Exercise 8.1 : Given a real sequence $(x_n)_{n\geqslant 0}$ with period $p \geqslant 1$ and define the discrete probability measure
$$\mathbb{P} := \frac{1}{p}\sum_{k=0}^{p-1}\delta_{(x_{n+k})_{n\geqslant 0}}.$$
Show that $\mathbb{P}$ is an invariant measure for the shift operator θ.

Exercise 8.2 : Using Kolmogorov's extension theorem (Theorem 3.1.25), explain how to extend a stationary sequence $(X_n)_{n\geqslant 0}$ to a stationary sequence $(X_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$.

Exercise 8.3 (Question 8.1.9) **:** Check that the invariant set $\mathcal{I}$ defined in Definition 8.1.8 is indeed a σ-algebra.

Exercise 8.4 : Given a positive integer $n \geqslant 2$. Please prove the following statements or find counterexamples.

(1) If φ is ergodic, is φ^n ergodic? If φ^n is ergodic, is φ ergodic?

(2) Same questions as above with "ergodic" replaced by "a measure-preserving transformation".

Exercise 8.5 (Question 8.1.11) **:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which we consider a measure-preserving transformation φ.

(1) A random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $\mathcal{I}$-measurable if and only if $X \circ \varphi = X$.

(2) φ is ergodic if and only if all the $\mathcal{I}$-measurable random variables $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are $\mathbb{P}$-a.s. constant.

Exercise 8.6 (Bernoulli system) **:** Given a positive integer $N \geqslant 1$ and $p_1, \dots, p_N \geqslant 0$ such that $\sum_k p_k = 1$. Let $E = \{1, \dots, N\}$ be a discrete state space, $\Omega = E^{\mathbb{Z}}$ be the sample space and $\mathcal{F} = \mathcal{P}(E)^{\mathbb{Z}}$, where $\mathcal{P}(E)$ is the set consisting of all the subsets of E. Define the shift operator θ as $\theta((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$ and the probability measure $\mathbb{P}$ satisfying
$$\mathbb{P}(x_k = n_k, \dots, x_l = n_l) = \prod_{i=k}^{l} p_{n_i}, \qquad \forall k \leqslant l, \quad \forall n_k, \dots, n_l \in E.$$

(1) Explain why the probability measure $\mathbb{P}$ is well-defined.

(2) Show that θ is a measure-preserving transformation on $(\Omega, \mathcal{F}, \mathbb{P})$.

(3) Show that θ is mixing.

(4) If we change the state space to $\Omega = E^{\mathbb{Z}_{\geqslant 0}}$, explain how to adapt the proofs above or disprove the above statements.

習題 8.1 ： 給定週期為 $p \geqslant 1$ 的實數序列 $(x_n)_{n\geqslant 0}$ 並定義離散機率測度
$$\mathbb{P} := \frac{1}{p}\sum_{k=0}^{p-1}\delta_{(x_{n+k})_{n\geqslant 0}}.$$
證明 $\mathbb{P}$ 是推移算子 θ 的不變測度。

習題 8.2 ： 利用 Kolmogorov 拓延定理（定理 3.1.25），解釋如何將給定的平穩序列 $(X_n)_{n\geqslant 0}$，拓延成下標在 $\mathbb{Z}$ 上的平穩序列 $(X_n)_{n\in\mathbb{Z}}$。

習題 8.3 【問題 8.1.9】 **：** 驗證定義 8.1.8 中定義的不變集合 $\mathcal{I}$ 的確是個 σ 代數。

習題 8.4 ： 給定正整數 $n \geqslant 2$。請證明下列敘述或找出反例。

(1) 若 φ 有遍歷性，是否 φ^n 也有遍歷性？若 φ^n 有遍歷性，是否 φ 也有遍歷性？

(2) 同上題，但將「遍歷性」改作「測度守恆變換」。

習題 8.5 【問題 8.1.11】 **：** 令 $(\Omega, \mathcal{F}, \mathbb{P})$ 為機率空間，φ 是個在他上面的測度守恆函數。

(1) 給定隨機變數 $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$。若且唯若 $X \circ \varphi = X$，則 X 會對 $\mathcal{I}$ 可測。

(2) 若且唯若所有 $\mathcal{I}$ 可測的隨機變數 $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ 皆 $\mathbb{P}$-a.s. 為常數，則 φ 有遍歷性。

