Exercise 9.1 : Let B be the one-dimensional standard Brownian motion. Fix $t > s \geqslant 0$ and find the value of $\mathbb{P}(B_s > 0, B_t > 0)$.

Exercise 9.2 (Time reversal) **:** Let $B = (B_t)_{t\in[0,1]}$ be a one-dimensional Brownian motion defined on the time interval $[0,1]$. Define the random process B' by setting $B'_t = B_1 - B_{1-t}$ for all $t \in [0,1]$. Show that B and B' are equal in distribution (in the space of functions $\mathcal{C}([0,1],\mathbb{R})$).

Exercise 9.3 : Let B be the one-dimensional standard Brownian motion. Define the random process $W = (W_t)_{t\geqslant 0}$ as

$$\forall t \geqslant 0, \qquad W_t = \int_0^t B_s \,\mathrm{d}s.$$

Fix $t > 0$. Compute $\mathbb{E}[W_t]$ and $\mathbb{E}[W_t^2]$ then find the distribution of W_t.

Exercise 9.4 : Given a random process $(X_t)_{0\leqslant t\leqslant 1}$. Assume that there exist $\alpha, \beta > 0$ and $K > 0$ such that

$$\mathbb{E}[|X_s - X_t|^\beta] \leqslant K|t-s|^{1+\alpha}, \qquad \forall s,t \in [0,1].$$

Recall the definition of the dyadic set in Theorem 9.2.2,

$$\mathcal{D} = \bigcup_{n\geqslant 0} \mathcal{D}_n, \qquad \mathcal{D}_n = \{\tfrac{k}{2^n} : 0 \leqslant k \leqslant 2^n\}.$$

Given $\gamma < \frac{\alpha}{\beta}$. Define the following events,

$$\forall n \geqslant 1, \qquad G_n = \{|X_{(i+1)/2^n} - X_{i/2^n}| \leqslant 2^{-\gamma n} \text{ for all } 0 \leqslant i \leqslant 2^n - 1\}.$$

(1) Show that there exists $\lambda > 0$ such that $\mathbb{P}(G_n^c) \leqslant K2^{-n\lambda}$.

(2) Fix a positive integer $N \geqslant 1$. Let $H_N = \cap_{n=N}^\infty G_n$. Prove that on H_N, for all $q, r \in \mathcal{D}$ satisfying $|q-r| < 2^{-N}$, we have
$$|X_q - X_r| \leqslant \frac{3}{1-2^{-\gamma}} |q-r|^\gamma.$$

(3) Deduce from the previous question that almost surely there exists a constant $C(\omega)$ such that
$$|X_q - X_r| \leqslant C|q-r|^\gamma, \qquad \forall q,r \in \mathcal{D}.$$

(4) Show that the Brownian motion is γ–Hölder continuous for all $\gamma < \frac{1}{2}$.

Exercise 9.5 : Let $S_1 = \sup_{0\leqslant t\leqslant 1} B_t$. Show that the following convergence in distribution holds,

$$\Big(\int_0^t e^{B_s}\,\mathrm{d}s\Big)^{1/\sqrt{t}} \xrightarrow[t\to\infty]{} e^{S_1}.$$

習題 9.1 ： 令 B 為一維的標準布朗運動。固定 $t > s \geqslant 0$ 並求 $\mathbb{P}(B_s > 0, B_t > 0)$。

習題 9.2 【時間反轉】： 令 $B = (B_t)_{t\in[0,1]}$ 為在時間 $[0,1]$ 上的一維布朗運動。定義隨機過程 B' 為對於所有 $t \in [0,1]$，我們設 $B'_t = B_1 - B_{1-t}$。證明 B 與 B' 有相同的分佈（在函數空間 $\mathcal{C}([0,1],\mathbb{R})$ 中）。

