Chapter 9: Brownian Motion

Exercise 9.1 : Let B be the one-dimensional standard Brownian motion. Fix $t > s \geqslant 0$ and find the value of $\mathbb{P}(B_s > 0, B_t > 0)$.

Exercise 9.2 (Time reversal) **:** Let $B = (B_t)_{t\in[0,1]}$ be a one-dimensional Brownian motion defined on the time interval $[0, 1]$. Define the random process B' by setting $B'_t = B_1 - B_{1-t}$ for all $t \in [0, 1]$. Show that B and B' are equal in distribution (in the space of functions $\mathcal{C}([0, 1], \mathbb{R})$).

Exercise 9.3 : Let B be the one-dimensional standard Brownian motion. Define the random process $W = (W_t)_{t\geqslant 0}$ as

$$\forall t \geqslant 0, \qquad W_t = \int_0^t B_s \, \mathrm{d}s.$$

Fix $t > 0$. Compute $\mathbb{E}[W_t]$ and $\mathbb{E}[W_t^2]$ then find the distribution of W_t.

Exercise 9.4 : Given a random process $(X_t)_{0\leqslant t\leqslant 1}$. Assume that there exist $\alpha, \beta > 0$ and $K > 0$ such that

$$\mathbb{E}[|X_s - X_t|^\beta] \leqslant K|t - s|^{1+\alpha}, \qquad \forall s, t \in [0, 1].$$

Recall the definition of the dyadic set in Theorem 9.2.2,

$$\mathcal{D} = \bigcup_{n\geqslant 0} \mathcal{D}_n, \qquad \mathcal{D}_n = \{\tfrac{k}{2^n} : 0 \leqslant k \leqslant 2^n\}.$$

Given $\gamma < \frac{\alpha}{\beta}$. Define the following events,

$$\forall n \geqslant 1, \qquad G_n = \{|X_{(i+1)/2^n} - X_{i/2^n}| \leqslant 2^{-\gamma n} \text{ for all } 0 \leqslant i \leqslant 2^n - 1\}.$$

(1) Show that there exists $\lambda > 0$ such that $\mathbb{P}(G_n^c) \leqslant K2^{-n\lambda}$.

(2) Fix a positive integer $N \geqslant 1$. Let $H_N = \cap_{n=N}^{\infty} G_n$. Prove that on H_N, for all $q, r \in \mathcal{D}$ satisfying $|q - r| < 2^{-N}$, we have

$$|X_q - X_r| \leqslant \frac{3}{1 - 2^{-\gamma}} |q - r|^\gamma.$$

(3) Deduce from the previous question that almost surely there exists a constant $C(\omega)$ such that

$$|X_q - X_r| \leqslant C|q - r|^\gamma, \qquad \forall q, r \in \mathcal{D}.$$

(4) Show that the Brownian motion is γ–Hölder continuous for all $\gamma < \frac{1}{2}$.

Exercise 9.5 : Let $S_1 = \sup_{0\leqslant t\leqslant 1} B_t$. Show that the following convergence in distribution holds,

$$\Big(\int_0^t e^{B_s} \, \mathrm{d}s\Big)^{1/\sqrt{t}} \xrightarrow[t\to\infty]{} e^{S_1}.$$

Exercise 9.6 (Brownian motion is not with finite variation) **:** Given a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$. We say that f is a function with finite variation (**有限變異函數**) if for any closed interval $[a, b]$, the sum

$$\sum_{i=0}^{p-1} |f(t_{i+1}) - f(t_i)|$$

is bounded for any positive integer $p \geqslant 1$ and any subdivirion (**子分割**) $a = t_0 < t_1 < \cdots < t_p = b$. Consider the one-dimensional standard Brownian motion, real numbers $b > a \geqslant 0$ and let

$$\forall n \geqslant 0, \qquad X_n = \sum_{k=1}^{2^n} (B_{a+k(b-a)2^{-n}} - B_{a+(k-1)(b-a)2^{-n}})^2.$$

Compute the expectation and the variance of the random variable X_n, find the a.s. limit of the sequence $(X_n)_{n\geqslant 0}$ and deduce that the function $t \mapsto B_t$ is not of finite variation on any non-empty interval.