習題 8.6 【伯努力系統】 **：** 給定正整數 $N \geqslant 1$，及 $p_1, \dots, p_N \geqslant 0$ 使得 $\sum_k p_k = 1$。令 $E = \{1, \dots, N\}$ 為離散狀態空間，$\Omega = E^{\mathbb{Z}}$ 為樣本空間，$\mathcal{F} = \mathcal{P}(E)^{\mathbb{Z}}$，其中 $\mathcal{P}(E)$ 為所有由 E 的子集合構成的集合。定義推移算子 θ 為 $\theta((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$ 及機率測度 $\mathbb{P}$ 滿足
$$\mathbb{P}(x_k = n_k, \dots, x_l = n_l) = \prod_{i=k}^{l} p_{n_i}, \qquad \forall k \leqslant l, \quad \forall n_k, \dots, n_l \in E.$$

(1) 解釋為何機率測度 $\mathbb{P}$ 是定義良好的。

(2) 證明 θ 是個在 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的測度守恆變換。

(3) 證明 θ 具有混合性。

(4) 若我們將狀態空間更改為 $\Omega = E^{\mathbb{Z}_{\geqslant 0}}$，請解釋如何修改上面證明或反證上面三個問題。

Exercise 8.7 : Two functions $f, g : [0,1] \longrightarrow [0,1]$ are said to be conjugated (共軛) if there exists a bijective (雙射) function $h : [0,1] \longrightarrow [0,1]$ such that

$$h \circ f = g \circ h. \tag{8.1}$$

(1) If f and g are conjugated and h satisfies Eq. (8.1), show $h \circ f^n = g^n \circ h$ for all $n \geqslant 1$.

Consider the tent map (帳篷函數),

$$g(x) = \begin{cases} 2x & \text{for } x < 1/2, \\ 2 - 2x & \text{for } x \geqslant 1/2, \end{cases}$$

and define the logistic map (單峰函數),

$$f(x) = 4x(1-x).$$

(2) Show that for the Lebesgue measure λ, g is measure-preserving.

(3) Show that the logistic function and the tent function are conjugated. Hint: consider $h(x) = \frac{1}{2}(1 - \cos \pi x)$.

(4) Define the measure μ as follows,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}.$$

Show that f is measure-preserving for μ.

Exercise 8.8 (Question 8.1.14) **:** Please find an example with $\mathcal{I} \subsetneq \mathcal{T}$.

Exercise 8.9 (Question 8.2.3) **:** We keep the notations in Theorem 8.2.1 and assume additionally that $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$. Prove the L^p convergence of the ergodic theorem.

Exercise 8.10 (Wiener's maximal inequality) **:** Let φ be a measure-preserving transformation on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and X be an integrable random variable in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. For all $n \geqslant 0$, define

$$X_n(\omega) = X(\varphi^n(\omega)), \qquad S_n(\omega) = \sum_{k=0}^{n-1} X_k(\omega),$$
$$A_n(\omega) = \frac{S_n(\omega)}{n}, \qquad D_n = \max\{A_1, \ldots, A_n\}.$$

Show that for all $\alpha > 0$ we have

$$\mathbb{P}(D_n > \alpha) \leqslant \frac{\mathbb{E}[|X|]}{\alpha}.$$

習題 8.7 ： 給定兩個函數 $f, g : [0,1] \longrightarrow [0,1]$，若存在雙射 (bijective) 函數 $h : [0,1] \longrightarrow [0,1]$ 使得

$$h \circ f = g \circ h, \tag{8.1}$$

我們說 f 與 g 互為共軛 (conjugated)。

(1) 若 f 與 g 共軛且 h 滿足式 (8.1)，證明對於所有 $n \geqslant 1$，我們有 $h \circ f^n = g^n \circ h$。

考慮帳篷函數 (tent map)：

$$g(x) = \begin{cases} 2x & \text{對於 } x < 1/2, \\ 2 - 2x & \text{對於 } x \geqslant 1/2, \end{cases}$$

並定義單峰函數 (logistic map)：

$$f(x) = 4x(1-x).$$

(2) 證明對勒貝格測度 λ 來說，g 是個測度守恆變換。

(3) 證明單峰函數與帳篷函數是共軛的。提示：可以考慮 $h(x) = \frac{1}{2}(1 - \cos \pi x)$。

(4) 定義測度 μ 如下：

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}.$$

證明 f 對於 μ 是個測度守恆變換。

習題 8.8 【問題 8.1.14】 **：** 請找出滿足 $\mathcal{I} \subsetneq \mathcal{T}$ 的例子。

習題 8.9 【問題 8.2.3】 **：** 我們沿用定理 8.2.1 中的記號，若假設 $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$，試證 L^p 收斂的遍歷定理。