習題 9.3 ： 令 B 為一維的標準布朗運動。定義隨機過程 $W = (W_t)_{t\geqslant 0}$，定義做

$$\forall t \geqslant 0, \qquad W_t = \int_0^t B_s \,\mathrm{d}s.$$

固定 $t > 0$，求 $\mathbb{E}[W_t]$ 及 $\mathbb{E}[W_t^2]$ 以及 W_t 的分佈。

習題 9.4 ： 給定隨機過程 $(X_t)_{0\leqslant t\leqslant 1}$。假設存在 $\alpha, \beta > 0$ 及 $K > 0$ 使得

$$\mathbb{E}[|X_s - X_t|^\beta] \leqslant K|t-s|^{1+\alpha}, \qquad \forall s,t \in [0,1].$$

回顧我們在定理 9.2.2 中二元集合的定義

$$\mathcal{D} = \bigcup_{n\geqslant 0} \mathcal{D}_n, \qquad \mathcal{D}_n = \{\tfrac{k}{2^n} : 0 \leqslant k \leqslant 2^n\}.$$

給定 $\gamma < \frac{\alpha}{\beta}$。定義下列事件

$$\forall n \geqslant 1, \qquad G_n = \{|X_{(i+1)/2^n} - X_{i/2^n}| \leqslant 2^{-\gamma n} \text{ 對於所有 } 0 \leqslant i \leqslant 2^n - 1\}.$$

(1) 證明存在 $\lambda > 0$ 使得 $\mathbb{P}(G_n^c) \leqslant K2^{-n\lambda}$。

(2) 固定正整數 $N \geqslant 1$，令 $H_N = \cap_{n=N}^\infty G_n$，求證在 H_N 之上，對於所有 $q, r \in \mathcal{D}$ 滿足 $|q-r| < 2^{-N}$，我們有
$$|X_q - X_r| \leqslant \frac{3}{1-2^{-\gamma}} |q-r|^\gamma.$$

(3) 由上個小題推得下列敘述：幾乎必然存在常數 $C(\omega)$ 使得
$$|X_q - X_r| \leqslant C|q-r|^\gamma, \qquad \forall q,r \in \mathcal{D}.$$

(4) 證明布朗運動對於所有 $\gamma < \frac{1}{2}$ 皆是 γ–Hölder 連續的。

習題 9.5 ： 令 $S_1 = \sup_{0\leqslant t\leqslant 1} B_t$。證明我們有下列的分佈收斂：

$$\Big(\int_0^t e^{B_s}\,\mathrm{d}s\Big)^{1/\sqrt{t}} \xrightarrow[t\to\infty]{} e^{S_1}.$$

Exercise 9.6 (Brownian motion is not with finite variation) : Given a function $f:\mathbb{R}_+\longrightarrow\mathbb{R}$. We say that f is a function with finite variation (有限變異函數) if for any closed interval $[a,b]$, the sum

$$\sum_{i=0}^{p-1}|f(t_{i+1})-f(t_i)|$$

is bounded for any positive integer $p\geqslant 1$ and any subdivirion (子分割) $a=t_0<t_1<\cdots<t_p=b$. Consider the one-dimensional standard Brownian motion, real numbers $b>a\geqslant 0$ and let

$$\forall n\geqslant 0,\qquad X_n=\sum_{k=1}^{2^n}(B_{a+k(b-a)2^{-n}}-B_{a+(k-1)(b-a)2^{-n}})^2.$$

Compute the expectation and the variance of the random variable X_n, find the a.s. limit of the sequence $(X_n)_{n\geqslant 0}$ and deduce that the function $t\mapsto B_t$ is not of finite variation on any non-empty interval.

Exercise 9.7 (Question 9.2.10) : How to modify the proof of Theorem 9.2.9 to deduce the following properties?

(1) For any function $f:[0,1)\longrightarrow\mathbb{R}$, we define its upper right derivative (上右微分) and lower right derivative (下右微分) as

$$\forall t\in[0,1),\qquad D^*f(t)=\limsup_{h\downarrow 0}\frac{f(t+h)-f(t)}{h},$$
$$D_*f(t)=\liminf_{h\downarrow 0}\frac{f(t+h)-f(t)}{h}.$$

Show that for the Brownian motion, almost surely for all $t\in[0,1)$, we have

$$D^*B_t=+\infty\qquad\text{or}\qquad D_*B_t=-\infty.$$

(2) Show that for any $k\geqslant 3$, for all $\gamma>\frac{1}{2}+\frac{1}{k}$, the trajectory of the Brownian motion is almost surely not γ–Hölder continuous.