Exercise 9.7 (Question 9.2.10) **:** How to modify the proof of Theorem 9.2.9 to deduce the following properties?

(1) For any function $f : [0, 1) \longrightarrow \mathbb{R}$, we define its upper right derivative (**上右微分**) and lower right derivative (**下右微分**) as

$$\forall t \in [0, 1), \qquad D^* f(t) = \limsup_{h \downarrow 0} \frac{f(t+h) - f(t)}{h},$$
$$D_* f(t) = \liminf_{h \downarrow 0} \frac{f(t+h) - f(t)}{h}.$$

Show that for the Brownian motion, almost surely for all $t \in [0, 1)$, we have

$$D^* B_t = +\infty \qquad \text{or} \qquad D_* B_t = -\infty.$$

(2) Show that for any $k \geqslant 3$, for all $\gamma > \frac{1}{2} + \frac{1}{k}$, the trajectory of the Brownian motion is almost surely not γ–Hölder continuous.

Exercise 9.8 (Question 9.2.11) **:** Let us define the following random set

$$\mathcal{H}_\gamma(\omega) = \{t \geqslant 0 : s \mapsto B_s(\omega) \text{ is } \gamma\text{–Hölder continuous at } t\}.$$

(1) Show that $\mathbb{P}(\mathcal{H}_\gamma = [0, \infty)) = 1$ for all $\gamma < \frac{1}{2}$.

(2) Show that $\mathbb{P}(\mathcal{H}_\gamma = \varnothing) = 1$ for all $\gamma > \frac{1}{2}$.

(3) Show that $\mathbb{P}(t \in \mathcal{H}_{1/2}) = 0$ for all $t \geqslant 0$.

(4) Burgess Davis proved in 1983 that $\mathbb{P}(\mathcal{H}_{1/2} \neq \varnothing) = 1$. Please explain why it is not contradictory to (3)?

Exercise 9.9 (Time inversion) **:** If B is a standard Brownian motion started from 0, show that the random process

$$X_0 = 0 \qquad \text{and} \qquad X_t = tB_{1/t}, \quad \forall t > 0,$$

is also a Brownian motion with the same property.

Exercise 9.10 (Law of the iterated logarithm) **:** Let B be the one-dimensional standard Brownian motion. Our goal is to prove the following result,

$$\limsup_{t\to\infty} \frac{B_t}{h(t)} = 1 \quad \text{a.s.,} \qquad h(t) = \sqrt{2t\ln\ln(t)}. \tag{9.1}$$

For all $t \geqslant 0$, let us define $S_t = \sup_{0\leqslant s\leqslant t} B_s$.

(1) Show that for all $t > 0$, we have

$$\mathbb{P}(S_t > u\sqrt{t}) \sim \sqrt{\frac{2}{\pi}}\frac{e^{-u^2/2}}{u}, \quad u \to +\infty.$$

(2) Let real numbers r and c such that $1 < r < c^2$. Observe the behavior of

$$\mathbb{P}(S_{r^n} > ch(r^{n-1})), \quad n \to \infty$$

and deduce that

$$\limsup_{t\to\infty} \frac{B_t}{h(t)} \leqslant 1.$$

(3) Show that almost surely there exists an infinite values of n such that

$$B_{r^n} - B_{r^{n-1}} \geqslant \sqrt{\frac{r-1}{r}} h(r^n).$$

Deduce Eq. (9.1) from this.

(4) Prove that $B_t/h(t)$ does not converge almost surely but does converge in probability. Find its limit for the convergence in probability. Compare to the law of large numbers and the central limit theorem, what can you say about this result?

(5) Show the following result on the local modulus of continuity of the Brownian motion,

$$\forall t \geqslant 0, \qquad \limsup_{h\downarrow 0} \frac{|B_{t+h} - B_t|}{\sqrt{2h\ln\ln(1/h)}} = 1 \quad \text{a.s.}$$