習題 8.10 【Wiener 極大不等式】 **：** 令 φ 為在機率空間 $(\Omega, \mathcal{F}, \mathbb{P})$ 上的測度守恆變換，X 為在 $L^1(\Omega, \mathcal{F}, \mathbb{P})$ 中的可積隨機變數。對於所有 $n \geqslant 0$，定義

$$X_n(\omega) = X(\varphi^n(\omega)), \qquad S_n(\omega) = \sum_{k=0}^{n-1} X_k(\omega),$$
$$A_n(\omega) = \frac{S_n(\omega)}{n}, \qquad D_n = \max\{A_1, \ldots, A_n\}.$$

證明對於所有 $\alpha > 0$，我們有

$$\mathbb{P}(D_n > \alpha) \leqslant \frac{\mathbb{E}[|X|]}{\alpha}.$$

Exercise 8.11 (Normal numbers) **:** For any $x \in [0,1]$, we can find a unique sequence $(a_n(x))_{n\geqslant 0}$ such that

(a) $a_n(x) \in \{0,1\}$ for all $n \geqslant 0$ and $a_n(x)$ does not stabilize at 1;

(b) the equality $x = \sum_{k\geqslant 0} 2^{-k} a_k(x)$ holds.

We say that $(a_n(x))_{n\geqslant 0}$ is the binary expansion (二元展開式) of x. Given a real number $x \in [0,1]$, if the frequency of 0's in its binary expansion satisfies

$$\frac{1}{n}\operatorname{Card}\{1 \leqslant k \leqslant n : a_k(x) = 0\} \xrightarrow[n\to\infty]{} \frac{1}{2},$$

then we say that x is a normal number (正規數). Show that $\lambda(\mathrm{d}x)$-a.s., the real number $x \in [0,1]$ is normal.

Exercise 8.12 : Consider the unit circle $\mathbb{S}^1 = \{x \in \mathbb{C} : |x| = 1\}$ and the uniform probability distribution μ defined above. For any $\beta \in \mathbb{R}$, we can define the rotation operator

$$\theta_\beta : e^{2\pi \mathrm{i}\,\alpha} \mapsto e^{2\pi \mathrm{i}(\alpha+\beta)}.$$

(1) Show that if β is rational, then θ_β is not ergodic.

(2) Assume that β is irrational and consider $f \in L^2(\mathbb{S}^1, \mathcal{B}(\mathbb{S}^1), \mu)$, show the ergodicity of θ_β using the uniqueness of the Fourier series of f. (Hint: use Exercise 8.5.)

Next, we will use a more direct method to prove that θ_β is ergodic when β is irrational,. First, we know the following property holds. Given $A \in \mathcal{B}(\mathbb{S}^1)$, for any $\varepsilon > 0$, we can find a countable disjoint union $(J_k)_{k\geqslant 0}$ such that

$$\mu(A\Delta J) < \varepsilon, \qquad J = \bigsqcup_{k\geqslant 0} J_k.$$

Given an irrational number β.

(3) Show that the sequence $(x_n = n\beta \mod 1)_{n\geqslant 0}$ is dense in $[0,1)$.

(4) Given $A \in \mathcal{B}(\mathbb{S}^1)$ such that $\mu(A) > 0$. Show that for all $\delta > 0$, there exists an interval J of $\mathbb{S}^1$ with $\mu(A \cap J) > (1-\delta)\mu(J)$.

(5) Deduce that for $A \in \mathcal{I}_{\theta_\beta}$, if $\mu(A) > 0$, then $\mu(A) = 1$.

習題 8.11 【正規數】**：** 對於任意 $x \in [0,1]$，我們可以找到唯一的序列 $(a_n(x))_{n\geqslant 0}$ 使得

(a) 對於所有 $n \geqslant 0$，我們有 $a_n(x) \in \{0,1\}$ 且 $a_n(x)$ 不收斂於 1；

(b) 我們有 $x = \sum_{k\geqslant 0} 2^{-k} a_k(x)$。

我們稱 $(a_n(x))_{n\geqslant 0}$ 為 x 的二元展開式 (binary expansion)。給定實數 $x \in [0,1]$，若二元展開式中 0 出現的頻率滿足

$$\frac{1}{n}\operatorname{Card}\{1 \leqslant k \leqslant n : a_k(x) = 0\} \xrightarrow[n\to\infty]{} \frac{1}{2},$$

則我們說 x 是個正規數 (normal number)。證明 $\lambda(\mathrm{d}x)$-a.s.，實數 $x \in [0,1]$ 是個正規數。