Exercise 9.8 (Question 9.2.11) : Let us define the following random set

$$\mathcal{H}_\gamma(\omega)=\{t\geqslant 0: s\mapsto B_s(\omega)\text{ is }\ \gamma\text{–Hölder continuous at }\ t\}.$$

(1) Show that $\mathbb{P}(\mathcal{H}_\gamma=[0,\infty))=1$ for all $\gamma<\frac{1}{2}$.

(2) Show that $\mathbb{P}(\mathcal{H}_\gamma=\varnothing)=1$ for all $\gamma>\frac{1}{2}$.

(3) Show that $\mathbb{P}(t\in\mathcal{H}_{1/2})=0$ for all $t\geqslant 0$.

(4) Burgess Davis proved in 1983 that $\mathbb{P}(\mathcal{H}_{1/2}\neq\varnothing)=1$. Please explain why it is not contradictory to (3)?

習題 9.6 【布朗運動不是有限變異函數】： 給定函數 $f:\mathbb{R}_+\longrightarrow\mathbb{R}$，若對於任意閉區間 $[a,b]$，任意正整數 $p\geqslant 1$ 及任意子分割 (subdivirion) $a=t_0<t_1<\cdots<t_p=b$，下列和為有界

$$\sum_{i=0}^{p-1}|f(t_{i+1})-f(t_i)|$$

則我們說 f 是個有限變異函數 (function with finite variation)。考慮一維的標準布朗運動，實數 $b>a\geqslant 0$ 並令

$$\forall n\geqslant 0,\qquad X_n=\sum_{k=1}^{2^n}(B_{a+k(b-a)2^{-n}}-B_{a+(k-1)(b-a)2^{-n}})^2.$$

計算隨機變數 X_n 的期望值及變異數並求序列 $(X_n)_{n\geqslant 0}$ 的 a.s. 極限，接著由此推得函數 $t\mapsto B_t$ 在任意非空區間上，皆不是個有限變異函數。

習題 9.7 【問題 9.2.10 】： 如何修改定理 9.2.9 的證明，進而推得下列性質。

(1) 對於任意函數 $f:[0,1)\longrightarrow\mathbb{R}$，我們定義上右微分 (upper right derivative) 及下右微分 (lower right derivative)：

$$\forall t\in[0,1),\qquad D^*f(t)=\limsup_{h\downarrow 0}\frac{f(t+h)-f(t)}{h},$$
$$D_*f(t)=\liminf_{h\downarrow 0}\frac{f(t+h)-f(t)}{h}.$$

試證對布朗運動來說，幾乎必然，對於所有 $t\in[0,1)$，我們有

$$D^*B_t=+\infty\qquad\text{或}\qquad D_*B_t=-\infty.$$

(2) 證明對於任意 $k\geqslant 3$，對所有 $\gamma>\frac{1}{2}+\frac{1}{k}$，布朗運動的軌跡幾乎必然不會是 γ–Hölder 連續的。

習題 9.8 【問題 9.2.11 】： 我們可以定義下列隨機集合

$$\mathcal{H}_\gamma(\omega)=\{t\geqslant 0: s\mapsto B_s(\omega)\text{ 在 }t\text{ 是 }\gamma\text{–Hölder 連續的}\}.$$

(1) 證明對於所有 $\gamma<\frac{1}{2}$，我們有 $\mathbb{P}(\mathcal{H}_\gamma=[0,\infty))=1$。

(2) 證明對於所有 $\gamma>\frac{1}{2}$，我們有 $\mathbb{P}(\mathcal{H}_\gamma=\varnothing)=1$。

(3) 證明對於所有 $t\geqslant 0$，我們有 $\mathbb{P}(t\in\mathcal{H}_{1/2})=0$。

(4) Burgess Davis 在 1983 年證明了 $\mathbb{P}(\mathcal{H}_{1/2}\neq\varnothing)=1$，請解釋為何與 (3) 沒有矛盾？

Exercise 9.9 (Time inversion) : If B is a standard Brownian motion started from 0, show that the random process

$$X_0 = 0 \qquad \text{and} \qquad X_t = tB_{1/t}, \quad \forall t > 0,$$

is also a Brownian motion with the same property.

