Chapter 8: Ergodic Theorem

Exercise 8.1 : Given a real sequence $(x_n)_{n\geqslant 0}$ with period $p \geqslant 1$ and define the discrete probability measure

$$\mathbb{P} := \frac{1}{p}\sum_{k=0}^{p-1} \delta_{(x_{n+k})_{n\geqslant 0}}.$$

Show that $\mathbb{P}$ is an invariant measure for the shift operator θ.

Exercise 8.2 : Using Kolmogorov's extension theorem (Theorem 3.1.25), explain how to extend a stationary sequence $(X_n)_{n\geqslant 0}$ to a stationary sequence $(X_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$.

Exercise 8.3 (Question 8.1.9) **:** Check that the invariant set $\mathcal{I}$ defined in Definition 8.1.8 is indeed a σ-algebra.

Exercise 8.4 : Given a positive integer $n \geqslant 2$. Please prove the following statements or find counterexamples.

(1) If φ is ergodic, is φ^n ergodic? If φ^n is ergodic, is φ ergodic?

(2) Same questions as above with "ergodic" replaced by "a measure-preserving transformation".

Exercise 8.5 (Question 8.1.11) **:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which we consider a measure-preserving transformation φ.

(1) A random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $\mathcal{I}$-measurable if and only if $X \circ \varphi = X$.

(2) φ is ergodic if and only if all the $\mathcal{I}$-measurable random variables $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are $\mathbb{P}$-a.s. constant.

Exercise 8.6 (Bernoulli system) **:** Given a positive integer $N \geqslant 1$ and $p_1, \ldots, p_N \geqslant 0$ such that $\sum_k p_k = 1$. Let $E = \{1, \ldots, N\}$ be a discrete state space, $\Omega = E^{\mathbb{Z}}$ be the sample space and $\mathcal{F} = \mathcal{P}(E)^{\mathbb{Z}}$, where $\mathcal{P}(E)$ is the set consisting of all the subsets of E. Define the shift operator θ as $\theta((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$ and the probability measure $\mathbb{P}$ satisfying

$$\mathbb{P}(x_k = n_k, \ldots, x_l = n_l) = \prod_{i=k}^{l} p_{n_i}, \qquad \forall k \leqslant l, \quad \forall n_k, \ldots, n_l \in E.$$

(1) Explain why the probability measure $\mathbb{P}$ is well-defined.

(2) Show that θ is a measure-preserving transformation on $(\Omega, \mathcal{F}, \mathbb{P})$.

(3) Show that θ is mixing.

(4) If we change the state space to $\Omega = E^{\mathbb{Z}_{\geqslant 0}}$, explain how to adapt the proofs above or disprove the above statements.

Exercise 8.7 : Two functions $f, g : [0,1] \longrightarrow [0,1]$ are said to be conjugated (共軛) if there exists a bijective (雙射) function $h : [0,1] \longrightarrow [0,1]$ such that

$$h \circ f = g \circ h. \tag{8.1}$$

(1) If f and g are conjugated and h satisfies Eq. (8.1), show $h \circ f^n = g^n \circ h$ for all $n \geqslant 1$.

Consider the tent map (帳篷函數),

$$g(x) = \begin{cases} 2x & \text{for } x < 1/2, \\ 2 - 2x & \text{for } x \geqslant 1/2, \end{cases}$$

and define the logistic map (單峰函數),

$$f(x) = 4x(1-x).$$

(2) Show that for the Lebesgue measure λ, g is measure-preserving.

(3) Show that the logistic function and the tent function are conjugated. Hint: consider $h(x) = \frac{1}{2}(1 - \cos \pi x)$.

(4) Define the measure μ as follows,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}.$$

Show that f is measure-preserving for μ.

Exercise 8.8 (Question 8.1.14) **:** Please find an example with $\mathcal{I} \subsetneq \mathcal{T}$.

Exercise 8.9 (Question 8.2.3) **:** We keep the notations in Theorem 8.2.1 and assume additionally that $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$. Prove the L^p convergence of the ergodic theorem.