習題 8.12 ： 考慮單位圓 $\mathbb{S}^1 = \{x \in \mathbb{C} : |x| = 1\}$ 以及定義在上面的均勻分佈 μ。對於任意 $\beta \in \mathbb{R}$，我們可以定義旋轉算子

$$\theta_\beta : e^{2\pi \mathrm{i}\,\alpha} \mapsto e^{2\pi \mathrm{i}(\alpha+\beta)}.$$

(1) 證明若 β 為有理數，則 θ_β 不具有遍歷性。

(2) 假設 β 為無理數並考慮 $f \in L^2(\mathbb{S}^1, \mathcal{B}(\mathbb{S}^1), \mu)$，使用 f 的傅立葉級數的唯一性來證明 θ_β 具有遍歷性。（提示：使用習題 8.5 。）

接著，我們嘗試用比較直接的方法，來證明當 β 為無理數時，θ_β 具有遍歷性。首先，我們知道下列性質成立：給定 $A \in \mathcal{B}(\mathbb{S}^1)$，對於任意 $\varepsilon > 0$，我們可以找到可數互斥區間 $(J_k)_{k\geqslant 0}$，使得

$$\mu(A\Delta J) < \varepsilon, \qquad J = \bigsqcup_{k\geqslant 0} J_k.$$

給定無理數 β。

(3) 證明數列 $(x_n = n\beta \mod 1)_{n\geqslant 0}$ 在 $[0,1)$ 上是稠密的。

(4) 給定 $A \in \mathcal{B}(\mathbb{S}^1)$ 滿足 $\mu(A) > 0$。證明對於所有 $\delta > 0$，存在 $\mathbb{S}^1$ 上的區間 J 使得 $\mu(A \cap J) > (1-\delta)\mu(J)$。

(5) 由此推得對於所有 $A \in \mathcal{I}_{\theta_\beta}$，若 $\mu(A) > 0$，則 $\mu(A) = 1$。

Exercise 8.13 : Let $U \subset \mathbb{R}$ be an open set and $f : U \longrightarrow U$ be a local $\mathcal{C}^1$ diffeomorphism (微分同胚). Given a non-negative continuous function $\rho : U \longrightarrow \mathbb{R}_{\geqslant 0}$ on U, we define a measure μ on U using the Radon-Nikodym derivative,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = \rho.$$

Show that μ is an invariant measure for f if and only if

$$\sum_{x \in f^{-1}(y)} \frac{\rho(x)}{\det Df(x)} = \rho(y), \quad \forall y \in U.$$

Exercise 8.14 : Let $E = [0, 1)$. Consider a probability space $(E, \mathcal{B}(E), \mu)$. Let $B \in \mathcal{B}(E)$ be such that $\mu(B) > 0$ and assume that there exists a collection of subintervals of E denoted $\mathcal{C}$ such that

(1) for all $\varepsilon > 0$ and $A \in \mathcal{B}(E)$, there exist a countable union of disjoint elements of $\mathcal{C}$, denoted $J = \sqcup_{k \geqslant 0} J_k$, such that $\mu(A \Delta J) < \varepsilon$);

(2) there exists $\gamma > 0$ such that $\mu(C \cap B) \geqslant \gamma\mu(C)$ for all $C \in \mathcal{C}$.

Prove that $\mu(B) = 1$.

Exercise 8.15 (Continued fraction) **:** For any $x \in (0, 1)$, define

$$A(x) = \lfloor \tfrac{1}{x} \rfloor \quad \text{and} \quad T(x) = \tfrac{1}{x} - A(x).$$

Given $x \in (0, 1)$ and define $a_n = A(T^{n-1}(x))$ for all $n \geqslant 0$, then we obtain the representation of x in continued fraction,

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \dots}}. \tag{8.2}$$

(1) We start by investigating some properties of continued fractions and show that Eq. (8.2) converges. First, for any positive integer $n \geqslant 1$ and positive real numbers $a_1, \dots, a_n > 0$, we define

$$[a_1, \dots, a_n] := \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}}.$$

Given $x \in (0, 1)$.

(a) Assume the following holds for a given positive integer $n \geqslant 1$,

$$T^k(x) \neq 0, \qquad 1 \leqslant k \leqslant n.$$

Prove that

$$x = [a_1 + T(x)] = [a_1, a_2 + T^2(x)] = \dots = [a_1, \dots, a_{n-1}, a_n + T^n(x)].$$

(b) Show that there exists $n \geqslant 1$ such that $T^n(x) = 0$ if and only if x is rational.

