c) Show that $\mathbb{P}(\widetilde{S}_n\geqslant a, S_n\leqslant b) = \mathbb{P}(S_n\geqslant 2a-b)$.
   (d) Deduce that $\mathbb{P}(\widetilde{S}_n\geqslant a) = 2\,\mathbb{P}(S_n\geqslant a) - \mathbb{P}(S_n = a) = \mathbb{P}(|S_n|\geqslant a) - \mathbb{P}(S_n=a)$.

(2) Suppose that μ is only a symmetric distribution and $\mu\neq\delta_0$. Show that $\mathbb{P}(\widetilde{S}_n\geqslant a)\leqslant 2\,\mathbb{P}(S_n\geqslant a)$.

Exercise 7.5 : Let $(X_n)_{n\geqslant 0}$ be a Markov chain defined on the discrete space E with transition matrix Q. Show that there does not exist a non-empty set $F \subsetneq E$ such that

$$\forall x \in F, \forall y \in F^c, \quad Q(x,y) = 0,$$

if and only if $(X_n)_{n\geqslant 0}$ is irreducible.

Exercise 7.6 (Question 7.3.4) **:** Check the relation $U = (I-Q)^{-1}$ holds where U is the potential kernel and Q is the transition matrix of a given Markov chain.

Exercise 7.7 : Let $(X_n)_{n\geqslant 0}$ be a Markov chain defined on the discrete space E with transition matrix Q. Denote by U its potential kernel and assume that $U(x,y) < \infty$ for all $x, y \in E$. Let $f : E \longrightarrow \mathbb{R}_{\geqslant 0}$ be a non-negative function on E.

(1) Show that

$$\mathbb{E}_x\Big[\sum_{n\geqslant 0} f(X_n) \sum_{p\geqslant n} f(X_p)\Big] = U(f \cdot Uf)(x).$$

(2) Rewrite the following expression in terms of U,

$$\mathbb{E}_x\Big[\Big(\sum_{n\geqslant 0} f(X_n)\Big)^2\Big].$$

(3) Find an expression for $\mathbb{E}_x[N_y^2]$ using $U(x,y)$ and $U(y,y)$.

Exercise 7.8 : We write E for a finite or countable state space, Q for a transition matrix and $(X_n)_{n\geqslant 0}$ for the corresponding Markov chain. Given $x \in E$, define $N_x = \sum_{n\geqslant 0} \mathbb{1}_{X_n = x}$ to be the number of times that the Markov chain returns to x.

(1) Does there exist a Markov chain departing from x such that the set $V_x = \{X_n : n \geqslant 0\}$ of the vertices it visits is not a deterministic set, that is not almost surely a constant?

(2) If $x, y \in E$, is the following statement true? "y is recurrent and there exists n such that $Q^n(x,y) > 0$, then we have $N_y = \infty$, $\mathbb{P}_x$-a.s."

(3) Given $x, y \in E$. Show that if $\mathbb{E}_x[N_y] = \infty$, then y is recurrent. Does the converse hold?

(4) Given $x, y \in E$. Is it possible to have $0 < \mathbb{E}_x[N_y] < \infty$ with y being a recurrent state?

(5) Given $x, y \in E$. If $\mathbb{E}_x[N_y] = \infty$, what are the possible values of $\mathbb{E}_y[N_x]$?

(6) Suppose that the set $V_x = \{y \in E : \exists n \text{ s.t. } Q^n(x,y) > 0\}$ is finite for all $x \in E$, show that there exists a recurrent state.

(7) Suppose that there exists $x_0 \in E$ such that $\sum Q^n(x_0, x) > 0$ and $\mathbb{P}_x(H_{x_0} < \infty) = 1$ for all $x \in E \backslash \{x_0\}$. Does there exist a recurrent state?

Exercise 7.9 : Consider the random walk on $\mathbb{Z}$ defined by

$$Y_n = Y_0 + \sum_{i=1}^{n} \xi_i,$$

where ξ_i are i.i.d. random variables taking values in $\mathbb{Z}$ with distribution μ. Assume that all the ξ_i are independent of Y_0 and write $m = \mathbb{E}[\xi_1]$ for the expectation of μ.

(1) Show that all the states $x \in \mathbb{Z}$ have the same recurrence type, that is either all of them are recurrent or all of them are transient.

Suppose that $\mathbb{E}[|\xi_1|] < \infty$ and define $m = \mathbb{E}[\xi_1]$.

(2) If $m \neq 0$, show that all the states are transient.

(3) If $m = 0$, we want to show that all the states are recurrent. By contradiction, assume that $Y_0 = 0$ is transient.

  (a) Show that there exists $C > 0$ such that for all $n \geqslant 1$, we have

$$\sum_{|x| \leqslant n} U(0, x) \leqslant Cn.$$

  (b) Given $\varepsilon > 0$. Show that $\frac{Y_n}{n}$ converges almost surely to 0 and that there exists a large enough N such that for all $n \geqslant p \geqslant N$, we have

$$\sum_{|x| \leqslant \varepsilon n} Q_p(0, x) \geqslant \sum_{|x| \leqslant \varepsilon p} Q_p(0, x) > \frac{1}{2}.$$

  (c) Conclude by contradiction.