Exercise 8.10 (Wiener's maximal inequality) **:** Let φ be a measure-preserving transformation on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and X be an integrable random variable in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. For all $n \geqslant 0$, define

$$X_n(\omega) = X(\varphi^n(\omega)), \qquad S_n(\omega) = \sum_{k=0}^{n-1} X_k(\omega),$$
$$A_n(\omega) = \frac{S_n(\omega)}{n}, \qquad D_n = \max\{A_1, \ldots, A_n\}.$$

Show that for all $\alpha > 0$ we have

$$\mathbb{P}(D_n > \alpha) \leqslant \frac{\mathbb{E}[|X|]}{\alpha}.$$

Exercise 8.11 (Normal numbers) **:** For any $x \in [0,1]$, we can find a unique sequence $(a_n(x))_{n \geqslant 0}$ such that

(a) $a_n(x) \in \{0,1\}$ for all $n \geqslant 0$ and $a_n(x)$ does not stabilize at 1;

(b) the equality $x = \sum_{k \geqslant 0} 2^{-k} a_k(x)$ holds.

We say that $(a_n(x))_{n \geqslant 0}$ is the binary expansion (二元展開式) of x. Given a real number $x \in [0,1]$, if the frequency of 0's in its binary expansion satisfies

$$\frac{1}{n} \operatorname{Card}\{1 \leqslant k \leqslant n : a_k(x) = 0\} \xrightarrow[n \to \infty]{} \frac{1}{2},$$

then we say that x is a normal number (正規數). Show that $\lambda(\mathrm{d}x)$-a.s., the real number $x \in [0,1]$ is normal.

Exercise 8.12 : Consider the unit circle $\mathbb{S}^1 = \{x \in \mathbb{C} : |x| = 1\}$ and the uniform probability distribution μ defined above. For any $\beta \in \mathbb{R}$, we can define the rotation operator

$$\theta_\beta : e^{2\pi \mathrm{i} \alpha} \mapsto e^{2\pi \mathrm{i}(\alpha + \beta)}.$$

(1) Show that if β is rational, then θ_β is not ergodic.

(2) Assume that β is irrational and consider $f \in L^2(\mathbb{S}^1, \mathcal{B}(\mathbb{S}^1), \mu)$, show the ergodicity of θ_β using the uniqueness of the Fourier series of f. (Hint: use Exercise 8.5.)

Next, we will use a more direct method to prove that θ_β is ergodic when β is irrational,. First, we know the following property holds. Given $A \in \mathcal{B}(\mathbb{S}^1)$, for any $\varepsilon > 0$, we can find a countable disjoint union $(J_k)_{k \geqslant 0}$ such that

$$\mu(A \Delta J) < \varepsilon, \qquad J = \bigsqcup_{k \geqslant 0} J_k.$$

Given an irrational number β.

(3) Show that the sequence $(x_n = n\beta \mod 1)_{n \geqslant 0}$ is dense in $[0,1)$.

(4) Given $A \in \mathcal{B}(\mathbb{S}^1)$ such that $\mu(A) > 0$. Show that for all $\delta > 0$, there exists an interval J of $\mathbb{S}^1$ with $\mu(A \cap J) > (1-\delta)\mu(J)$.

(5) Deduce that for $A \in \mathcal{I}_{\theta_\beta}$, if $\mu(A) > 0$, then $\mu(A) = 1$.

Exercise 8.13 : Let $U \subset \mathbb{R}$ be an open set and $f : U \longrightarrow U$ be a local $\mathcal{C}^1$ diffeomorphism (微分同胚). Given a non-negative continuous function $\rho : U \longrightarrow \mathbb{R}_{\geqslant 0}$ on U, we define a measure μ on U using the Radon-Nikodym derivative,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = \rho.$$

Show that μ is an invariant measure for f if and only if

$$\sum_{x \in f^{-1}(y)} \frac{\rho(x)}{\det Df(x)} = \rho(y), \quad \forall y \in U.$$

Exercise 8.14 : Let $E = [0,1)$. Consider a probability space $(E, \mathcal{B}(E), \mu)$. Let $B \in \mathcal{B}(E)$ be such that $\mu(B) > 0$ and assume that there exists a collection of subintervals of E denoted $\mathcal{C}$ such that

(1) for all $\varepsilon > 0$ and $A \in \mathcal{B}(E)$, there exist a countable union of disjoint elements of $\mathcal{C}$, denoted $J = \sqcup_{k \geqslant 0} J_k$, such that $\mu(A \Delta J) < \varepsilon$;

(2) there exists $\gamma > 0$ such that $\mu(C \cap B) \geqslant \gamma \mu(C)$ for all $C \in \mathcal{C}$.

Prove that $\mu(B) = 1$.