習題 8.13 ： 令 $U \subset \mathbb{R}$ 為開集及 $f : U \longrightarrow U$ 為局部 $\mathcal{C}^1$ 微分同胚 (diffeomorphism)。給定在 U 上的非負連續函數 $\rho : U \longrightarrow \mathbb{R}_{\geqslant 0}$，我們使用 Radon-Nikodym 微分來定義在 U 上的測度 μ：

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = \rho.$$

證明若且唯若

$$\sum_{x \in f^{-1}(y)} \frac{\rho(x)}{\det Df(x)} = \rho(y), \quad \forall y \in U,$$

則 μ 對於 f 來說是個不變測度。

習題 8.14 ： 令 $E = [0, 1)$。考慮機率空間 $(E, \mathcal{B}(E), \mu)$。令 $B \in \mathcal{B}(E)$ 滿足 $\mu(B) > 0$，並假設存在由 E 的子區間構成的集合 $\mathcal{C}$ 使得：

(1) 對於所有 $\varepsilon > 0$ 及 $A \in \mathcal{B}(E)$ 皆存在由 $\mathcal{C}$ 中元素構成的可數互斥聯集 $J = \sqcup_{k \geqslant 0} J_k$ 使得 $\mu(A \Delta J) < \varepsilon$)；

(2) 存在 $\gamma > 0$ 使得對於所有 $C \in \mathcal{C}$，我們有 $\mu(C \cap B) \geqslant \gamma\mu(C)$。

證明 $\mu(B) = 1$。

習題 8.15 【連分數】： 對於任意 $x \in (0, 1)$，我們定義

$$A(x) = \lfloor \tfrac{1}{x} \rfloor \quad 且 \quad T(x) = \tfrac{1}{x} - A(x).$$

給定 $x \in (0, 1)$，對於所有 $n \geqslant 1$，我們定義 $a_n = a_n(x) := A(T^{n-1}(x))$，我們得到的是 x 的連分數表示式：

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \dots}}. \tag{8.2}$$

(1) 我們先來討論連分數的性質，並證明式 (8.2) 會收斂。首先，對於任意正整數 $n \geqslant 1$ 及正實數 $a_1, \dots, a_n > 0$，我們定義

$$[a_1, \dots, a_n] := \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}}.$$

給定 $x \in (0, 1)$。

(a) 假設對於正整數 $n \geqslant 1$，下列成立

$$T^k(x) \neq 0, \qquad 1 \leqslant k \leqslant n.$$

試證

$$x = [a_1 + T(x)] = [a_1, a_2 + T^2(x)] = \dots = [a_1, \dots, a_{n-1}, a_n + T^n(x)].$$

(b) 證明若且唯若 x 為有理數，則存在 $n \geqslant 1$ 使得 $T^n(x) = 0$。

Suppose that x is irrational. Define

$$\frac{p_n(x)}{q_n(x)} := [a_1(x), \dots, a_n(x)], \qquad \forall n \geqslant 1, \tag{8.3}$$

where $p_k(x)$ and $q_k(x)$ are coprime positive integers.

(c) Show the following recurrence relations,

$$\forall n \geqslant 1, \quad p_{n+1}(x) = q_n(Tx),$$
$$q_{n+1}(x) = a_1(x)q_n(Tx) + p_n(Tx).$$

Given a sequence $(a_n)_{n\geqslant 1}$ of positive integers, define the following matrices and integers $r_n, s_n, p_n, q_n \in \mathbb{Z}$ by

$$\forall n \geqslant 1, \qquad A_n := \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$
$$M_n := A_1 \dots A_n = \begin{pmatrix} r_n & p_n \\ s_n & q_n \end{pmatrix}, \tag{8.4}$$

For a matrix $M \in \mathrm{SL}_2(\mathbb{Z})$, its Möbius transformation (莫比烏斯變換) is defined by

$$M(x) = \frac{rx+p}{sx+q}, \qquad M = \begin{pmatrix} r & p \\ s & q \end{pmatrix}, \tag{8.5}$$

where M needs to be understood as a function from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$.

(d) Show that p_n and q_n are coprime and the following recurrence relations hold for all $n \geqslant 1$,

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \qquad p_0 = 0,$$
$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \qquad q_0 = 1.$$

(e) Show that $A_1 \dots A_n = (A_n \dots A_1)^T$ for all $n \geqslant 1$ and deduce that

$$\frac{q_{n-1}}{q_n} = [a_n, \dots, a_1].$$

(f) Prove that for a given irrational $x \in (0,1)$, the pair $(p_n(x), q_n(x))$ defined in Eq. (8.3) and the pair (p_n, q_n) defined by $(a_n = a_n(x))_{n\geqslant 1}$ in Eq. (8.4) are the same.