Exercise 9.10 (Law of the iterated logarithm) : Let B be the one-dimensional standard Brownian motion. Our goal is to prove the following result,

$$\limsup_{t\to\infty} \frac{B_t}{h(t)} = 1 \quad \text{a.s.}, \qquad h(t) = \sqrt{2t\ln\ln(t)}. \tag{9.1}$$

For all $t \geqslant 0$, let us define $S_t = \sup_{0\leqslant s\leqslant t} B_s$.

(1) Show that for all $t > 0$, we have

$$\mathbb{P}(S_t > u\sqrt{t}) \sim \sqrt{\frac{2}{\pi}}\frac{e^{-u^2/2}}{u}, \quad u \to +\infty.$$

(2) Let real numbers r and c such that $1 < r < c^2$. Observe the behavior of

$$\mathbb{P}(S_{r^n} > ch(r^{n-1})), \quad n \to \infty$$

and deduce that

$$\limsup_{t\to\infty} \frac{B_t}{h(t)} \leqslant 1.$$

(3) Show that almost surely there exists an infinite values of n such that

$$B_{r^n} - B_{r^{n-1}} \geqslant \sqrt{\frac{r-1}{r}}h(r^n).$$

Deduce Eq. (9.1) from this.

(4) Prove that $B_t/h(t)$ does not converge almost surely but does converge in probability. Find its limit for the convergence in probability. Compare to the law of large numbers and the central limit theorem, what can you say about this result?

(5) Show the following result on the local modulus of continuity of the Brownian motion,

$$\forall t \geqslant 0, \qquad \limsup_{h\downarrow 0} \frac{|B_{t+h} - B_t|}{\sqrt{2h\ln\ln(1/h)}} = 1 \quad \text{a.s.}$$

習題 9.9 【時間倒轉】：若 B 是個一維由 0 出發的標準布朗運動，證明定義如下的隨機過程也是個有相同性質的布朗運動：

$$X_0 = 0 \qquad \text{以及} \qquad X_t = tB_{1/t}, \quad \forall t > 0.$$

習題 9.10 【對數迭代律】：令 B 為一維的標準布朗運動，我們的目的是證明下列結果

$$\limsup_{t\to\infty} \frac{B_t}{h(t)} = 1 \quad \text{a.s.}, \qquad h(t) = \sqrt{2t\ln\ln(t)}. \tag{9.1}$$

對於所有 $t \geqslant 0$，我們定義 $S_t = \sup_{0\leqslant s\leqslant t} B_s$。

(1) 證明對於所有 $t > 0$，我們有

$$\mathbb{P}(S_t > u\sqrt{t}) \sim \sqrt{\frac{2}{\pi}}\frac{e^{-u^2/2}}{u}, \quad u \to +\infty.$$

(2) 令實數 r 及 c 滿足 $1 < r < c^2$，觀察

$$\mathbb{P}(S_{r^n} > ch(r^{n-1})), \quad n \to \infty$$

的行為，並推得

$$\limsup_{t\to\infty} \frac{B_t}{h(t)} \leqslant 1.$$

(3) 證明幾乎必然存在無窮多個 n 使得

$$B_{r^n} - B_{r^{n-1}} \geqslant \sqrt{\frac{r-1}{r}}h(r^n).$$

由此推得式 (9.1)。

(4) 證明 $B_t/h(t)$ 的 a.s. 極限不存在，但會機率收斂，求其機率收斂的極限。想想看此結果與大數法則還有中央極限定理的關係，並針對此結果做評論。

(5) 證明下列關於布朗運動局部連續性尺度的結果：

$$\forall t \geqslant 0, \qquad \limsup_{h\downarrow 0} \frac{|B_{t+h} - B_t|}{\sqrt{2h\ln\ln(1/h)}} = 1 \quad \text{a.s.}$$