Exercise 8.15 (Continued fraction) **:** For any $x \in (0,1)$, define

$$A(x) = \lfloor \tfrac{1}{x} \rfloor \quad \text{and} \quad T(x) = \tfrac{1}{x} - A(x).$$

Given $x \in (0,1)$ and define $a_n = A(T^{n-1}(x))$ for all $n \geqslant 0$, then we obtain the representation of x in continued fraction,

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \dots}}. \tag{8.2}$$

(1) We start by investigating some properties of continued fractions and show that Eq. (8.2) converges. First, for any positive integer $n \geqslant 1$ and positive real numbers $a_1, \dots, a_n > 0$, we define

$$[a_1, \dots, a_n] := \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}}.$$

Given $x \in (0,1)$.

(a) Assume the following holds for a given positive integer $n \geqslant 1$,

$$T^k(x) \neq 0, \qquad 1 \leqslant k \leqslant n.$$

Prove that

$$x = [a_1 + T(x)] = [a_1, a_2 + T^2(x)] = \dots = [a_1, \dots, a_{n-1}, a_n + T^n(x)].$$

(b) Show that there exists $n \geqslant 1$ such that $T^n(x) = 0$ if and only if x is rational.

Suppose that x is irrational. Define

$$\frac{p_n(x)}{q_n(x)} := [a_1(x), \dots, a_n(x)], \qquad \forall n \geqslant 1, \tag{8.3}$$

where $p_k(x)$ and $q_k(x)$ are coprime positive integers.

(c) Show the following recurrence relations,

$$\begin{aligned} \forall n \geqslant 1, \quad & p_{n+1}(x) = q_n(Tx), \\ & q_{n+1}(x) = a_1(x) q_n(Tx) + p_n(Tx). \end{aligned}$$

Given a sequence $(a_n)_{n \geqslant 1}$ of positive integers, define the following matrices and integers

$r_n, s_n, p_n, q_n \in \mathbb{Z}$ by

$$\forall n \geqslant 1, \qquad A_n := \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$
$$M_n := A_1 \dots A_n = \begin{pmatrix} r_n & p_n \\ s_n & q_n \end{pmatrix}, \tag{8.4}$$

For a matrix $M \in \mathrm{SL}_2(\mathbb{Z})$, its Möbius transformation (莫比烏斯變換) is defined by

$$M(x) = \frac{rx+p}{sx+q}, \qquad M = \begin{pmatrix} r & p \\ s & q \end{pmatrix}, \tag{8.5}$$

where M needs to be understood as a function from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$.

(d) Show that p_n and q_n are coprime and the following recurrence relations hold for all $n \geqslant 1$,

$$\begin{aligned} p_{n+1} &= a_{n+1}p_n + p_{n-1}, \qquad & p_0 &= 0, \\ q_{n+1} &= a_{n+1}q_n + q_{n-1}, \qquad & q_0 &= 1. \end{aligned}$$

(e) Show that $A_1 \dots A_n = (A_n \dots A_1)^T$ for all $n \geqslant 1$ and deduce that

$$\frac{q_{n-1}}{q_n} = [a_n, \dots, a_1].$$

(f) Prove that for a given irrational $x \in (0,1)$, the pair $(p_n(x), q_n(x))$ defined in Eq. (8.3) and the pair (p_n, q_n) defined by $(a_n = a_n(x))_{n\geqslant 1}$ in Eq. (8.4) are the same.

(g) Show that for all $n \geqslant 1$, we have $x = M_n(T^n(x))$ and

$$x - \frac{p_n}{q_n} = \frac{(-1)^n T^n(x)}{q_n(q_n + T^n(x)q_{n-1})}.$$

Deduce from above that the sequences $(\frac{p_{2n}}{q_{2n}})_{n\geqslant 0}$ and $(\frac{p_{2n+1}}{q_{2n+1}})_{n\geqslant 0}$ are respectively increasing and decreasing and that both converges to x.

(2) Next, we will discuss the behavior of the real numbers in $(0,1)$ under the transformation T.

(a) Show that the Lebesgue measure λ is not invariant for T. Hint: one may show that $\lambda(T^{-1}(0,\frac{1}{2})) = 2 - \ln 4$.

Define the following probability measure on $(0,1)$,

$$\mu(B) = \frac{1}{\log 2}\int_B \frac{\mathrm{d}x}{1+x}, \qquad \forall B \in \mathcal{B}((0,1)).$$

We want to understand the properties of T with respect to the measure μ.

(b) Use Exercise 8.13 to show that T is measure-preserving for μ.