(g) Show that for all $n \geqslant 1$, we have $x = M_n(T^n(x))$ and

$$x - \frac{p_n}{q_n} = \frac{(-1)^n T^n(x)}{q_n(q_n + T^n(x)q_{n-1})}.$$

Deduce from above that the sequences $(\frac{p_{2n}}{q_{2n}})_{n\geqslant 0}$ and $(\frac{p_{2n+1}}{q_{2n+1}})_{n\geqslant 0}$ are respectively increasing and decreasing and that both converges to x.

假設 x 為無理數，我們定義

$$\frac{p_n(x)}{q_n(x)} := [a_1(x), \dots, a_n(x)], \qquad \forall n \geqslant 1, \tag{8.3}$$

其中 $p_k(x)$ 與 $q_k(x)$ 為互質的正整數。

(c) 證明下列遞迴關係：

$$\forall n \geqslant 1, \quad p_{n+1}(x) = q_n(Tx),$$
$$q_{n+1}(x) = a_1(x)q_n(Tx) + p_n(Tx).$$

給定正整數序列 $(a_n)_{n\geqslant 1}$，我們定義下列矩陣及整數 $r_n, s_n, p_n, q_n \in \mathbb{Z}$：

$$\forall n \geqslant 1, \qquad A_n := \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$
$$M_n := A_1 \dots A_n = \begin{pmatrix} r_n & p_n \\ s_n & q_n \end{pmatrix}, \tag{8.4}$$

對於任意矩陣 $M \in \mathrm{SL}_2(\mathbb{Z})$，我們定義莫比烏斯變換 (Möbius transformation)：

$$M(x) = \frac{rx+p}{sx+q}, \qquad M = \begin{pmatrix} r & p \\ s & q \end{pmatrix}, \tag{8.5}$$

其中我們將 M 視為由 $\overline{\mathbb{R}}$ 映射至 $\overline{\mathbb{R}}$ 的函數。

(d) 對於所有 $n \geqslant 1$，證明 p_n 及 q_n 互質以及下列遞迴關係：

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \qquad p_0 = 0,$$
$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \qquad q_0 = 1.$$

(e) 對於所有 $n \geqslant 1$，證明 $A_1 \dots A_n = (A_n \dots A_1)^T$ 並推得

$$\frac{q_{n-1}}{q_n} = [a_n, \dots, a_1].$$

(f) 證明在給定無理數 $x \in (0,1)$ 時，由式 (8.3) 所定義出來的數對 $(p_n(x), q_n(x))$ 及由 $(a_n = a_n(x))_{n\geqslant 1}$ 在式 (8.4) 中定義出來的數對 (p_n, q_n) 相同。

(g) 證明對於所有 $n \geqslant 1$，我們有 $x = M_n(T^n(x))$ 以及

$$x - \frac{p_n}{q_n} = \frac{(-1)^n T^n(x)}{q_n(q_n + T^n(x)q_{n-1})}.$$

由此推得 $(\frac{p_{2n}}{q_{2n}})_{n\geqslant 0}$ 及 $(\frac{p_{2n+1}}{q_{2n+1}})_{n\geqslant 0}$ 分別為遞增及遞減數列，且皆收斂至 x。

(2) Next, we will discuss the behavior of the real numbers in $(0,1)$ under the transformation T.

(a) Show that the Lebesgue measure λ is not invariant for T. Hint: one may show that $\lambda(T^{-1}(0,\frac{1}{2})) = 2 - \ln 4$.

Define the following probability measure on $(0,1)$,

$$\mu(B) = \frac{1}{\log 2}\int_B \frac{\mathrm{d}x}{1+x}, \qquad \forall B \in \mathcal{B}((0,1)).$$

We want to understand the properties of T with respect to the measure μ.

(b) Use Exercise 8.13 to show that T is measure-preserving for μ.

Then, for all postivie integers $n \geqslant 1$ and $a_1, \dots, a_n \geqslant 1$, we define

$$\Delta(a_1,\dots,a_n) = \{x \in [0,1) : a_1(x) = a_1, \dots, a_n(x) = a_n\}.$$

At the same time, we also adapt the definitions from (1) to define coprime integers p_k and q_k (Eq. (8.3)), matrices A_n and M_n (Eq. (8.4)) and their corresponding Möbius transformation (Eq. (8.5)). Below, we fix $n \geqslant 1$ and $a_1, \dots, a_n \geqslant 1$.

(c) Show that $\Delta(a_1,\dots,a_n)$ is an interval of $[0,1)$ with endpoints

$$\frac{p_n}{q_n} \quad \text{and} \quad \frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

Which one is the left endpoint and which one is the right endpoint? Compute $\lambda(\Delta(a_1,\dots,a_n))$.