(4) We keep considering the case $m = 0$. Show that the Markov chain is irreducible if and only if the subgroup generated by $\{y \in \mathbb{Z} : \mu(y) > 0\}$ is the whole $\mathbb{Z}$.

Exercise 7.10 : Let us consider the branching process defined in (4) of Example 7.1.11.

(1) Explain why when $\mu = \delta_1$, all the states are recurrent. In what follows, we will assume $\mu \neq \delta_1$.

(2) Show that 0 is a recurrent state.

(3) We show that 0 is the *only* recurrent state in two steps.

  (a) Suppose that $\mu(0) > 0$, show that $U(x, 0) > 0$ and $U(0, x) = 0$ for any $x \geqslant 1$ and conclude.

  (b) Suppose that $\mu(0) = 0$, for any $x \geqslant 1$, find $y \geqslant 1$ such that $U(x, y) > 0$ and $U(y, x) = 0$. Conclude.

(4) Show that the Markov chain defined by the branching process satisfies one of the two following properties.

  - There exists $N \geqslant 0$ such that $X_n = 0$ for all $n \geqslant N$.
  - We have $X_n \longrightarrow +\infty$ when $n \longrightarrow \infty$.

Exercise 7.11 (Wright-Fischer model) **:** Fix a positive integer $N \geqslant 1$ and consider the following model for the evolution of the gene pool in the population of a species. Assume that there are N haploid (單倍體) individuals in the population and that the allele (等位基因) is either A or a. From the generation n to the following generation $n+1$, we make the following assumptions.

- The population size stays constant, meaning that there are N individuals in each generation.
- Each gene from the new generation is chosen uniformly at random from the old generation.

Thus, this model is a Markov chain defined on $E = \{0, \dots, N\}$ with transition matrix

$$Q(i,j) = \binom{N}{j} \Big(\frac{i}{N}\Big)^j \Big(1 - \frac{i}{N}\Big)^{N-j}, \quad \forall i, j \in E.$$

(1) Which states in E are recurrent? Which are transient?

(2) Given $k \in E$, show that $(X_n)_{n \geqslant 0}$ is a martingale under $\mathbb{P}_k$. Prove that the limit $X_\infty = \lim_{n \to \infty} X_n$ exists almost surely. Find the distribution of X_∞ under $\mathbb{P}_k$.

Exercise 7.12 (Question 7.4.6) **:** Does there exist an invariant measure which is not reversible?

Exercise 7.13 : Does any general Markov chain have a non-zero invariant measure? How about in the case where it is irreducible? If a non-zero invariant measure exists, does this imply the uniqueness?

Exercise 7.14 : Given a Markov chain living on a finite state space. We know that if it is irreducible, then its invariant measure exists and is unique from Theorem 7.4.10 and Theorem 7.4.12. Please find the statement of Perron-Frobenius Theorem and compare to the Markov chain mentioned above.

Exercise 7.15 (Question 7.4.9) **:** Given $p \in (0,1)$ and consider the random walk on $\mathbb{Z}$ defined by the following transition matrix,

$$\begin{aligned} Q(i, i+1) &= p, \\ Q(i, i-1) &= 1 - p. \end{aligned}$$

In the case where $p \neq \frac{1}{2}$, construct two different invariant measures (two measures defined up to a multiplicative constant are considered as the same).

Exercise 7.16 (Kolmogorov's condition) **:** Consider a Markov chain on E with transition matrix Q. Suppose that it is irreducible. Prove that there exists a reversible measure (but not necessarily a probability measure) if and only if the two following conditions are satisfied.

- For all $(x, y) \in E^2$, if $Q(x,y) > 0$, then we also have $Q(y,x) > 0$.
- For all $x_0, x_1, \dots, x_n = x_0$, if $\prod_{i=1}^n Q(x_i, x_{i-1}) > 0$, then we have

$$\prod_{i=1}^n \frac{Q(x_{i-1}, x_i)}{Q(x_i, x_{i-1})} = 1.$$

Exercise 7.17 (Ehrenfest model) **:** Given an integer $N \geqslant 1$ and consider a Markov chain defined on $\{0, 1, \ldots, N\}$ with transition matrix given by

$$\begin{aligned} Q(j, j+1) &= \tfrac{N-j}{N} \quad \text{if } 0 \leqslant j \leqslant N-1, \\ Q(j, j-1) &= \tfrac{j}{N} \quad \text{if } 1 \leqslant j \leqslant N. \end{aligned}$$

This model describes a system consisting of a container on the left and another one on the left, connected by a tube with N particles inside. At each step, a particle is picked up uniformly at random then transported through the tube to the other container. The Markov chain $(X_k)_{k \geqslant 0}$ counts the number of particles in the left container during this evolution.

(1) Please find at least one invariant measure of this Markov chain. Is there uniqueness?

(2) Show that there exists a unique invariant probability measure denoted by π. Find π.

Define the following entropy (熵) with respect to the Markov chain at time $k \geqslant 0$,

$$\forall k \geqslant 0, \quad H(X_k) = -\sum_{i=0}^{N} \mathbb{P}(X_k = i) \ln\Big(\frac{\mathbb{P}(X_k = i)}{\pi_i}\Big).$$

(3) Using Jensen's inequality, show that this entropy increases in time and find its limit.

(4) Please describe its relation with the second law of thermodynamics.