Then, for all postivie integers $n \geqslant 1$ and $a_1, \dots, a_n \geqslant 1$, we define

$$\Delta(a_1, \dots, a_n) = \{x \in [0,1) : a_1(x) = a_1, \dots, a_n(x) = a_n\}.$$

At the same time, we also adapt the definitions from (1) to define coprime integers p_k and

q_k (Eq. (8.3)), matrices A_n and M_n (Eq. (8.4)) and their corresponding Möbius transformation (Eq. (8.5)). Below, we fix $n \geqslant 1$ and $a_1, \dots, a_n \geqslant 1$.

(c) Show that $\Delta(a_1, \dots, a_n)$ is an interval of $[0, 1)$ with endpoints

$$\frac{p_n}{q_n} \quad \text{and} \quad \frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

Which one is the left endpoint and which one is the right endpoint? Compute $\lambda(\Delta(a_1, \dots, a_n))$.

(d) Use (1) (e) to prove that $\mu(\Delta(a_n, \dots, a_1)) = \mu(\Delta(a_1, \dots, a_n))$.

(e) Given $0 \leqslant \alpha < \beta \leqslant 1$ and write $I = [\alpha, \beta)$. Show that we have, according to the parity of n, that

$$T^{-n}(I) \cap \Delta(a_1, \dots, a_n) = [M_n(\alpha), M_n(\beta)) \quad \text{or} \quad (M_n(\beta), M_n(\alpha)].$$

(f) Deduce from above that

$$\frac{\lambda(T^{-n}(I) \cap \Delta_n)}{\lambda(I)\lambda(\Delta_n)} = \frac{q_n(q_{n-1} + q_n)}{(q_{n-1}\beta + q_n)(q_{n-1}\alpha + q_n)} \in (\tfrac{1}{2}, 2).$$

(g) Show that for any $B \in \mathcal{B}([0, 1))$, we have

$$\mu(T^{-n}(B) \cap \Delta_n) \geqslant \frac{\ln 2}{4}\mu(B)\mu(\Delta_n)$$

(h) Use Exercise 8.14 to show that T is ergodic with respect to μ.

(i) Show that there exists a constant $K > 0$ such that the following convergence holds $\lambda(\mathrm{d}x)$-a.s.,

$$\Big(\prod_{k=1}^{n} a_k(x)\Big)^{1/n} \xrightarrow[n\to\infty]{} K.$$

This constant is called Khintchine's constant. Compute its value.

(3) In the end (not a part of the exercise), we can use the properties obtained above to show the following results proven by Paul Lévy in 1929. The following convergences hold $\lambda(\mathrm{d}x)$-a.s.,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \ln q_n(x) &= \frac{\pi^2}{12 \ln 2}, \\
\lim_{n\to\infty} \frac{1}{n} \ln \lambda(\Delta_n) &= -\frac{\pi^2}{6 \ln 2}, \\
\lim_{n\to\infty} \frac{1}{n} \ln \Big| x - \frac{p_n}{q_n} \Big| &= -\frac{\pi^2}{6 \ln 2}.
\end{aligned}$$

Exercise 8.16 (First-passage percolation) **:** Consider the lattice $\mathbb{Z}^d$ in dimension $d \geqslant 1$. Denote its vertex set by V and its edge set by E. We are given an i.i.d. family $(\tau(e))_{e\in E}$ of non-negative and integrable random variables, and we write $\tau(x,y) = \tau(y,x) = \tau(e)$ for any edge $e = \{x,y\}$. Given $x,y \in \mathbb{Z}^d$ and a path $\gamma : x_0 = x, \dots, x_n = y$ connecting them where by path we mean $\{x_i, x_{i+1}\} \in E$ for all $0 \leqslant i \leqslant n-1$. We define the travel time of the path

$$\tau(\gamma) = \sum_{i=0}^{n-1} \tau(x_i, x_{i+1}).$$

Given $x, y \in \mathbb{Z}^d$, we can define the first-passage time (首次通行時間)

$$T(x,y) = \inf\{\tau(\gamma) : \gamma \text{ is a path connecting } x \text{ to } y\}.$$

(1) Given $x \in \mathbb{Q}^d$, show that the following limit exists almost surely and in L^1,

$$\mu(x) := \lim_{n\to\infty} \frac{T(0, \lfloor nx \rfloor)}{n}.$$

(2) Show that $\mu(x+y) \leqslant \mu(x) + \mu(y)$ for all $x, y \in \mathbb{Q}^d$.

(3) Show that $\mu(cx) = |c|\mu(x)$ for all $c \in \mathbb{Q}$ and $x \in \mathbb{Q}^d$.