(d) Use (1) (e) to prove that $\mu(\Delta(a_n,\dots,a_1)) = \mu(\Delta(a_1,\dots,a_n))$.

(e) Given $0 \leqslant \alpha < \beta \leqslant 1$ and write $I = [\alpha,\beta)$. Show that we have, according to the parity of n, that

$$T^{-n}(I) \cap \Delta(a_1,\dots,a_n) = [M_n(\alpha), M_n(\beta)) \quad \text{or} \quad (M_n(\beta), M_n(\alpha)].$$

(f) Deduce from above that

$$\frac{\lambda(T^{-n}(I)\cap\Delta_n)}{\lambda(I)\lambda(\Delta_n)} = \frac{q_n(q_{n-1}+q_n)}{(q_{n-1}\beta + q_n)(q_{n-1}\alpha + q_n)} \in (\tfrac{1}{2}, 2).$$

(g) Show that for any $B \in \mathcal{B}([0,1))$, we have

$$\mu(T^{-n}(B)\cap\Delta_n) \geqslant \frac{\ln 2}{4}\mu(B)\mu(\Delta_n)$$

(h) Use Exercise 8.14 to show that T is ergodic with respect to μ.

(i) Show that there exists a constant $K > 0$ such that the following convergence holds $\lambda(\mathrm{d}x)$-a.s.,

$$\Big(\prod_{k=1}^n a_k(x)\Big)^{1/n} \xrightarrow[n\to\infty]{} K.$$

This constant is called Khintchine's constant. Compute its value.

(2) 接著，我們要討論在 $(0,1)$ 上的實數在變換 T 之下的行為。

(a) 證明勒貝格測度 λ 對 T 並不是個不變測度。提示：可以證明 $\lambda(T^{-1}(0,\frac{1}{2})) = 2 - \ln 4$。

定義在 $(0,1)$ 上的機率測度

$$\mu(B) = \frac{1}{\log 2}\int_B \frac{\mathrm{d}x}{1+x}, \qquad \forall B \in \mathcal{B}((0,1)).$$

我們想要了解 T 對於測度 μ 的性質。

(b) 利用習題 8.13 ，證明 T 對於 μ 是個測度守恆變換。

接著，對於所有正整數 $n \geqslant 1$ 及正整數 $a_1, \dots, a_n \geqslant 1$，我們定義

$$\Delta(a_1,\dots,a_n) = \{x \in [0,1) : a_1(x) = a_1, \dots, a_n(x) = a_n\}.$$

我們也以第 (1) 題方式，來定義互質的正整數 p_k 與 q_k（式 (8.3)）、矩陣 A_n 及 M_n（式 (8.4)）與相對應的莫比烏斯變換（式 (8.5)）。在下面，我們固定 $n \geqslant 1$ 及 $a_1, \dots, a_n \geqslant 1$。

(c) 證明 $\Delta(a_1,\dots,a_n)$ 是個在 $[0,1)$ 中的區間，且其兩端點為

$$\frac{p_n}{q_n} \quad \text{以及} \quad \frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

何者是左端點，何者是右端點？計算 $\lambda(\Delta(a_1,\dots,a_n))$。

(d) 利用 (1) (e) 小題，證明 $\mu(\Delta(a_n,\dots,a_1)) = \mu(\Delta(a_1,\dots,a_n))$。

(e) 給定 $0 \leqslant \alpha < \beta \leqslant 1$，記 $I = [\alpha,\beta)$。證明根據 n 的奇偶性，我們有

$$T^{-n}(I) \cap \Delta(a_1,\dots,a_n) = [M_n(\alpha), M_n(\beta)) \quad \text{或} \quad (M_n(\beta), M_n(\alpha)].$$

(f) 從上題推得

$$\frac{\lambda(T^{-n}(I)\cap\Delta_n)}{\lambda(I)\lambda(\Delta_n)} = \frac{q_n(q_{n-1}+q_n)}{(q_{n-1}\beta + q_n)(q_{n-1}\alpha + q_n)} \in (\tfrac{1}{2}, 2).$$

(g) 證明對於任意 $B \in \mathcal{B}([0,1))$，我們有

$$\mu(T^{-n}(B)\cap\Delta_n) \geqslant \frac{\ln 2}{4}\mu(B)\mu(\Delta_n)$$

(h) 使用習題 8.14 證明 T 對於 μ 具有遍歷性。

(i) 證明存在常數 $K > 0$ 使得下列收斂 $\lambda(\mathrm{d}x)$-a.s. 成立：

$$\Big(\prod_{k=1}^n a_k(x)\Big)^{1/n} \xrightarrow[n\to\infty]{} K.$$

此常數稱作 Khintchine 常數，並計算他的值。

(3) In the end (not a part of the exercise), we can use the properties obtained above to show the following results proven by Paul Lévy in 1929. The following convergences hold $\lambda(\mathrm{d}x)$-a.s.,

$$\lim_{n\to\infty}\frac{1}{n}\ln q_n(x)=\frac{\pi^2}{12\ln 2},$$
$$\lim_{n\to\infty}\frac{1}{n}\ln\lambda(\Delta_n)=-\frac{\pi^2}{6\ln 2},$$
$$\lim_{n\to\infty}\frac{1}{n}\ln\left|x-\frac{p_n}{q_n}\right|=-\frac{\pi^2}{6\ln 2}.$$

Exercise 8.16 (First-passage percolation) **:** Consider the lattice $\mathbb{Z}^d$ in dimension $d\geqslant 1$. Denote its vertex set by V and its edge set by E. We are given an i.i.d. family $(\tau(e))_{e\in E}$ of non-negative and integrable random variables, and we write $\tau(x,y)=\tau(y,x)=\tau(e)$ for any edge $e=\{x,y\}$. Given $x,y\in\mathbb{Z}^d$ and a path $\gamma: x_0=x,\dots,x_n=y$ connecting them where by path we mean $\{x_i,x_{i+1}\}\in E$ for all $0\leqslant i\leqslant n-1$. We define the travel time of the path

$$\tau(\gamma)=\sum_{i=0}^{n-1}\tau(x_i,x_{i+1}).$$

Given $x,y\in\mathbb{Z}^d$, we can define the first-passage time (首次通行時間)

$$T(x,y)=\inf\{\tau(\gamma):\gamma \text{ is a path connecting } x \text{ to } y\}.$$

(1) Given $x\in\mathbb{Q}^d$, show that the following limit exists almost surely and in L^1,

$$\mu(x):=\lim_{n\to\infty}\frac{T(0,\lfloor nx\rfloor)}{n}.$$

(2) Show that $\mu(x+y)\leqslant\mu(x)+\mu(y)$ for all $x,y\in\mathbb{Q}^d$.

(3) Show that $\mu(cx)=|c|\mu(x)$ for all $c\in\mathbb{Q}$ and $x\in\mathbb{Q}^d$.

(3) 最後（非習題），利用我們在前兩小題得到性質，可以證明下列由 Paul Lévy 在 1929 年得到的結果：$\lambda(\mathrm{d}x)$-a.s.，我們有

$$\lim_{n\to\infty}\frac{1}{n}\ln q_n(x)=\frac{\pi^2}{12\ln 2},$$
$$\lim_{n\to\infty}\frac{1}{n}\ln\lambda(\Delta_n)=-\frac{\pi^2}{6\ln 2},$$
$$\lim_{n\to\infty}\frac{1}{n}\ln\left|x-\frac{p_n}{q_n}\right|=-\frac{\pi^2}{6\ln 2}.$$

習題 8.16 【首次通過滲透模型】： 考慮 $d\geqslant 1$ 維度的網格 $\mathbb{Z}^d$，將其頂點集記作 V，邊集記作 E。我們給定 i.i.d. 非負可積的隨機變數序列 $(\tau(e))_{e\in E}$，並對於任意邊 $e=\{x,y\}$，我們記 $\tau(x,y)=\tau(y,x)=\tau(e)$。給定 $x,y\in\mathbb{Z}^d$ 及連接他們的路徑 $\gamma: x_0=x,\dots,x_n=y$ 滿足對於所有 $0\leqslant i\leqslant n-1$，我們有 $\{x_i,x_{i+1}\}\in E$，我們定義路徑旅行時間

$$\tau(\gamma)=\sum_{i=0}^{n-1}\tau(x_i,x_{i+1}).$$

給定 $x,y\in\mathbb{Z}^d$，我們可以定義首次通行時間 (first-passage time)

$$T(x,y)=\inf\{\tau(\gamma):\gamma \text{ 是條連接 } x \text{ 至 } y \text{ 的路徑}\}.$$

(1) 給定 $x\in\mathbb{Q}^d$，證明下列極限會 a.s. 收斂及在 L^1 中收斂：

$$\mu(x):=\lim_{n\to\infty}\frac{T(0,\lfloor nx\rfloor)}{n}.$$

(2) 證明對於所有 $x,y\in\mathbb{Q}^d$，我們有 $\mu(x+y)\leqslant\mu(x)+\mu(y)$。

(3) 證明對於所有 $c\in\mathbb{Q}$ 及 $x\in\mathbb{Q}^d$，我們有 $\mu(cx)=|c|\mu(x)$。